

EXTREME LOCALISATION OF EIGENFUNCTIONS TO ONE-DIMENSIONAL HIGH-CONTRAST PERIODIC PROBLEMS WITH A DEFECT

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Abstract

Following a number of recent studies of resolvent and spectral convergence of non-uniformly elliptic families of differential operators describing the behaviour of periodic composite media with high contrast, we study the corresponding one-dimensional version that includes a “defect”: an inclusion of fixed size with a given set of material parameters. It is known that the spectrum of the purely periodic case without the defect and its limit, as the period ε goes to zero, has a band-gap structure. We consider a sequence of eigenvalues λ_ε that are induced by the defect and converge to a point λ_0 located in a gap of the limit spectrum for the periodic case. We show that the corresponding eigenfunctions are “extremely” localised to the defect, in the sense that the localisation exponent behaves as $\exp(-\nu/\varepsilon)$, $\nu > 0$, which has not been observed in the existing literature. In two- and three-dimensional configurations, whose one-dimensional cross-sections are described by the setting considered, this implies the existence of propagating waves that are localised to a vicinity of the defect. We also show that the unperturbed operators are norm-resolvent close to a degenerate operator on the real axis, which is described explicitly.

Keywords High-contrast homogenisation · Wave localisation · Spectrum · Decay estimates

1 Introduction

The question of whether a macroscopic perturbation of material properties in a periodic medium or structure (periodic composite) induces the existence of a localised solution (bound state) to the time-harmonic version of the equations of motion is of special importance from the physics, engineering and mathematical points of view. Depending on the application context, such a solution can have either an advantageous or undesirable effect on the behaviour of systems containing the related composite medium as a component. For example, in the context of photonic (phononic) crystal fibres, perturbations of this kind have been exploited for the transport of electromagnetic (elastic) energy over large distances with little loss into the surrounding space, see *e.g.* [14, 18]. In the mathematics literature, proofs of the existence or non-existence of such a localised solution have been carried out using the tools of the classical asymptotic analysis of the governing equations and spectral analysis of operators generated by the governing equations in various “natural” function spaces. The choice of the concrete class of equations and functions under study is usually motivated by the applications in mind, and several works that have marked the development of the related analytical techniques cover a wide range of operators and their relatively compact perturbations, *e.g.* [23], [3], [2], [10].

The present work is a study of localisation properties for a class of composite media that has been the subject of increasing interest in the mathematics and physics literature recently, in view of its relation to the so-called metamaterials, *e.g.* manufactured composites possessing negative refraction properties. It has been shown in [8] that the spectrum of a stratified high-contrast composite, represented mathematically by a one-dimensional periodic second-order differential equation, has an infinitely increasing number of gaps (lacunae) opening in the spectrum, in the limit of the small ratio ε between the period and the overall size of the composite. This analytical feature, analogous to the spectral property of multi-dimensional high-contrast periodic composites shown in [20], provides a mathematical recipe for the use of such materials in physics context or technologies where the presence of localised modes (generated by defects in the medium) has important practical implications. In the physical context of photonic crystal fibres and within the mathematical setup of multi-dimensional high-contrast media, this link has been studied in [13], [5], [6]. In the paper [13], a two-scale asymptotics for eigenfunctions of a high-contrast second-order elliptic differential operator with a finite-size perturbation (defect) was derived. It was shown that for eigenvalues λ in gaps of the spectrum of the (two-scale) operator representing the leading-order term of this asymptotics, there are sequences of eigenvalues of the finite-period problems that converge to λ as $\varepsilon \rightarrow 0$. The subsequent works [5], [6] developed a multiscale version of Agmon's approach [1] and proved that the corresponding eigenfunctions of the limit operator decay exponentially fast away from the defect. An important technical assumption in all these works is that the low-modulus inclusions in the composite have a positive distance to the boundary of the period cell, which is not possible to satisfy in one dimension.

In the more recent paper [8], a family of non-uniformly elliptic periodic one-dimensional problems with high contrast was studied, which in practically relevant situations corresponds to a stratified composite with alternating layers of homogeneous media with highly contrasting material properties. It was shown that the spectra of the corresponding operators converge, as $\varepsilon \rightarrow 0$, to the band-gap spectrum of a two-scale operator described explicitly in terms of the original material parameters. Introducing a finite-size defect D into the setup of [8], one is led to consider the operator

$$-(a_D^\varepsilon u)', \quad a_D^\varepsilon > 0,$$

where a_D^ε takes values of order one on D and is ε -periodic ($\varepsilon > 0$) in $\mathbb{R} \setminus D$ with alternating values of order one and ε^2 . As was mentioned in [8, Section 5.1], a formal analysis suggests that the rate of decay of eigenfunctions localised in the vicinity of the perturbation D is "accelerated exponential", rather than just exponential as in [6], in the sense that the decay exponent increases in absolute value as $\varepsilon \rightarrow 0$. The goal of the present work is to provide a rigorous proof of this property, formulated below as Theorem 2.4. In view of the above discussion, this new localisation property can be seen as a consequence of two features of the underlying periodic composite: loss of uniform ellipticity (via the presence of soft inclusions in a moderately stiff material) and the one-dimensional nature of the problem.

In addition to our main result, we formulate (Section 3) a new characterisation of the limit spectrum for the unperturbed family of problems in the whole space discussed in [8] and strengthen (Section 6) the result of [8] by proving an order-sharp norm-resolvent convergence estimate for this family (Theorem 2.2). In particular, this new estimate implies order-sharp uniform asymptotic estimates, as $\varepsilon \rightarrow 0$, for the related family of evolution semigroups, *cf. e.g.* [21] for a strong-convergence version of this kind of result.

2 Problem formulation and main results

For $\varepsilon, h \in (0, 1)$, we introduce the sets

$$\Omega_0^\varepsilon := \bigcup_{z \in \mathbb{Z}} (\varepsilon z, \varepsilon z + \varepsilon h), \quad \text{and} \quad \Omega_1^\varepsilon := \bigcup_{z \in \mathbb{Z}} (\varepsilon z + \varepsilon h, \varepsilon z + \varepsilon) = \mathbb{R} \setminus \overline{\Omega_0^\varepsilon},$$

and denote $Y_0 := (0, h)$, $Y_1 := (h, 1)$, $Y := (0, 1)$. We define the ε -periodic functions

$$a^\varepsilon(x) := \begin{cases} \varepsilon^2 a_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon, \\ a_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon, \end{cases} \quad \rho^\varepsilon(x) = \rho(\frac{x}{\varepsilon}), \quad \rho(y) := \begin{cases} \rho_0(y), & y \in Y_0, \\ \rho_1(y), & y \in Y_1, \end{cases} \quad (2.1)$$

for $a_j, a_j^{-1}, \rho_j, \rho_j^{-1} \in L^\infty(Y_j)$, $j = 0, 1$, periodic with period 1. It is convenient to set $a_0 \equiv 0$ on Y_1 and $a_1 \equiv 0$ and Y_0 , thus we can write, for example, $a^\varepsilon(x) = \varepsilon^2 a_0(x/\varepsilon) + a_1(x/\varepsilon)$. We denote $\Omega_0 := \bigcup_{z \in \mathbb{Z}} (Y_0 + z)$, $\Omega_1 := \bigcup_{z \in \mathbb{Z}} (Y_1 + z)$ and reserve the notation z for an integer, unless stated otherwise. We will refer to the sets $\Omega_0^\varepsilon, \Omega_0$ and $\Omega_1^\varepsilon, \Omega_1$ as the soft and stiff component, respectively.

For a positive Lebesgue-measurable function w on a Borel set $B \subset \mathbb{R}$, such that $w, w^{-1} \in L^\infty(B)$, we employ the notation $L_w^2(B)$ to indicate that the space $L^2(B)$ is equipped with the inner product

$$(u, v)_w := \int_B w u \bar{v}, \quad u, v \in L^2(B).$$

For a closed and semi-bounded sesquilinear form $\beta : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{C}$, the (self-adjoint) operator A associated to β is densely defined in $L_w^2(\mathbb{R})$ by the action $Au = f$, where for a given $f \in L_w^2(\mathbb{R})$, the function $u \in H^1(\mathbb{R})$ is the solution to the integral identity

$$\beta(u, v) = \int_{\mathbb{R}} w f \bar{v} \quad \forall v \in H^1(\mathbb{R}).$$

Henceforth, all function spaces we employ consist of complex-valued functions and are over \mathbb{C} .

For the sesquilinear form

$$\beta^\varepsilon(u, v) := \int_{\mathbb{R}} a^\varepsilon u' \bar{v}', \quad u, v \in H^1(\mathbb{R}),$$

we consider A^ε , the operator defined in $L_{\rho^\varepsilon}^2(\mathbb{R})$ and associated to β^ε . The spectrum $\sigma(A^\varepsilon)$ of A^ε is absolutely continuous and, by introducing the rescaled Floquet-Bloch transform \mathcal{U}_ε , see (6.82), we note that $\sigma(A^\varepsilon)$ admits the representation

$$\sigma(A^\varepsilon) = \bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta^\varepsilon),$$

where $\sigma(A_\theta^\varepsilon)$ is the spectrum of the $L_\rho^2(Y)$ densely-defined self-adjoint operator A_θ^ε associated to the form

$$\beta_\theta^\varepsilon(u, v) := \int_Y (a_0 + \varepsilon^{-2} a_1) u' \bar{v}',$$

acting in the space $H_\theta^1(Y)$ of functions $u \in H^1(Y)$ that are θ -quasiperiodic, *i.e.* such that $u(y) = \exp(i\theta y)v(y)$, $y \in Y$, for some 1-periodic function $v \in H^1(Y)$. For each ε, θ , the operator A_θ^ε has compact resolvent and consequently its spectrum $\sigma(A_\theta^\varepsilon)$ is discrete.

Consider the space

$$V_\theta := \{u \in H_\theta^1(Y) : u' = 0 \text{ on } Y_1\} \quad (2.2)$$

and its closure in $L_\rho^2(Y)$, which we denote by \overline{V}_θ , which we also equip with the norm of $L_\rho^2(Y)$. We introduce the densely defined operators A_θ in \overline{V}_θ given by $A_\theta u = f$, where for all $f \in \overline{V}_\theta$ the function u in the domain of A_θ is such that

$$\int_{Y_0} a_0 u' \bar{v}' = \int_Y \rho f \bar{v} \quad \forall v \in V_\theta. \quad (2.3)$$

For each θ , the operator A_θ has compact resolvent and so $\sigma(A_\theta)$ is discrete. In a recent work [8], see also Section 6 of the present manuscript, the spectrum $\sigma(A^\varepsilon)$ was shown to converge in the Hausdorff sense to the union of the spectra of the operators A_θ , *i.e.*

$$\lim_{\varepsilon \rightarrow 0} \sigma(A^\varepsilon) = \bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta). \quad (2.4)$$

Remark 2.1. $\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta)$ can be seen as the spectrum of a certain operator A^0 unitary equivalent to the direct integral of operators $\int^\oplus A_\theta$, see Appendix A for the details.

In Section 6, we construct infinite-order asymptotics (as $\varepsilon \rightarrow 0$) for the resolvents of A_θ^ε , uniform in θ , with respect to the H^1 -norm and, in particular, prove the following refinement of the result established in [8]:

Theorem 2.2. *The operator A_θ^ε norm-resolvent converges to A_θ , uniformly in θ , at the rate ε^2 . More precisely, there exists a constant $C > 0$ such that*

$$\|(A_\theta^\varepsilon + 1)^{-1}f - (A_\theta + 1)^{-1}P_\theta f\|_{L_\rho^2(Y)} \leq C\varepsilon^2\|f\|_{L_\rho^2(Y)} \quad \forall \theta \in [0, 2\pi), \quad f \in L_\rho^2(Y),$$

where P_θ is the orthogonal projection of $L_\rho^2(Y)$ onto $\overline{V_\theta}$.

Consequently, since the spectra $\sigma(A_\theta^\varepsilon)$ and $\sigma(A_\theta)$ are discrete, we have the following result: for each $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that

$$|\lambda_n^\varepsilon(\theta) - \lambda_n(\theta)| \leq c_n\varepsilon^2 \quad \forall \theta \in [0, 2\pi), \quad \varepsilon \in (0, 1).$$

Here, $\{\lambda_n^\varepsilon(\theta)\}_{n \in \mathbb{N}}$, $\{\lambda_n(\theta)\}_{n \in \mathbb{N}}$ are the eigenvalue sequences of A_θ^ε , A_θ , respectively, labelled in the increasing order¹. It follows that for sufficiently small ε , the spectrum $\sigma(A^\varepsilon)$ has a gap if the set $\bigcup_\theta \sigma(A_\theta)$ contains a gap. In Section 3 we give an example of a class of coefficients for which this set contains infinitely many gaps. Furthermore, we demonstrate that $\lambda \in \bigcup_\theta \sigma(A_\theta)$ if and only if the inequality

$$\left| v_1(h) + (a_0 v_2')(h) - \lambda v_2(h) \int_{Y_1} \rho_1 \right| \leq 2$$

holds. Here v_1 and v_2 are the (λ -dependent) solutions of

$$-(a_0 v_j')' = \lambda \rho_0 v_j \quad \text{on } Y_0, \quad j = 1, 2,$$

subject to the conditions

$$\begin{pmatrix} v_1(0) & v_2(0) \\ (a_0 v_1')(0) & (a_0 v_2')(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 2.3. *Note that any solution u of $-(a_0 u')' = \lambda \rho_0 u$ is absolutely continuous and so is its co-derivative $a_0 u'$. Hence, their value at any point y is well defined (unlike the value of a_0 or u' in general). This explains the use of notation $(a_0 v_j')(y)$, which we will hold to throughout the paper.*

Next, we introduce $d_-, d_+ \in \mathbb{R}$ and on the set $D = (d_-, d_+)$ replace the coefficients (2.1) by some uniformly positive and bounded functions a_D, ρ_D , namely we consider

$$a_D^\varepsilon(x) := \begin{cases} a_D(x), & x \in D, \\ a_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon \setminus D, \\ \varepsilon^2 a_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon \setminus D, \end{cases} \quad \rho_D^\varepsilon(x) := \begin{cases} \rho_D(x), & x \in D, \\ \rho_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon \setminus D, \\ \rho_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon \setminus D. \end{cases}$$

We shall study the spectrum of the operator A_D^ε defined in $L_{\rho_D^\varepsilon}^2(\mathbb{R})$ and associated to the form

$$\beta_D^\varepsilon(u, v) := \int_{\mathbb{R}} a_D^\varepsilon u' \overline{v'}, \quad u, v \in H^1(\mathbb{R}). \quad (2.5)$$

As this operator arises from a compact perturbation of the coefficients of A^ε , it is well-known that the essential spectra of A_D^ε and A^ε coincide, see *e.g.* [10]. For eigenvalues situated, for small values of ε , in the gaps of the essential spectrum of A_D^ε (equivalently, in the gaps of the essential spectrum of A^ε), we expect the eigenfunctions to be localised around the defect. In view of the above observation about the spectra of

¹Notice that all the eigenvalues are simple due to the one-dimensional nature of the corresponding problem.

A^ε and A_θ , $\theta \in [0, 2\pi)$, we are therefore interested in the analysis of eigenfunctions of A_D^ε corresponding to eigenvalues that are located in the gaps of the limit spectrum $\bigcup_\theta \sigma(A_\theta)$.

Consider the operator $A_{N,D}$ defined in $L^2_{\rho_D}(D)$ and associated to the form

$$\beta_{N,D}(u, v) := \int_D a_D u' \overline{v'}, \quad u, v \in H^1(D). \quad (2.6)$$

acting in $H^1(D)$. The functions from the domain of $A_{N,D}$ satisfy the Neumann condition on the boundary of D . We show that if the defect D is chosen so that the spectrum $\sigma(A_{N,D})$ has a non-empty intersection with $\mathbb{R} \setminus \bigcup_\theta \sigma(A_\theta)$, then for sufficiently small ε the operator A_D^ε has non-empty point spectrum. Notice that we can always choose a_D , ρ_D , d_- and d_+ such that this is true. Moreover, we demonstrate that for eigenvalue sequences that converge to a point in $\mathbb{R} \setminus \bigcup_\theta \sigma(A_\theta)$ the corresponding eigenfunctions are localised to a small neighbourhood of the defect. Namely, the eigenfunctions u_ε exhibit accelerated exponential decay outside the defect in the sense that the function $\exp(\text{dist}(x, D)\nu/\varepsilon)u_\varepsilon(x)$, $x \in \mathbb{R}$, is an element of $L^2(\mathbb{R} \setminus D)$ for sufficiently small ε , where the value ν is determined by the distance of the limit point of λ_ε to the set $\bigcup_\theta \sigma(A_\theta)$. These results are collated in the following theorem, which we prove in Sections 4, 5.

Theorem 2.4.

1. For every $\lambda_0 \in \sigma(A_{N,D}) \setminus \bigcup_\theta \sigma(A_\theta)$ (which is always simple) there exists a sequence of simple eigenvalues λ_ε of A_D^ε converging to λ_0 and constants $C_1, C_2 > 0$ such that

$$\begin{aligned} |\lambda_\varepsilon - \lambda_0| &\leq C_1 \varepsilon^{1/2}, \\ \left\| \sum_{j \in J_\varepsilon} c_j^\varepsilon u_{\varepsilon,j} - u_0 \right\|_{L^2(D)} &\leq C_2 \varepsilon^{1/2}, \\ \left\| \sum_{j \in J_\varepsilon} c_j^\varepsilon u_{\varepsilon,j} \right\|_{L^2(\mathbb{R} \setminus D)} &\leq C_2 \varepsilon^{1/2}, \end{aligned} \quad (2.7)$$

where u_0 is a normalised eigenfunction of $A_{N,D}$ corresponding to the eigenvalue λ_0 , the set

$$J_\varepsilon := \{j : |\lambda_{\varepsilon,j} - \lambda_0| \leq C_2 \varepsilon^{1/2}\} \quad (2.8)$$

is finite, and for each $j \in J_\varepsilon$ the function $u_{\varepsilon,j}$ is the $L^2_{\rho_D}(\mathbb{R})$ -normalised eigenfunction of A_D^ε with eigenvalue $\lambda_{\varepsilon,j}$.

2. Suppose that λ_ε is an eigenvalue of A_D^ε for each ε and that $\lambda_\varepsilon \rightarrow \lambda_0 \notin \lim_{\varepsilon \rightarrow 0} \sigma(A^\varepsilon) = \bigcup_\theta \sigma(A_\theta)$. Then the $L^2(\mathbb{R})$ -normalised eigenfunctions u_ε of A_D^ε corresponding to the eigenvalues λ_ε are localised in the following sense.

For $\nu > 0$, let $g_{\nu/\varepsilon}$ denote the exponentially growing function

$$g_{\nu/\varepsilon}(x) := \begin{cases} 1, & x \in D, \\ \exp\left(\frac{\nu}{\varepsilon} \text{dist}(x, D)\right), & x \in \mathbb{R} \setminus D, \end{cases} \quad (2.9)$$

and take μ_1 to be the smallest by the absolute value root of the quadratic function

$$q(\mu) := \mu^2 - \left(v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \right) \mu + 1. \quad (2.10)$$

Then, for sufficiently small values of ε the function $g_{\nu/\varepsilon} u_\varepsilon$ is an element of $L^2(\mathbb{R})$ for all $\nu < |\ln|\mu_1||$.

Remark 2.5. One can improve the eigenvalues convergence rate at least to $|\lambda_\varepsilon - \lambda_0| \leq \tilde{C}\varepsilon$ and in a rather generic case even to $|\lambda_\varepsilon - \lambda_0| \leq \tilde{C}\varepsilon^2$ for some $\tilde{C} > 0$ (improving accordingly the convergence estimate for the eigenfunctions), by ‘‘attaching’’ the periodic structure to the defect D in a ‘‘correct’’ way, see the end of Section 4 and Theorem 4.2 for the details.

3 The limit spectrum of the unperturbed operator

Here we quantitatively characterise the limit spectrum (*cf.* (2.4))

$$\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta)$$

and establish criteria for the existence of spectral gaps. To this end we consider the eigenvalue problem: find $\lambda \in [0, \infty)$ and $u \in V_\theta = \{v \in H_\theta^1(Y) : v' \equiv 0 \text{ on } Y_1\}$ such that

$$\int_0^h a_0 u' \overline{v'} = \lambda \int_0^1 \rho u \overline{v} \quad \forall v \in V_\theta. \quad (3.11)$$

By taking test functions $v \in C_0^\infty(Y_0)$ we deduce that $u|_{Y_0}$ is a weak solution to the equation

$$-(a_0 u')' = \lambda \rho_0 u \quad (3.12)$$

on Y_0 . For L^∞ -functions a_0 and ρ_0 , the equation (3.12) holds pointwise almost everywhere and by integrating by parts in (3.11) we deduce that

$$(a_0 u')(h^-) \overline{v(h)} - (a_0 u')(0^+) \overline{v(0)} = \lambda \int_{Y_1} \rho_1 u \overline{v} \quad \forall v \in V_\theta.$$

Here $f(z^+) := \lim_{x \searrow z} f(x)$, and $f(z^-) := \lim_{x \nearrow z} f(x)$ for a function f , whenever the corresponding limit exists.

Since any element $v \in V_\theta$ satisfies $v(y) = e^{i\theta} v(0)$, $y \in Y_1$, the above observations imply that u satisfies (3.11) if and only if $w = u|_{Y_0} \in H^1(Y_0)$ is a weak solution of the problem

$$\begin{cases} -(a_0 u')' = \lambda \rho_0 u & \text{in } Y_0, \\ u(h) = e^{i\theta} u(0), \\ e^{-i\theta} (a_0 u')(h^-) - (a_0 u')(0^+) = \lambda u(0) \int_{Y_1} \rho_1. \end{cases} \quad (3.13)$$

We now describe the solutions to (3.13), equivalently (3.11).

3.1 Representation via a fundamental system

Due to the existence and uniqueness theorem for linear first-order systems with locally integrable coefficients, see *e.g.* [19], for all $\lambda \in \mathbb{R}$ the first-order system

$$U' = AU, \quad A := \begin{pmatrix} 0 & a_0^{-1} \\ -\lambda \rho_0 & 0 \end{pmatrix}, \quad (3.14)$$

has a fundamental solution system

$$V(\lambda, \cdot) := \begin{pmatrix} v_1(\lambda, \cdot) & v_2(\lambda, \cdot) \\ (a_0 v_1')(\lambda, \cdot) & (a_0 v_2')(\lambda, \cdot) \end{pmatrix},$$

so that any solution to $-(a_0 u')' = \lambda \rho_0 u$ in Y_0 is a linear combination of $v_1(\lambda, \cdot)$ and $v_2(\lambda, \cdot)$, and

$$V(\lambda, 0) = \begin{pmatrix} v_1(\lambda, 0) & v_2(\lambda, 0) \\ (a_0 v_1')(\lambda, 0) & (a_0 v_2')(\lambda, 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.15)$$

cf. Remark 2.3. Note that the associated Wronskian of the system is constant:

$$\det V(\lambda, y) = v_1(\lambda, y)(a_0 v_2')(\lambda, y) - v_2(\lambda, y)(a_0 v_1')(\lambda, y) = 1 \quad \forall y \in Y_0, \quad \lambda \in \mathbb{R}. \quad (3.16)$$

It follows from the above that all solutions u to (3.13) are of the form

$$u = c_1 v_1 + c_2 v_2 \quad (3.17)$$

for some $c_1, c_2 \in \mathbb{C}$. Substituting the representation (3.17) into the second and third equations of (3.13) leads to the system

$$\begin{pmatrix} v_1(\lambda, h) - e^{i\theta} & v_2(\lambda, h) \\ (a_0 v_1')(\lambda, h) - e^{i\theta} \lambda \int_{Y_1} \rho_1 & (a_0 v_2')(\lambda, h) - e^{i\theta} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.18)$$

For the existence of a non-trivial solution (c_1, c_2) to (3.18), and therefore non-trivial u in (3.13), the value λ must necessarily solve the equation

$$2 \cos \theta = D(\lambda), \quad D(\lambda) := v_1(\lambda, h) + (a_0 v_2')(\lambda, h) - \lambda v_2(\lambda, h) \int_{Y_1} \rho_1. \quad (3.19)$$

Hence, the set (cf. (2.4))

$$\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta)$$

consists of all non-negative λ such that

$$|D(\lambda)| \leq 2. \quad (3.20)$$

From the relation (3.19) we can deduce much more about the limit spectrum. Setting $\lambda_k(\theta)$, $k \in \mathbb{N}$, $\theta \in [0, 2\pi)$, to the k -th eigenvalue of A_θ labelled according to the min-max principle, we define $E_k : [0, 2\pi) \rightarrow [0, \infty)$ to be the k -th spectral band function given by $\theta \mapsto \lambda_k(\theta)$. The name spectral band function comes from the (clear) characterisation:

$$\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta) = \bigcup_{k \in \mathbb{N}} \text{Ran} E_k.$$

We shall prove below the following result about the nature of the spectral band functions.

Theorem 3.1.

1. The functions E_k , $k \in \mathbb{N}$, are continuous and even around $\theta = \pi$: $E_k(\theta) = E_k(2\pi - \theta)$, $\theta \in [0, \pi]$.
2. The functions $E_{2m-1}(\cdot)$, $m \in \mathbb{N}$, are strictly increasing on $(0, \pi)$.
3. The functions $E_{2m}(\cdot)$, $m \in \mathbb{N}$, are strictly decreasing on $(0, \pi)$.
4. The spectral bands are given by the following intervals:

$$\text{Ran } E_{2m-1} = [\lambda_{2m-1}(0), \lambda_{2m-1}(\pi)], \quad \text{Ran } E_{2m} = [\lambda_{2m}(\pi), \lambda_{2m}(0)], \quad m \in \mathbb{N}.$$

Let us focus on Claim 4 of the above theorem. It informs us that the interval $(\lambda_{2m-1}(\pi), \lambda_{2m}(\pi))$ (respectively, $(\lambda_{2m}(0), \lambda_{2m+1}(0))$) is a spectral gap if and only if $\lambda = \lambda_{2m-1}(\pi)$ (respectively, $\lambda = \lambda_{2m}(0)$) is a simple eigenvalue of the anti-periodic (respectively, periodic) limit problem (3.12).² We now characterise when the eigenvalues of the periodic, anti-periodic problems are degenerate, *i.e.* have multiplicity two, in terms of the fundamental system (v_1, v_2) . At such points λ the spectral bands touch and there are no gaps.

²It is straightforward to argue, cf. [17, Theorem XIII.89 (c)], that each eigenvalue $\lambda_k(\theta)$, $k \in \mathbb{N}$, is always simple when $\theta \in (0, \pi)$.

Proposition 3.2 (Condition for the absence of spectral gaps). *Fix $n = 2m$, $m \in \mathbb{N}$. Then $\lambda_n(0) = \lambda_{n+1}(0)$ if and only if*

$$v_2(\lambda_n(0), h) = 0, \quad (a_0 v'_1)(\lambda_n(0), h) = \lambda_n(0) v_1(\lambda_n(0), h) \langle \rho_1 \rangle, \quad \langle \rho_1 \rangle := \int_{Y_1} \rho_1. \quad (3.21)$$

Similarly, if $n = 2m - 1$, $m \in \mathbb{N}$, then $\lambda_n(\pi) = \lambda_{n+1}(\pi)$ if and only if

$$v_2(\lambda_n(\pi), h) = 0, \quad (a_0 v'_1)(\lambda_n(\pi), h) = \lambda_n(\pi) v_1(\lambda_n(\pi), h) \langle \rho_1 \rangle. \quad (3.22)$$

Remark 3.3. *Note that (3.21) is equivalent to*

$$v_2(\lambda_n(0), h) = 0, \quad (a_0 v'_2)(\lambda_n(0), h) = 1, \quad v_1(\lambda_n(0), h) = 1, \quad (a_0 v'_1)(\lambda_n(0), h) = \lambda_n(0) \langle \rho_1 \rangle. \quad (3.23)$$

and that (3.22) is equivalent to

$$v_2(\lambda_n(\pi), h) = 0, \quad (a_0 v'_2)(\lambda_n(\pi), h) = -1, \quad v_1(\lambda_n(\pi), h) = -1, \quad (a_0 v'_1)(\lambda_n(\pi), h) = -\lambda_n(\pi) \langle \rho_1 \rangle. \quad (3.24)$$

Sufficiency is obvious for both cases. For necessity, we shall consider $\lambda = \lambda_n(0)$. The case $\lambda = \lambda_n(\pi)$ is similar. We shall drop the argument $(\lambda_n(0), h)$ to neaten up the derivation. By (3.16) and (3.21) we have

$$1 = v_1 a_0 v'_2 - v_2 a_0 v'_1 = v_1 a_0 v'_2.$$

Furthermore, since $D(\lambda_n(0)) = 2$, cf. (3.19), then multiplying (3.19) by $a_0 v'_2$ (which is seen to be nonzero from the above equation) we compute

$$\begin{aligned} 2a_0 v'_2 &= a_0 v'_2 v_1 + (a_0 v'_2)^2 - a_0 v'_2 \lambda_n(0) v_2 \langle \rho_1 \rangle \\ &= 1 + (a_0 v'_2)^2. \end{aligned}$$

Equivalently, $(a_0 v'_2 - 1)^2 = 0$. Hence, $(a_0 v'_2)(\lambda_n(0), h) = 1$ and therefore $v_1(\lambda_n(0), h) = 1$, and (3.23) follows.

Proof of Proposition 3.2. Let us consider $\lambda_n(0)$, the case $\lambda_n(\pi)$ is similar. Necessity follows by noting that if $\lambda = \lambda_n(0)$ has multiplicity 2 then $v_i(\lambda, \cdot)$, $i = 1, 2$, are linear combinations of the orthogonal periodic eigenfunctions to the limit problem (3.11). In particular the conditions $v_i(\lambda, h) = v_i(\lambda, 0)$ and $(a_0 v'_i)(\lambda, h) - (a_0 v'_i)(\lambda, 0) = \lambda v_i(\lambda, 0) \langle \rho_1 \rangle$, $i = 1, 2$, hold. Then (3.21) follows from the initial values $v_2(\lambda, 0) = 0$ and $(a_0 v'_1)(\lambda, 0) = 0$.

For sufficiency, by (3.23) both the fundamental solutions satisfy the limit spectral problem, *i.e.* they are linearly independent eigenfunctions of $\lambda = \lambda_n(0)$. \square

The proof of Theorem 3.1 readily follows, by arguing for example as in [9, Chapter 2.3], from the following further analysis on the function D given by (3.19).

Lemma 3.4. *The function D is analytic. Furthermore, the following assertions hold.*

- (a) $D'(\lambda) \neq 0$ when $|D(\lambda)| < 2$.
- (b) $D'(\lambda_n(0)) = 0$ if and only if (3.21) holds.
- (c) If $D'(\lambda_n(0)) = 0$ then $D''(\lambda_n(0)) < 0$.
- (d) $D'(\lambda_n(\pi)) = 0$ if and only if (3.22) holds.
- (e) If $D'(\lambda_n(\pi)) = 0$ then $D''(\lambda_n(\pi)) > 0$.

Proof. The analyticity of D follows from that of $\lambda \mapsto V(\lambda, h)$, which is well-known to hold, see for example [19, Chapter 2].

Proof of a). Differentiating D , cf. (3.19), gives

$$D'(\lambda) = \partial_\lambda v_1(\lambda, h) + \partial_\lambda (a_0 v_2')(\lambda, h) - v_2(\lambda, h) \langle \rho_1 \rangle - \lambda \partial_\lambda v_2(\lambda, h) \langle \rho_1 \rangle. \quad (3.25)$$

Now if we differentiate both sides of (3.14) with respect to λ we get

$$\partial_\lambda u' = A \partial_\lambda u + (0, -\rho_0 u_1)^\top.$$

Furthermore, if $u = (v_i, a_0 v_i')^\top$, then $\partial_\lambda u(\lambda, 0) = 0$. Therefore by the variation of constants method we deduce (recalling V is the fundamental system of (3.14)) that

$$\begin{pmatrix} \partial_\lambda v_i(\lambda, h) \\ \partial_\lambda (a_0 v_i')(\lambda, h) \end{pmatrix} = \begin{pmatrix} \int_0^h \rho_0(s) v_i(\lambda, s) [v_1(\lambda, h) v_2(\lambda, s) - v_2(\lambda, h) v_1(\lambda, s)] ds \\ \int_0^h \rho_0(s) v_i(\lambda, s) [(a_0 v_1')(\lambda, h) v_2(\lambda, s) - (a_0 v_2')(\lambda, h) v_1(\lambda, s)] ds \end{pmatrix}. \quad (3.26)$$

Therefore, from (3.25), (3.26), after some algebra, we deduce that

$$\begin{aligned} D'(\lambda) &= \alpha(\lambda, h) \int_0^h \rho_0(s) v_1(\lambda, s) v_2(\lambda, s) ds - v_2(\lambda, h) \int_0^h \rho_0(s) v_1^2(\lambda, s) ds \\ &\quad + [(a_0 v_1')(\lambda, h) - \lambda \langle \rho_1 \rangle v_1(\lambda, h)] \int_0^h \rho_0(s) v_2^2(\lambda, s) ds - v_2(\lambda, h) \langle \rho_1 \rangle. \end{aligned} \quad (3.27)$$

and, from (3.16), (3.19), we readily compute

$$D(\lambda)^2 = 4 + \alpha(\lambda, h)^2 + 4(v_2(a_0 v_1') - \lambda v_1 v_2 \langle \rho_1 \rangle)(\lambda, h), \quad (3.28)$$

$$\alpha(\lambda, s) := (v_1 - a_0 v_2' + \lambda v_2 \langle \rho_1 \rangle)(\lambda, s). \quad (3.29)$$

Multiplying (3.27) by $4v_2(\lambda, h)$ and using (3.28), after some more algebra, we determine that

$$\begin{aligned} 4v_2(\lambda, h) D'(\lambda) &= -(4 - D(\lambda)^2) \int_0^h \rho_0(s) v_2^2(\lambda, s) ds - 4v_2^2(\lambda, h) \langle \rho_1 \rangle \\ &\quad - \int_0^h \rho_0(s) (\alpha(\lambda, h) v_2(\lambda, s) - 2v_2(\lambda, h) v_1(\lambda, s))^2 ds. \end{aligned} \quad (3.30)$$

It follows from (3.30) that if $|D(\lambda)| < 2$ then $4v_2(\lambda, h) D'(\lambda) < 0$. In particular, $D'(\lambda) \neq 0$, *i.e.* a) holds.

Proof of b). Henceforth, we shall use the notation f for the value of a function $f(\lambda, s)$ at $(\lambda_n(0), h)$. Suppose that (3.21) holds. By Remark 3.3, equivalently (3.23) holds. Then, by (3.29) we compute $\alpha = 0$ and, therefore, from (3.27) it follows that $D'(\lambda_n(0)) = 0$.

Next, we suppose that $D'(\lambda_n(0)) = 0$ and prove that (3.21) holds. As $\lambda_n(0)$ is a root of $D(\cdot)^2 - 4$, we deduce from (3.30) that

$$0 = -4v_2^2 \langle \rho_1 \rangle - \int_0^h \rho_0(s) (\alpha v_2(\lambda_n(0), s) - 2v_2 v_1(\lambda_n(0), s))^2 ds.$$

Therefore $v_2^2 = v_2(\lambda_n(0), h)^2 = 0$, and

$$\rho_0(s) (\alpha v_2(\lambda_n(0), \cdot) - 2v_2 v_1(\lambda_n(0), \cdot))^2 = 0.$$

Setting $v_2 = 0$, above gives $\alpha(\lambda_n(0), h) = 0$. It remains to show that $[(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0) \langle \rho_1 \rangle v_1(\lambda_n(0), h)] = 0$. Upon setting $v_2 = \alpha = 0$ and $\lambda = \lambda_n(0)$ in (3.27) we conclude

$$0 = [(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0) \langle \rho_1 \rangle v_1(\lambda_n(0), h)] \int_0^h \rho_0(s) v_2^2(\lambda_n(0), s) ds.$$

Since $v_2(\lambda_n(0), \cdot) \neq 0$ on $(0, h)$ we deduce $[(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0) \langle \rho_1 \rangle v_1(\lambda_n(0), h)] = 0$, *i.e.* (3.21) holds.

Proof of c). We shall prove that

$$\begin{aligned} \frac{1}{2} D''(\lambda_n(0)) \leq & \left(\int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s) ds \right)^2 \\ & - \left(\int_0^h \rho_0(s) v_1^2(\lambda_n(0), s) ds \right) \left(\int_0^h \rho_0(s) v_2^2(\lambda_n(0), s) ds \right). \end{aligned} \quad (3.31)$$

Then, as $v_1(\lambda_n(0), \cdot)$ and $v_2(\lambda_n(0), \cdot)$ are linearly independent it follows from (3.31) and the Hölder inequality that

$$\frac{1}{2} D''(\lambda_n(0)) < 0.$$

Let us prove (3.31). First we make some preliminary calculations. By (3.26) and (3.23), which holds since $D'(\lambda_n(0)) = 0$, *cf.* b) and Remark 3.3, we compute (dropping the argument $(\lambda_n(0), h)$ as above)

$$\begin{aligned} \partial_\lambda v_1 &= \int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s) ds, \\ \partial_\lambda v_2 &= \int_0^h \rho_0(s) v_2^2(\lambda_n(0), s) ds, \\ \partial_\lambda (a_0 v_1') &= \lambda_n(0) \langle \rho_1 \rangle \partial_\lambda v_1 - \int_0^h \rho_0(s) v_1^2(\lambda_n(0), s) ds, \\ \partial_\lambda (a_0 v_2') &= \lambda_n(0) \langle \rho_1 \rangle \partial_\lambda v_2 - \partial_\lambda v_1. \end{aligned} \quad (3.32)$$

We now are going to differentiate both sides of (3.27) with respect to λ . Half of the terms will immediately been seen to be zero. Indeed, if we take the first term in the right-hand side of (3.27), differentiate it and evaluate it at $(\lambda_n(0), h)$, then because $\alpha = 0$, *cf.* (3.29) and (3.23), we deduce that

$$\partial_\lambda \left(\alpha(\lambda, h) \int_0^h \rho_0(s) v_1(\lambda, s) v_2(\lambda, s) ds \right) = \partial_\lambda \alpha \int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s) ds.$$

The same is true for all the other terms. Therefore, differentiating (3.27) and bearing in mind (3.32), we compute (after a little bit more algebra) that

$$D''(\lambda_n(0)) = \partial_\lambda \alpha \partial_\lambda v_1 - 2 \partial_\lambda v_2 \left(\int_0^h \rho_0(s) v_1^2(\lambda_n(0), s) ds + \langle \rho_1 \rangle \right). \quad (3.33)$$

From (3.32) we see that $\partial_\lambda v_2 \geq 0$ and so it follows from (3.33) that

$$D''(\lambda_n(0)) \leq \partial_\lambda \alpha \partial_\lambda v_1 - 2 \partial_\lambda v_2 \int_0^h \rho_0(s) v_1^2(\lambda_n(0), s) ds.$$

We complete the proof of (3.31), *cf.* (3.32), if we can prove that

$$\partial_\lambda \alpha = 2 \partial_\lambda v_1. \quad (3.34)$$

Multiplying (3.28) by $v_1(\lambda, s)$ and utilising (3.16) gives

$$v_1(\lambda, s)\alpha(\lambda, s) = v_1^2(\lambda, s) - 1 - v_2(\lambda, s)(a_0 v_1')(\lambda, s) + \lambda v_1(\lambda, s)v_2(\lambda, s)\langle \rho_1 \rangle.$$

Then, differentiating both sides of the above equation with respect to λ and evaluating at $(\lambda_n(0), h)$ gives

$$\alpha \partial_\lambda v_1 + v_1 \partial_\lambda \alpha = 2v_1 \partial_\lambda v_1 - (a_0 v_1') \partial_\lambda v_2 - v_2 \partial_\lambda (a_0 v_1') + v_1 v_2 \langle \rho_1 \rangle + \lambda_n(0) \langle \rho_1 \rangle (v_2 \partial_\lambda v_1 + v_1 \partial_\lambda v_2).$$

Upon utilising (3.23), we deduce that (3.34) holds, and the proof of c) follows.

The proofs of d) and e) are similar to that of b) and c). \square

Example 3.5. We end the subsection with the following simple example. Suppose that a_0, ρ_0 and ρ_1 are equal to unity on their support, then $v_1(\lambda, y) = \cos(\sqrt{\lambda}y)$, $v_2(\lambda, y) = (1/\sqrt{\lambda}) \sin(\sqrt{\lambda}y)$, and

$$D(\lambda) = 2 \cos(\sqrt{\lambda}h) - \sqrt{\lambda} \sin(\sqrt{\lambda}h)(1-h).$$

In particular, we see that conditions (3.21) and (3.22) never hold and consequently the (infinitely many) intervals

$$(\lambda_{2m-1}(\pi), \lambda_{2m}(\pi)), \quad (\lambda_{2m}(0), \lambda_{2m+1}(0)), \quad m \in \mathbb{N},$$

are gaps. Here $\lambda_k(0)$ (respectively, $\lambda_k(\pi)$) is the k -th zero of $D - 2$ (respectively, $D + 2$). These gaps become wider as $k \rightarrow \infty$.

3.2 Representation via a spectral decomposition

Consider the operator \tilde{A}_θ defined on $L^2_{\rho_0}(Y_0)$ and associated to the form

$$\tilde{\beta}_\theta(u, v) := \int_{Y_0} a_0 u' \bar{v}', \quad u, v \in H^1_\theta(Y_0),$$

in the sense of procedure described in Section 2. By virtue of the fact that the operator \tilde{A}_θ has compact resolvent, its $L^2_{\rho_0}(Y_0)$ -orthonormal sequence of eigenfunctions $\{\Phi_\theta^{(n)}\}_{n \in \mathbb{N}}$ is complete in the space $L^2_{\rho_0}(Y_0)$. We denote by $\mu_n(\theta)$, $n \in \mathbb{N}$, the eigenvalues of $\Phi_\theta^{(n)} \in H^1_\theta(Y_0)$:

$$\int_{Y_0} a_0 (\Phi_\theta^{(n)})' \bar{v}' = \mu_n(\theta) \int_{Y_0} \rho_0 \Phi_\theta^{(n)} \bar{v} \quad \forall v \in H^1_\theta(Y_0). \quad (3.35)$$

Multiplying the first equation in (3.13) by $\overline{\Phi_\theta^{(n)}}$ and integrating by parts we have

$$\begin{aligned} \lambda \int_{Y_0} \rho_0 u \overline{\Phi_\theta^{(n)}} &= - \int_{Y_0} (a_0 u')' \overline{\Phi_\theta^{(n)}} = - ((a_0 u')(h^-) \overline{\Phi_\theta^{(n)}(h)} - (a_0 u')(0^+) \overline{\Phi_\theta^{(n)}(0)}) + \int_{Y_0} a_0 u' (\overline{\Phi_\theta^{(n)}})' \\ &= - (e^{-i\theta} (a_0 u')(h^-) - (a_0 u')(0^+)) \overline{\Phi_\theta^{(n)}(0)} + \mu_n(\theta) \int_{Y_0} \rho_0 u \overline{\Phi_\theta^{(n)}}. \end{aligned}$$

The third equation in (3.13) implies

$$(\mu_n(\theta) - \lambda) \int_{Y_0} \rho_0 u \overline{\Phi_\theta^{(n)}} = \lambda u(0) \overline{\Phi_\theta^{(n)}(0)} \int_{Y_1} \rho_1.$$

Therefore, upon performing a spectral decomposition of u in terms of $\Phi_\theta^{(n)}$, i.e. setting

$$u = \sum_{n \in \mathbb{N}} \zeta_n \Phi_\theta^{(n)}, \quad \zeta_n = \int_{Y_0} \rho_0 u \overline{\Phi_\theta^{(n)}},$$

we see that

$$\zeta_n = \frac{\lambda}{\mu_n(\theta) - \lambda} u(0) \overline{\Phi_\theta^{(n)}(0)} \int_{Y_1} \rho_1, \quad n \in \mathbb{N}.$$

In particular, one has $u(0) = \sum_{n \in \mathbb{N}} \zeta_n \Phi_\theta^{(n)}(0)$. Thus, we arrive at the statement: if $\lambda \in \bigcup_\theta \sigma(A_\theta)$ then there exists $\theta \in [0, 2\pi)$ such that

$$\sum_{n \in \mathbb{N}} \frac{\lambda}{\mu_n(\theta) - \lambda} |\Phi_\theta^{(n)}(0)|^2 = \left(\int_{Y_1} \rho_1 \right)^{-1}. \quad (3.36)$$

The converse statement is also true. Indeed, suppose that for some $\theta \in [0, 2\pi)$ the value λ satisfies (3.36), then we find that

$$\zeta_n := \frac{\lambda}{\mu_n(\theta) - \lambda} u(0) \overline{\Phi_\theta^{(n)}(0)} \int_{Y_1} \rho_1, \quad n \in \mathbb{N},$$

satisfy

$$\limsup_n \frac{|\zeta_n|^2}{b_n} = 0, \quad 0 \leq \limsup_n \mu_n(\theta) \frac{|\zeta_n|^2}{b_n} < \infty, \quad b_n := \frac{\lambda}{\mu_n(\theta) - \lambda} |\Phi_\theta^{(n)}(0)|^2.$$

Since, by assumption,

$$\sum_n b_n = \left(\int_{Y_1} \rho_1 \right)^{-1},$$

it follows that

$$\sum_n |\zeta_n|^2 < \infty, \quad \sum_n \mu_n(\theta) |\zeta_n|^2 < \infty,$$

that is the function $u = \sum_{n \in \mathbb{N}} \zeta_n \Phi_\theta^{(n)}$ belongs to $H_\theta^1(Y_0)$ and, consequently, to V_θ when extended by the constant $u(h)$ into Y_1 . Moreover, direct calculation shows that λ and u satisfy (3.11). Hence, we have shown that $\lambda \in \bigcup_\theta \sigma(A_\theta)$.

4 Asymptotics of the defect eigenvalue problem

Suppose $\lambda_\varepsilon, u_\varepsilon$ is an eigenvalue-eigenfunction pair for the defect problem, that is

$$-(a_D^\varepsilon u_\varepsilon)' = \lambda_\varepsilon \rho_D^\varepsilon u_\varepsilon \quad \text{on } \mathbb{R}, \quad (4.37)$$

where u_ε is continuous, subject to the interface conditions

$$a_D u_\varepsilon|_D = a_D^\varepsilon u_\varepsilon|_{\mathbb{R} \setminus D} \quad \text{on } \{d_-, d_+\}. \quad (4.38)$$

and

$$a_1 u_\varepsilon'|_{\Omega_1^\varepsilon \setminus D} = \varepsilon^2 a_0 u_\varepsilon'|_{\Omega_0^\varepsilon \setminus D} \quad \text{on } \{x \in \mathbb{R} \setminus D : x = \varepsilon(z+h) \text{ or } x = \varepsilon z \text{ for some } z \in \mathbb{Z}\}. \quad (4.39)$$

In this section we study the behaviour with respect to ε of the eigenvalues λ_ε and eigenfunctions u_ε , using asymptotic expansions. We show that, up to the leading order, the values of λ_ε are described by an eigenvalue of the weighted Neumann-Laplacian on the defect D , see (4.42) below. More precisely, we show that for each eigenvalue λ_0 of (4.42) in a gap of $\bigcup_\theta \sigma(A_\theta)$, there exists a sequence of eigenvalues λ_ε of (4.37) converging to λ_0 . However, it remains unclear whether every accumulation point of λ_ε inside a gap of $\bigcup_\theta \sigma(A_\theta)$ belongs to the spectrum of (4.42).

We seek asymptotic expansions for the eigenvalues λ_ε and eigenfunctions u_ε of (4.37)–(4.39) in the form

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots, \quad (4.40)$$

with

$$u_\varepsilon(x) = \begin{cases} u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \dots, & x \in (d_-, d_+), \\ w_0(\frac{x}{\varepsilon}) + \varepsilon^2 w_2(\frac{x}{\varepsilon}) + \dots, & x \in (-\infty, d_-) \cup (d_+, \infty). \end{cases} \quad (4.41)$$

We assume that functions $w_i, u_i, i = 0, 1, 2, \dots$, are continuous.

4.1 Governing equations

Substituting (4.40), (4.41) into (4.37) and (4.38) and equating the ε^0 -coefficient on the defect gives

$$\begin{cases} -(a_D u'_0)' = \lambda_0 \rho_D u_0 & \text{on } (d_-, d_+), \\ a_D u'_0|_D = 0 & \text{on } \{d_-, d_+\}, \end{cases} \quad (4.42)$$

that is, λ_0 is an eigenvalue of the weighted Neumann-Laplace operator $A_{N,D}$ on the defect, *cf.* (2.6). Note that this is true regardless of whether d_-, d_+ belong to Ω_1^ε or Ω_0^ε . We fix u_0 by setting $\|u_0\|_{L^2_{\rho_D}(D)} = 1$.

For $c \in \mathbb{R}$, let $\lfloor c \rfloor_\varepsilon$ and $\lceil c \rceil_\varepsilon$ denote the largest integer z such that $\varepsilon z \leq c$ and the smallest integer z such that $c \leq \varepsilon z$, respectively. Substituting (4.40), (4.41) into (4.37), (4.39) and comparing the coefficients for different powers of ε in the resulting expression yields

$$\begin{cases} -(a_1 w'_0)' = 0 & \text{on } Y_1 + z, \\ (a_1 w'_0)((z+h)^+) = 0, \\ (a_1 w'_0)((z+1)^-) = 0, \end{cases} \quad (4.43)$$

and

$$\begin{cases} -(a_0 w'_0)' = \lambda_0 \rho_0 w_0 & \text{on } Y_0 + z, \\ -(a_1 w'_2)' = \lambda_0 \rho_1 w_0 & \text{on } Y_1 + z, \\ (a_1 w'_2)((z+h)^+) = (a_0 w'_0)((z+h)^-), \\ (a_1 w'_2)((z+1)^-) = (a_0 w'_0)((z+1)^+), \end{cases} \quad (4.44)$$

for all

$$z \in \mathcal{I}_\varepsilon := \{z \in \mathbb{Z} : z \geq \lceil d_+ \rceil_\varepsilon \text{ or } z \leq \lfloor d_- \rfloor_\varepsilon - 1\} \quad (4.45)$$

(that is, $z \in \mathcal{I}_\varepsilon$ if and only if the intersection of $\varepsilon(Y+z)$ and D is empty). The assertion (4.43) implies that $a_1 w'_0 \equiv 0$ on $Y_1 + z$ and therefore w_0 is constant on each such interval. By the second equation of (4.44), and the fact w_0 is constant on each interval $Y_1 + z$, the function $a_1 w'_2$ has the form

$$(a_1 w'_2)(y) = (a_1 w'_2)((z+h)^+) - \lambda_0 w_0(z+h) \int_{z+h}^y \rho_1, \quad y \in Y_1 + z. \quad (4.46)$$

Combining (4.46), the fact that w_0 is constant on $Y_1 + z$ and the first and last equations of (4.44) implies that for all $z \in \mathcal{I}_\varepsilon$ one has

$$\begin{cases} -(a_0 w'_0)' = \lambda_0 \rho_0 w_0 & \text{on } Y_0 + z, \\ w_0 \equiv w_0(z+h) = w_0(z+1) & \text{on } Y_1 + z, \\ (a_0 w'_0)((z+1)^+) - (a_0 w'_0)((z+h)^-) = -\lambda_0 w_0(z+h) \int_{Y_1} \rho_1. \end{cases} \quad (4.47)$$

The problem (4.47) fully governs the behaviour of w_0 in $\mathbb{R} \setminus ([d_-]_\varepsilon - 1, \lceil d_+ \rceil_\varepsilon)$. We can utilise the fundamental system (v_1, v_2) from Section 3.1 to quantitatively characterise w_0 . Indeed, since in each cell $Y+z$ any solution to the first equation in (4.47) is a linear combination of v_1 and v_2 , one has

$$w_0(y) = \begin{cases} l_z v_1(y-z) + m_z v_2(y-z), & y \in Y_0 + z, \\ l_z v_1(h) + m_z v_2(h), & y \in Y_1 + z, \end{cases} \quad (4.48)$$

for constants $l_z, m_z, z \in \mathcal{I}_\varepsilon$, where the expression on $Y_1 + z$ follows from the second condition in (4.47). Using (3.15), the continuity of w_0 and the jump of the co-derivative condition from (4.47), it is not difficult to derive the following recurrence relation:

$$\begin{pmatrix} l_{z+1} \\ m_{z+1} \end{pmatrix} = \begin{pmatrix} v_1(h) & v_2(h) \\ (a_0 v_1')(h) - \lambda_0 v_1(h) \int_{Y_1} \rho_1 & (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \end{pmatrix} \begin{pmatrix} l_z \\ m_z \end{pmatrix}. \quad (4.49)$$

Now, recalling the Wronskian property (3.16), we find that the characteristic polynomial q of the matrix in (4.49) is (cf. (2.10))

$$q(\mu) = \mu^2 - \left(v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \right) \mu + 1. \quad (4.50)$$

The roots μ_1, μ_2 of q satisfy the identity $\mu_1 \mu_2 = 1$ and the nature of w_0 as it varies from one period to the next is determined by the quantity $v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1$. Namely, if (cf. (3.20))

$$\left| v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \right| \leq 2,$$

then the roots μ_1, μ_2 are complex conjugate with $|\mu_1| = |\mu_2| = 1$ and solutions w_0 are described by the linear span of two quasi-periodic functions with phase difference π . In Section 3 we demonstrated that λ_0 satisfies this constraint if and only if λ_0 belongs to the limit spectrum

$$\lim_{\varepsilon \rightarrow 0} \sigma(A^\varepsilon) = \bigcup_{\theta} \sigma(A_\theta).$$

4.2 Construction of an approximate solution

For λ_0 in the gaps of this limit spectrum, *i.e.* when λ_0 satisfies the inequality

$$\left| v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \right| > 2,$$

the roots μ_1, μ_2 of q , see (4.50), satisfy $|\mu_1| < 1$ and $|\mu_2| > 1$. For such λ_0 , we can construct “unstable” solutions, one of which decays at $+\infty$ and the other at $-\infty$. Indeed, denoting by \varkappa_1 and \varkappa_2 the eigenvectors of the matrix in (4.49) corresponding to μ_1 and μ_2 , respectively, we find in the interval $[[d_+]_\varepsilon, \infty)$ that w_0 given by (4.48), (4.49) satisfies $w_0(y+1) = \mu_j w_0(y)$ if $(l_{[d_+]_\varepsilon}, m_{[d_+]_\varepsilon})^\top = \varkappa_j, j = 1, 2$. Similarly, in the interval $(-\infty, [d_-]_\varepsilon]$, one has $w_0(y) = \mu_j w_0(y-1)$ if $(l_{[d_-]_\varepsilon-1}, m_{[d_-]_\varepsilon-1})^\top = \varkappa_j, j = 1, 2$. For w_0 to decay to the left and right of the defect, we set $(l_{[d_+]_\varepsilon}, m_{[d_+]_\varepsilon})^\top = \varkappa_1$ and $(l_{[d_-]_\varepsilon-1}, m_{[d_-]_\varepsilon-1})^\top = \varkappa_2$. In this way we ensure that

$$\begin{aligned} w_0(y+1) &= \mu_1 w_0(y) && \text{for } y \in [d_+/\varepsilon, \infty), \\ w_0(y-1) &= \mu_2^{-1} w_0(y) = \mu_1 w_0(y) && \text{for } y \in (-\infty, d_-/\varepsilon], \end{aligned} \quad (4.51)$$

where we have extended w_0 to the intervals $[d_+/\varepsilon, [d_+]_\varepsilon)$ and $([d_-]_\varepsilon, d_-/\varepsilon]$ by the formulae $w_0(y) = \mu_1^{-1} w_0(y+1)$ and $w_0(y) = \mu_1^{-1} w_0(y-1)$, respectively.

The function w_0 to the right and to the left from the defect is defined up to multiplication by a constant. The next natural step is to “attach” both parts of w_0 to the solution u_0 on the defect choosing the aforementioned constants appropriately. However, there is a possibility that for particular values ε one has $w_0(d_+/\varepsilon) = 0$ or $w_0(d_-/\varepsilon) = 0$. This requires that w_0 be redefined near the boundary of D , which we do next.

On each side of the defect D , there are two possibilities on the stiff component $\Omega_1 \cap [d_+/\varepsilon, \infty)$ (or $\Omega_1 \cap (-\infty, d_-/\varepsilon]$): either w_0 does not vanish or $w_0 \equiv 0$. In the latter case, since w_0 is not identically zero on

the whole interval $[d_+/\varepsilon, \infty)$ (or $(-\infty, d_-/\varepsilon]$), it necessarily has an extremum inside each soft interval $Y_0 + z$ and $a_0 w'_0 = 0$ at the points of extrema. If w_0 does not vanish on the stiff component, we set

$$w_0(y) = w_0(\lceil d_+ \rceil_\varepsilon + h), \quad y \in [d_+/\varepsilon, \lceil d_+ \rceil_\varepsilon + h], \quad (4.52a)$$

$$w_0(y) = w_0(\lfloor d_- \rfloor_\varepsilon), \quad y \in [\lfloor d_- \rfloor_\varepsilon, d_-/\varepsilon], \quad (4.52b)$$

while in the case when $w_0 \equiv 0$ on the stiff component there exist $y_+^*, y_-^* \in Y_0$, independent of ε , such that

$$\begin{aligned} |w_0(\lceil d_+ \rceil_\varepsilon + y_+^*)| &= \max_{Y_0 + \lceil d_+ \rceil_\varepsilon} |w_0| > 0, \\ |w_0(\lfloor d_- \rfloor_\varepsilon - 1 + y_-^*)| &= \max_{Y_0 + \lfloor d_- \rfloor_\varepsilon - 1} |w_0| > 0, \end{aligned}$$

and we set

$$w_0(y) = w_0(\lceil d_+ \rceil_\varepsilon + y_+^*), \quad y \in [d_+/\varepsilon, \lceil d_+ \rceil_\varepsilon + y_+^*], \quad (4.53a)$$

$$w_0(y) = w_0(\lfloor d_- \rfloor_\varepsilon - 1 + y_-^*), \quad y \in [\lfloor d_- \rfloor_\varepsilon - 1 + y_-^*, d_-/\varepsilon]. \quad (4.53b)$$

We then choose \varkappa_1 and \varkappa_2 so that the modified w_0 matches the value of u_0 at the end-points of D :

$$w_0(d_+/\varepsilon) = u_0(d_+), \quad w_0(d_-/\varepsilon) = u_0(d_-).$$

Putting together (4.48), (4.52a)–(4.53b), it follows that the vectors \varkappa_1, \varkappa_2 do not depend on ε . Hence, it is not difficult to see that so constructed w_0 is bounded in L^2_ρ uniformly in ε ,

$$\|w_0\|_{L^2_\rho(\mathbb{R} \setminus \varepsilon^{-1}D)} \leq C. \quad (4.54)$$

Next, we construct the corrector w_2 , treating first the right side of the defect. According to the two possibilities above, we start by assuming that (4.52a) holds. The second and the third equations in (4.44) determine w_2 up to an arbitrary additive constant in each interval

$$Y_1 + z \subset [\lceil d_+ \rceil_\varepsilon + 1 + h, +\infty). \quad (4.55)$$

The choice of these constants is not important, except that w_2 should remain “controlled”, and in what follows, for simplicity, we set

$$w_2(z + h) = 0. \quad (4.56)$$

Note that the existence of w_2 satisfying the second and third equations in (4.44) follows from the last identity in (4.47). On the interval $Y_1 + \lceil d_+ \rceil_\varepsilon$ we only require w_2 to satisfy the following conditions at its boundary:

$$\begin{aligned} (a_1 w'_2)((\lceil d_+ \rceil_\varepsilon + 1)^-) &= (a_0 w'_0)((\lceil d_+ \rceil_\varepsilon + 1)^+), \\ (a_1 w'_2)((\lceil d_+ \rceil_\varepsilon + h)^+) &= w_2(\lceil d_+ \rceil_\varepsilon + h) = 0. \end{aligned}$$

To this end we fix two smooth functions f_1 and f_2 such that $0 \leq f_1 \leq 1$, $f_1(1) = 1$, $f_1(y) = 0$ for $y \in [h, (h+1)/2]$, $f_2(y) = 0$ for $y = h$ and $y \in [(h+1)/2, 1]$, $f_2(y) < 0$ for $y \in (h, (h+1)/2)$. We define w_2 on $Y_1 + \lceil d_+ \rceil_\varepsilon$ by

$$w_2(\lceil d_+ \rceil_\varepsilon + y) := -(a_0 w'_0)((\lceil d_+ \rceil_\varepsilon + 1)^+) \int_y^1 a_1^{-1} (f_1 + c f_2), \quad y \in Y_1, \quad (4.57)$$

choosing the constant c so that $w_2(\lceil d_+ \rceil_\varepsilon + h) = 0$. Moreover, we have $w_2(\lceil d_+ \rceil_\varepsilon + 1) = 0$. We set $w_2 = 0$ on $[d_+/\varepsilon, \lceil d_+ \rceil_\varepsilon + h] \cup [\lceil d_+ \rceil_\varepsilon + 1, \lceil d_+ \rceil_\varepsilon + 1 + h]$. Finally, in the intervals $Y_0 + z \subset [\lceil d_+ \rceil_\varepsilon + 2, +\infty)$ we do not

require w_2 to satisfy any equation. Instead we make a specific choice of w_2 as follows. For a non-negative function $f \in C_0^\infty(Y_0)$, $f \not\equiv 0$, we define

$$w_2(z+y) := w_2(z) + c_z \int_0^y \frac{f}{a_0}, \quad y \in Y_0, \quad z \geq [d_+]_\varepsilon + 2, \quad (4.58)$$

where the coefficient c_z is chosen so that w_2 is continuous on $[d_+/\varepsilon, +\infty)$, namely,

$$c_z = -w_2(z) \left(\int_{Y_0} \frac{f}{a_0} \right)^{-1}. \quad (4.59)$$

In particular, we have

$$(a_0 w_2')(z^+) = (a_0 w_2')((z+h)^-) = 0.$$

Moving on to the second possibility, we assume that (4.53a) holds. Then on the intervals $Y_1 + z \subset [[d_+]_\varepsilon + h, +\infty)$ we choose w_2 to satisfy the second and third equations in (4.44) and the condition $w_2(z+h) = 0$. We extend w_2 by zero on $[d_+/\varepsilon, [d_+]_\varepsilon + h)$, and on the intervals $Y_0 + z \subset [[d_+]_\varepsilon + 1, +\infty)$ we define w_2 as in (4.58).

We define w_2 to the left of the defect in a similar way. Namely, we assume first that (4.52b) holds and define w_2 according to (4.44) in the intervals $Y_1 + z \subset (-\infty, [d_-]_\varepsilon - 1]$, requiring $w_2(z+1) = 0$. On $[[d_-]_\varepsilon - 1 + h, [d_-]_\varepsilon]$ we define w_2 by a formula analogous to (4.57) so that it satisfies the conditions

$$\begin{aligned} (a_1 w_2')(([d_-]_\varepsilon - 1 + h)^+) &= (a_0 w_0')(([d_-]_\varepsilon - 1 + h)^-), \\ (a_1 w_2')(([d_-]_\varepsilon)^-) &= w_2([d_-]_\varepsilon) = 0. \end{aligned}$$

We then extend w_2 by zero on $([d_-]_\varepsilon, d_-/\varepsilon]$ and define it on the intervals $Y_0 + z \subset (-\infty, [d_-]_\varepsilon - 1 + h]$ according to (4.58).

Finally, if (4.53b) holds we define w_2 according to (4.44) on the intervals $Y_1 + z \subset (-\infty, [d_-]_\varepsilon - 1]$, additionally requiring that $w_2(z+1) = 0$, extend w_2 by zero into $([d_-]_\varepsilon - 1, d_-/\varepsilon]$, and use (4.58) to define w_2 on the intervals $Y_0 + z \subset (-\infty, [d_-]_\varepsilon - 1 - h]$.

4.3 Justification of asymptotics

First we estimate the term w_2 . Assume that (4.52a) holds, and consider w_2 on $[d_+/\varepsilon, +\infty)$. A straightforward calculation gives, cf. (4.44), (4.56),

$$w_2(z+y) = (a_0 w_0')((z+h)^-) \int_h^y a_1^{-1} - \lambda_0 w_0(z+h) \int_h^y \left(a_1^{-1}(\cdot) \int_h^\cdot \rho_1 \right), \quad z \geq [d_+]_\varepsilon + 1, \quad y \in Y_1. \quad (4.60)$$

It follows from (4.58), (4.59), (4.60), and (4.51) that

$$w_2(y+1) = \mu_1 w_2(y) \quad \text{for } y \in [[d_+]_\varepsilon + 1 + h, +\infty),$$

and, thereupon,

$$((a_0 + a_1)w_2')'(y+1) = \mu_1((a_0 + a_1)w_2')'(y) \quad \text{for } y \in [[d_+]_\varepsilon + 1 + h, +\infty).$$

With w_2 and $((a_0 + a_1)w_2')'$ clearly bounded in $L_\rho^2(d_+/\varepsilon, [d_+]_\varepsilon + 1 + h)$ independently of ε , cf. (4.57), we conclude that

$$\|((a_0 + a_1)w_2')'\|_{L_\rho^2(d_+/\varepsilon, +\infty)} + \|w_2\|_{L_\rho^2(d_+/\varepsilon, +\infty)} \leq C$$

for a suitable constant C independent of ε .

In an analogous way we derive the estimates for w_2 on $[d_+/\varepsilon, +\infty)$ in the case (4.53a) and on $(-\infty, d_-/\varepsilon]$ in the cases (4.52b), (4.53b). Thus we assert

$$\|((a_0 + a_1)w_2)'\|_{L^2_\rho(\mathbb{R} \setminus \varepsilon^{-1}D)} + \|w_2\|_{L^2_\rho(\mathbb{R} \setminus \varepsilon^{-1}D)} \leq C. \quad (4.61)$$

Suppose now that $\lambda_0 \in \sigma(A_{N,D}) \setminus (\bigcup_\theta \sigma(A_\theta))$. The construction described above guarantees that the function

$$u_{\varepsilon, \text{ap}}(x) := \begin{cases} u_0(x), & x \in D, \\ w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon), & x \in \mathbb{R} \setminus D, \end{cases} \quad (4.62)$$

is continuous and has a continuous co-derivative $a_D^\varepsilon u'_{\varepsilon, \text{ap}}$, implying that $u_{\varepsilon, \text{ap}}$ belongs to the domain of the operator A_D^ε .

It follows from the spectral theorem for self-adjoint operators (see *e.g.* [4]) that for all functions $f \in \text{dom}(A_D^\varepsilon) \subset L^2_{\rho_D^\varepsilon}(\mathbb{R})$ such that $\|f\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} = 1$, one has

$$\text{dist}(\lambda_0, \sigma(A_D^\varepsilon)) \leq \|(A_D^\varepsilon - \lambda_0)f\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})}.$$

Straightforward calculations show that, except for the small regions near the boundary of the defect, we have

$$\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}} = \begin{cases} 0, & x \in D, \\ -\varepsilon^2 (a_0 w_2)'(x/\varepsilon) - \varepsilon^2 \lambda_0 \rho_0(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_0^\varepsilon \setminus \varepsilon[[d_-]_\varepsilon - 1, [d_+]_\varepsilon + 1], \\ -\varepsilon^2 \lambda_0 \rho_1(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_1^\varepsilon \setminus \varepsilon[[d_-]_\varepsilon - 1, [d_+]_\varepsilon + 1]. \end{cases} \quad (4.63)$$

Near the boundary we need to consider each of the cases (4.52a) – (4.53b) separately. Assume first that (4.52a) holds. Then we have

$$\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}} = \begin{cases} -\lambda_0 \rho(x/\varepsilon) w_0(x/\varepsilon), & x/\varepsilon \in [d_+/\varepsilon, [d_+]_\varepsilon + h], \\ -\lambda_0 \rho_1(x/\varepsilon) w_0(x/\varepsilon) - (a_1 w_2)'(x/\varepsilon) \\ \quad - \varepsilon^2 \lambda_0 \rho_1(x/\varepsilon) w_2(x/\varepsilon), & x/\varepsilon \in [[d_+]_\varepsilon + h, [d_+]_\varepsilon + 1]. \end{cases}$$

By construction, w_0, w_2 and $(a_1 w_2)'$ are bounded continuous functions independent of ε . Hence $\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}}$ is bounded in $L^\infty(d_+, \varepsilon([d_+]_\varepsilon + 1))$. Performing direct calculations and applying a similar argument in the three remaining cases, we conclude that

$$\|\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}}\|_{L^\infty(\varepsilon[[d_-]_\varepsilon - 1, [d_+]_\varepsilon + 1] \setminus D)} \leq C.$$

Since the size of the region $\varepsilon[[d_-]_\varepsilon - 1, [d_+]_\varepsilon + 1] \setminus D$ is of order ε , the latter inequality together with (4.63) and (4.61) readily implies that

$$\|(A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}}\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} \leq C \varepsilon^{1/2}, \quad (4.64)$$

for some constant $C > 0$.

We establish the following result, which implies Claim 1 of Theorem 2.4. In particular, the second and third estimates in (2.7) follow from (4.54), (4.61), (4.62), the estimate (4.66) below and the identity

$$\|w_i(\cdot/\varepsilon)\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R} \setminus D)} = \varepsilon^{1/2} \|w_i\|_{L^2_\rho(\mathbb{R} \setminus \varepsilon^{-1}D)}, \quad i = 0, 2. \quad (4.65)$$

Theorem 4.1. *Suppose that $\lambda_0 \in \sigma(A_{N,D}) \setminus (\bigcup_\theta \sigma(A_\theta))$.*

1. *There exists $C_1 > 0$, independent of ε , such that*

$$\text{dist}(\lambda_0, \sigma(A_D^\varepsilon)) \leq C_1 \varepsilon^{1/2}.$$

2. For sufficiently small ε there exist (simple) eigenvalues λ_ε of A_D^ε such that $|\lambda_\varepsilon - \lambda_0| \leq C_1 \varepsilon^{1/2}$.
3. For sufficiently small ε the function $u_{\varepsilon, \text{ap}}$ is an approximate eigenfunction of A_D^ε , in the sense that there exists an ε -independent constant $C_2 > 0$ and $c_j^\varepsilon \in \mathbb{R}$ such that

$$\left\| u_{\varepsilon, \text{ap}} - \sum_{j \in J_\varepsilon} c_j^\varepsilon u_{\varepsilon, j} \right\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} \leq C_2 \varepsilon^{1/2}, \quad (4.66)$$

where the set J_ε is defined by (2.8), and $u_{\varepsilon, j}$ are appropriate eigenfunctions of A_D^ε .

Proof. Claim 1 of the theorem follows from (4.64) and the fact that

$$\lim_{\varepsilon \rightarrow 0} \|u_{\varepsilon, \text{ap}}\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} = \|u_0\|_{L^2_{\rho_D}(D)} = 1,$$

due to (4.54), (4.61) and (4.65). Claim 2 follows by noting that the essential spectra of A_D^ε and A^ε coincide, and that $\sigma(A^\varepsilon) = \sigma_{\text{ess}}(A^\varepsilon)$ converges to $\bigcup_\theta \sigma(A_\theta)$, as $\varepsilon \rightarrow 0$, to which λ_0 does not belong. To prove claim 3, one can argue as in [22], or [12, Section 11.1], using (4.64) and a spectral decomposition of $u_{\varepsilon, \text{ap}}$ with respect to the operator A_D^ε . \square

4.4 Improvement of the error bound

It is clear from the construction of $u_{\varepsilon, \text{ap}}$ that the main error term of order $\varepsilon^{1/2}$ comes from what is conventionally called boundary layer, near the endpoints of the defect D . In fact, one can improve the error bound (4.64) by “attaching” the ε -periodic structure to the defect in an appropriate way, thereby preventing the appearance of the boundary effect. Our approach is based on the behaviour of the function w_0 , see the observation made in the beginning of Section 4.2 preceding the adjustment of w_0 . We provide the detailed construction only at the right end of the defect D . The construction at the left end is completely analogous.

First, let us assume that w_0 in (4.51) has no extrema inside the soft intervals $[d_+/\varepsilon, \infty) \cap \Omega_0$. Then w_0 does not vanish on the stiff component $[d_+/\varepsilon, \infty) \cap \Omega_1$. In this case we “attach” the periodic structure to D so that it touches the soft component, *i.e.* we define the soft and stiff components to the right of D via $\Omega_0^+ := \bigcup_z (Y_0 + z + d_+/\varepsilon)$ and $\Omega_1^+ := \bigcup_z (Y_1 + z + d_+/\varepsilon)$, respectively, $z = 0, 1, \dots$. The definition of the relevant notation, such as coefficients a_D^ε , ρ_D^ε , *etc.*, should be adjusted in an obvious way, however, we will not dwell on this. We define w_0 on $[d_+/\varepsilon, \infty)$ according to (4.47), (4.51), requiring $w_0(d_+/\varepsilon) = u_0(d_+)$. We construct w_2 on $[d_+/\varepsilon + h, \infty)$ according to (4.44) and (4.58), requiring $w_2(d_+/\varepsilon + h + z) = 0$, $z = 0, 1, \dots$, and set $w_2 \equiv 0$ on $Y_0 + d_+/\varepsilon$. Now the co-derivative of $w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon)$ at d_+ is equal to $\varepsilon(a_0 w_0')((d_+/\varepsilon)^+)$. We define a corrector εu_1 in D by setting (*cf.* (4.57))

$$u_1(x) := -(a_0 w_0')((d_+/\varepsilon)^+) \int_x^{d_+} a_D^{-1}(f_1 + c f_2), \quad x \in D.$$

Here the smooth functions f_1 and f_2 are chosen in the following way. Let the points d_1 and d_2 be such that $(d_- + d_+)/2 \leq d_1 < d_2 < d_+$. We require that $0 \leq f_1 \leq 1$, $f_1(d_+) = 1$, $f_1(x) = 0$ for $x \in [d_-, d_2]$, $f_2(x) = 0$ for $x \in [d_-, d_1] \cup [d_2, d_+]$, $f_2(x) < 0$ for $x \in (d_1, d_2)$. Finally, the constant c is chosen so that $u_1 \equiv 0$ on $[d_-, d_1]$.

Now we assume that w_0 in (4.51) has extrema inside the soft component $[d_+/\varepsilon, \infty) \cap \Omega_0$, *i.e.* there exists a point $y_+^* \in Y_0$ such that $|w_0(y_+^* + z)| = \max_{Y_0 + z} |w_0|$ and $(a_0 w_0')(y_+^* + z) = 0$ for all z satisfying $Y_0 + z \subset [d_+/\varepsilon, \infty)$ (note that w_0 may or may not vanish on the stiff component — it is not important). This situation is rather generic, for example, for constant a_0 and ρ_0 this assumption is true for any $\lambda_0 > a_0 \pi^2 / (\rho_0 h^2)$. In this case we “attach” the periodic structure to D at the point where $|w_0|$ attains its maximum: we define the soft and stiff components to the right of D via $\Omega_0^+ := \bigcup_z (Y_0 - y_+^* + z + d_+/\varepsilon) \cap (d_+/\varepsilon, \infty)$, and $\Omega_1^+ := \bigcup_z (Y_1 - y_+^* + z + d_+/\varepsilon)$ respectively, $z = 0, 1, \dots$. Analogously to the above, w_0 is defined according to (4.44), (4.51), requiring $w_0(d_+/\varepsilon) = u_0(d_+)$, and w_2 is defined according to (4.47) and (4.58) on

$[d_+/\varepsilon + h - y_+^*, \infty)$, requiring $w_2(d_+/\varepsilon + h - y_+^* + z) = 0$, $z = 0, 1, \dots$, and we set $w_2 \equiv 0$ on $[0, h - y_+^*] + d_+/\varepsilon$. Since the co-derivative of $w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon)$ vanishes at d_+ by construction, we do not need any corrector term in D contrary to the previous case. Thus we set $u_1 \equiv 0$ in this case.

One can perform analogous construction on the left of D . In any case the new approximate solution

$$u_{\varepsilon, \text{ap}}(x) := \begin{cases} u_0(x) + \varepsilon u_1(x), & x \in D, \\ w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon), & x \in \mathbb{R} \setminus D, \end{cases}$$

belongs to the domain of A_D^ε and satisfies

$$\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}} = \begin{cases} -\varepsilon (a_D u_1')'(x) - \varepsilon \lambda_0 \rho_D u_1(x), & x \in D, \\ -\varepsilon^2 (a_0 w_2')'(x/\varepsilon) - \varepsilon^2 \lambda_0 \rho_0(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_0^\varepsilon \setminus D, \\ -\varepsilon^2 \lambda_0 \rho_1(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_1^\varepsilon \setminus D. \end{cases}$$

(We remind that the notation a_D^ε , ρ_D^ε , Ω_0^ε and Ω_1^ε has to be redefined accordingly to the above construction in each case).

In the case if w_0 has no extrema inside the soft component at least on one of the intervals $(-\infty, d_-/\varepsilon]$ or $[d_+/\varepsilon, \infty)$, the term u_1 is non-zero. Then, similarly to (4.64), we obtain an improved estimate

$$\|(A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}}\|_{L_{\rho_D^\varepsilon}^2(\mathbb{R})} \leq C\varepsilon.$$

However, if w_0 has extrema inside the soft component on each of the intervals $(-\infty, d_-/\varepsilon]$ and $[d_+/\varepsilon, \infty)$, the term $u_1 \equiv 0$, and we have

$$\|(A_D^\varepsilon - \lambda_0) u_{\varepsilon, \text{ap}}\|_{L_{\rho_D^\varepsilon}^2(\mathbb{R})} \leq C\varepsilon^2.$$

The improved estimates for the error term immediately imply the following statement.

Theorem 4.2. *Suppose that $\lambda_0 \in \sigma(A_{N,D}) \setminus (\bigcup_\theta \sigma(A_\theta))$ and additionally the solution w_0 in (4.51) has extrema inside the soft component on each of the intervals $(-\infty, d_-/\varepsilon]$ and $[d_+/\varepsilon, \infty)$ (respectively, has no extrema inside the soft component at least on one of the intervals $(-\infty, d_-/\varepsilon]$ or $[d_+/\varepsilon, \infty)$). Then one can attach composite structures on both sides of the defect D and define the approximate solution $u_{\varepsilon, \text{ap}}$ to the eigenvalue problem in a specific way described above, so that the assertions of Theorem 4.1 hold with the improved estimates*

$$\begin{aligned} \text{dist}(\lambda_0, \sigma(A_D^\varepsilon)) &\leq C_1 \varepsilon^2 \text{ (resp. } \leq C_1 \varepsilon), \\ |\lambda_\varepsilon - \lambda_0| &\leq C_1 \varepsilon^2 \text{ (resp. } \leq C_1 \varepsilon), \\ \left\| u_{\varepsilon, \text{ap}} - \sum_{j \in J_\varepsilon} c_j^\varepsilon u_{\varepsilon, j} \right\|_{L_{\rho_D^\varepsilon}^2(\mathbb{R})} &\leq C_2 \varepsilon^2 \text{ (resp. } \leq C_2 \varepsilon), \end{aligned} \tag{4.67}$$

for some $C_1, C_2 > 0$.

Remark 4.3. *In the above theorem the attached structures do not need to be periodic extensions of each other. In case of “non-matching” periodic structures on each side of the defect the essential spectrum of the resulting operator is exactly the same as in the purely periodic case without the defect. This can easily be seen by considering Weyl’s sequences in each of the cases.*

5 Extreme localisation of defect eigenfunctions

The method of asymptotic expansions allows us to show that for any eigenvalue λ_0 of $A_{N,D}$, cf. (2.6), (4.42), in a gap of $\bigcup_\theta \sigma(A_\theta)$ there exists a sequence of eigenvalues of A_D^ε converging to λ_0 . In this section we

provide a statement on the rate of decay of eigenfunctions of A_D^ε outside the defect. Namely, the fact that one-dimensional problems admit an explicit form of solutions in terms of the fundamental system allows us to show that the eigenfunctions u_ε decay at an accelerated exponential rate outside of the defect, which is Claim 2 of Theorem 2.4.

We assume a sequence of eigenvalues λ_ε of A_D^ε converges to $\lambda_0 \in \mathbb{R} \setminus \bigcup_\theta \sigma(A_\theta)$ as $\varepsilon \rightarrow 0$, and consider the corresponding sequence u_ε of $L^2(\mathbb{R})$ -normalised eigenfunctions, *i.e.*

$$\int_{\mathbb{R}} a_D^\varepsilon u_\varepsilon' \varphi' = \lambda_\varepsilon \int_{\mathbb{R}} \rho_D^\varepsilon u_\varepsilon \varphi \quad \forall \varphi \in H^1(\mathbb{R}).$$

Recalling the unitary operator $\mathcal{R}_\varepsilon : L_{\rho_\varepsilon}^2(\mathbb{R}) \rightarrow L_{\rho}^2(\mathbb{R})$ given by $\mathcal{R}_\varepsilon(f)(y) = \varepsilon^{1/2} f(\varepsilon y)$, we note that for all $z \in \mathcal{I}_\varepsilon$ (see (4.45)), the function $\tilde{u}_\varepsilon := \mathcal{R}_\varepsilon u_\varepsilon$ solves

$$-(a_0 \tilde{u}_\varepsilon')' = \lambda_\varepsilon \rho_0 \tilde{u}_\varepsilon \quad \text{on } Y_0 + z, \quad (5.68)$$

$$-\varepsilon^{-2} (a_1 \tilde{u}_\varepsilon')' = \lambda_\varepsilon \rho_1 \tilde{u}_\varepsilon \quad \text{on } Y_1 + z, \quad (5.69)$$

and satisfies the interface conditions

$$\begin{aligned} \tilde{u}_\varepsilon|_{Y_0+z}(z+h) &= \tilde{u}_\varepsilon|_{Y_1+z}(z+h), & (a_0 \tilde{u}_\varepsilon')((z+h)^-) &= \varepsilon^{-2} (a_1 \tilde{u}_\varepsilon')((z+h)^+), \\ \tilde{u}_\varepsilon|_{Y_0+z+1}(z+1) &= \tilde{u}_\varepsilon|_{Y_1+z}(z+1), & (a_0 \tilde{u}_\varepsilon')((z+1)^+) &= \varepsilon^{-2} (a_1 \tilde{u}_\varepsilon')((z+1)^-). \end{aligned} \quad (5.70)$$

There exist solutions $v_1^\varepsilon, v_2^\varepsilon$ to the equation $-(a_0 u')' = \lambda_\varepsilon \rho_0 u$, on Y_0 , and solutions $w_1^\varepsilon, w_2^\varepsilon$ to the equation $-\varepsilon^{-2} (a_1 u')' = \lambda_\varepsilon \rho_1 u$, on Y_1 , such that

$$\begin{pmatrix} v_1^\varepsilon & v_2^\varepsilon \\ a_0 v_1^{\varepsilon'} & a_0 v_2^{\varepsilon'} \end{pmatrix} \Big|_{y=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} w_1^\varepsilon & w_2^\varepsilon \\ a_1 w_1^{\varepsilon'} & a_1 w_2^{\varepsilon'} \end{pmatrix} \Big|_{y=h} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.71)$$

The solution \tilde{u}_ε to (5.68), (5.69), $z \in \mathcal{I}_\varepsilon$, admits the representation

$$\tilde{u}_\varepsilon(y) = \begin{cases} a_z^\varepsilon v_1^\varepsilon(y-z) + b_z^\varepsilon v_2^\varepsilon(y-z), & y \in Y_0 + z, \\ c_z^\varepsilon w_1^\varepsilon(y-z) + d_z^\varepsilon w_2^\varepsilon(y-z), & y \in Y_1 + z. \end{cases} \quad (5.72)$$

For all ε , the coefficients $a_z^\varepsilon, b_z^\varepsilon, c_z^\varepsilon$ and d_z^ε , $z \in \mathcal{I}_\varepsilon$, are related to each other by the conditions (5.70), as follows:

$$\begin{aligned} c_z^\varepsilon &= a_z^\varepsilon v_1^\varepsilon(h) + b_z^\varepsilon v_2^\varepsilon(h), & \varepsilon^{-2} d_z^\varepsilon &= a_z^\varepsilon (a_0 v_1^{\varepsilon'})(h) + b_z^\varepsilon (a_0 v_2^{\varepsilon'})(h), \\ a_{z+1}^\varepsilon &= c_z^\varepsilon w_1^\varepsilon(1) + d_z^\varepsilon w_2^\varepsilon(1), & \varepsilon^2 b_{z+1}^\varepsilon &= c_z^\varepsilon (a_1 w_1^{\varepsilon'})(1) + d_z^\varepsilon (a_1 w_2^{\varepsilon'})(1). \end{aligned}$$

Eliminating c_z^ε and d_z^ε gives the iterative system

$$\begin{pmatrix} a_{z+1}^\varepsilon \\ b_{z+1}^\varepsilon \end{pmatrix} = M_\varepsilon \begin{pmatrix} a_z^\varepsilon \\ b_z^\varepsilon \end{pmatrix}, \quad (5.73)$$

where the matrix M_ε is given by

$$M_\varepsilon = \begin{pmatrix} v_1^\varepsilon(h)w_1^\varepsilon(1) + \varepsilon^2 (a_0 v_1^{\varepsilon'})(h)w_2^\varepsilon(1) & v_2^\varepsilon(h)w_1^\varepsilon(1) + \varepsilon^2 (a_0 v_2^{\varepsilon'})(h)w_2^\varepsilon(1) \\ \varepsilon^{-2} v_1^\varepsilon(h)(a_1 w_1^{\varepsilon'})(1) + (a_0 v_1^{\varepsilon'})(h)(a_1 w_2^{\varepsilon'})(1) & \varepsilon^{-2} v_2^\varepsilon(h)(a_1 w_1^{\varepsilon'})(1) + (a_0 v_2^{\varepsilon'})(h)(a_1 w_2^{\varepsilon'})(1) \end{pmatrix}. \quad (5.74)$$

It follows from the property that the modified Wronskian is constant,

$$\det \begin{pmatrix} v_1^\varepsilon & v_2^\varepsilon \\ a_0 v_1^{\varepsilon'} & a_0 v_2^{\varepsilon'} \end{pmatrix} \equiv 1, \quad \det \begin{pmatrix} w_1^\varepsilon & w_2^\varepsilon \\ a_1 w_1^{\varepsilon'} & a_1 w_2^{\varepsilon'} \end{pmatrix} \equiv 1,$$

that the characteristic polynomial of M_ε is given by

$$\begin{aligned} \det(M_\varepsilon - \mu I) &= \mu^2 - \mu h_\varepsilon + 1, \\ h_\varepsilon &= v_1^\varepsilon(h)w_1^\varepsilon(1) + \varepsilon^2(a_0v_1^{\varepsilon'})(h)w_2^\varepsilon(1) + \varepsilon^{-2}v_2^\varepsilon(h)(a_1w_1^{\varepsilon'})(1) + (a_0v_2^{\varepsilon'})(h)(a_1w_2^{\varepsilon'})(1). \end{aligned} \quad (5.75)$$

Recalling, from Section 3.1, the fundamental solutions v_1, v_2 of (cf. (3.15))

$$-(a_0u')' = \lambda_0\rho_0u \quad \text{in } Y_0,$$

satisfying

$$\begin{pmatrix} v_1(0) & v_2(0) \\ (a_0v_1')(0) & (a_0v_2')(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we shall prove in the second half of this section the following property.

Lemma 5.1. *The following convergence holds:*

$$\lim_{\varepsilon \rightarrow 0} h_\varepsilon = v_1(h) + (a_0v_2')(h) - \lambda_0v_2(h) \int_{Y_1} \rho_1. \quad (5.76)$$

Assuming that (5.76) holds, since $\lambda_0 \in \mathbb{R} \setminus \bigcup_\theta \sigma(A_\theta)$, or equivalently (see Section 3.1) λ_0 is such that (cf. (3.20))

$$\left| v_1(h) + (a_0v_2')(h) - \lambda_0v_2(h) \int_{Y_1} \rho_1 \right| > 2,$$

for sufficiently small ε we find that $|h_\varepsilon| > 2$.

As per the discussion in Section 4, the eigenvalues $\mu_1^\varepsilon, \mu_2^\varepsilon$ of the matrix M_ε satisfy the identity $\mu_1^\varepsilon\mu_2^\varepsilon = 1$ and the nature of \tilde{u}_ε away from the defect is determined by the coefficient h_ε . In particular, if $|h_\varepsilon| > 2$ then the roots $\mu_1^\varepsilon, \mu_2^\varepsilon$ are such that $|\mu_1^\varepsilon| < 1$ and $|\mu_2^\varepsilon| > 1$ and there exist linearly independent functions v_g, v_d on $\mathbb{R} \setminus ([d_-]_\varepsilon, [d_+]_\varepsilon)$ that grow and decay respectively. In this case, for u_ε to be an element of $L^2(\mathbb{R})$ it is necessary that u_ε is proportional to the decaying solution v_d , which takes the form

$$v_d(x) = \begin{cases} \exp\left(\frac{\ln|\mu_1^\varepsilon|}{\varepsilon} \text{dist}(x, D)\right) p_1^\varepsilon(x/\varepsilon), & x \in [d_+, \infty), \\ \exp\left(\frac{\ln|\mu_1^\varepsilon|}{\varepsilon} \text{dist}(x, D)\right) p_2^\varepsilon(x/\varepsilon), & x \in (-\infty, d_-], \end{cases}$$

for some periodic (respectively, anti-periodic) functions $p_1^\varepsilon, p_2^\varepsilon$, when $h_\varepsilon > 2$ (respectively, when $h_\varepsilon < -2$). Therefore, for any ν satisfying $\nu < -\ln|\mu_1^\varepsilon| = |\ln|\mu_1^\varepsilon||$ the product $g_{\nu/\varepsilon}u_\varepsilon$ is in $L^2(\mathbb{R})$, where $g_{\nu/\varepsilon}$ is defined by (2.9). Then the third claim of Theorem 2.4 follows by noticing that by (5.76) μ_1^ε converges to μ_1 , the smallest by absolute value root of $\mu^2 - h\mu + 1$, where

$$h := v_1(h) + (a_0v_2')(h) - \lambda_0v_2(h) \int_{Y_1} \rho_1,$$

as $\varepsilon \rightarrow 0$.

It remains to prove the convergence (5.76).

Proof of Lemma 5.1.

The vector field

$$\eta_j^\varepsilon := \begin{pmatrix} v_j^\varepsilon - v_j \\ a_0v_j^{\varepsilon'} - a_0v_j' \end{pmatrix}, \quad j = 1, 2, \quad (5.77)$$

solves the initial-value problem

$$\eta_j^{\varepsilon'} = \Phi^\varepsilon \eta_j^\varepsilon + \Psi_j^\varepsilon \quad \text{in } Y_0, \quad \eta_j^\varepsilon(0) = 0, \quad j = 1, 2, \quad (5.78)$$

for the matrix Φ^ε and vector Ψ_j^ε , $j = 1, 2$, given by

$$\Phi^\varepsilon = \begin{pmatrix} 0 & a_0^{-1} \\ -\lambda_\varepsilon \rho_0 & 0 \end{pmatrix}, \quad \Psi_j^\varepsilon = \begin{pmatrix} 0 \\ (\lambda_0 - \lambda_\varepsilon) \rho_0 v_j \end{pmatrix}, \quad j = 1, 2.$$

Since $\lambda_\varepsilon \rightarrow \lambda_0$ the solutions to (5.78) converge uniformly on Y_0 to the trivial solution of

$$\eta' = \Phi \eta \quad \text{in } Y_0, \quad \eta(0) = 0,$$

where Φ is the limit of Φ_ε , as $\varepsilon \rightarrow 0$ (see *e.g.* [19, Theorem 1.6.1]). Namely, we have

$$|\eta_j^\varepsilon(y)| = |\eta_j^\varepsilon(y) - \eta(y)| \leq C |\lambda_\varepsilon - \lambda_0|, \quad j = 1, 2,$$

for some constant C independent of ε . In particular, (5.77) implies that

$$\lim_{\varepsilon \rightarrow 0} v_j^\varepsilon(h) = v_j(h), \quad \lim_{\varepsilon \rightarrow 0} (a_0 v_j^{\varepsilon'})'(h) = (a_0 v_j')'(h), \quad j = 1, 2. \quad (5.79)$$

Similarly, it is easy to see that w_j^ε and $a_1 w_j^{\varepsilon'}$ converge uniformly on Y_1 to w_j and $a_1 w_j'$, where w_j , $j = 1, 2$, are the solutions of $(a_1 w')' = 0$ satisfying

$$\begin{pmatrix} w_1(h) & w_2(h) \\ (a_1 w_1')'(h) & (a_1 w_2')'(h) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $w_1 \equiv 1$ and $a_1 w_2' \equiv 1$ on Y_1 we see that

$$\begin{pmatrix} w_1^\varepsilon & w_2^\varepsilon \\ a_1 w_1^{\varepsilon'} & a_1 w_2^{\varepsilon'} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \int_h^y a_1^{-1} \\ 0 & 1 \end{pmatrix} \quad \text{uniformly on } Y_1 \text{ as } \varepsilon \rightarrow 0. \quad (5.80)$$

Furthermore, by the fundamental theorem of calculus and the fact $-\varepsilon^{-2}(a_1 w_1^{\varepsilon'})' = \lambda_\varepsilon \rho_1 w_1^\varepsilon$, we have

$$\varepsilon^{-2}(a_1 w_1^{\varepsilon'})(1) - \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(h) = -\lambda_\varepsilon \int_h^1 \rho_1 w_1^\varepsilon,$$

and since

$$\int_h^1 \rho_1 w_1^\varepsilon - w_1^\varepsilon(h) \int_h^1 \rho_1 = \int_h^1 \rho_1 (w_1^\varepsilon - w_1^\varepsilon(h)) = \int_{Y_1} \rho_1(y) \left(\int_h^y w_1^{\varepsilon'} \right) dy,$$

it follows that

$$\begin{aligned} \left| \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(1) - \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(h) + \lambda_\varepsilon w_1^\varepsilon(h) \int_h^1 \rho_1 \right| &= \left| \lambda_\varepsilon \int_{Y_1} \rho_1(y) \left(\int_h^y w_1^{\varepsilon'} \right) dy \right| \\ &\leq |\lambda_\varepsilon| \|\rho_1\|_{L^\infty} \|w_1^{\varepsilon'}\|_{L^\infty}, \end{aligned}$$

which together with (5.80) implies

$$\lim_{\varepsilon \rightarrow 0} \left| \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(1) - \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(h) + w_1^\varepsilon(h) \lambda_\varepsilon \int_h^1 \rho_1 \right| = 0.$$

Taking into account the initial conditions (5.71) we obtain

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2}(a_1 w_1^{\varepsilon'})(1) = -\lambda_0 \int_{Y_1} \rho_1. \quad (5.81)$$

Finally, assertions (5.79), (5.80) and (5.81) imply (5.76), as required.

6 Resolvent estimates for the problem without defect

In this section we study the behaviour of the unperturbed periodic operator A^ε in the operator norm as $\varepsilon \rightarrow 0$. In particular, we construct a full asymptotic expansions for the resolvent of A^ε using a version of the asymptotic framework developed in [7], see Theorem 6.2 below. This directly implies the order-sharp operator norm resolvent convergence estimate, uniform in θ , formulated in Theorem 2.2. The latter, in turn, implies the uniform in θ convergence, as $\varepsilon \rightarrow 0$, of the spectral band functions $\lambda_n^\varepsilon(\theta)$ to $\lambda_n(\theta)$, $n \in \mathbb{N}$, which is also order-sharp.

Recall the operator A^ε in $L^2_{\rho^\varepsilon}(\mathbb{R})$ associated with the sesquilinear form

$$\beta^\varepsilon(u, v) = \int_{\Omega_1^\varepsilon} a_1(\frac{\cdot}{\varepsilon})u'\overline{v'} + \int_{\Omega_0^\varepsilon} \varepsilon^2 a_0(\frac{\cdot}{\varepsilon})u'\overline{v'}, \quad u, v \in H^1(\mathbb{R}).$$

By a scaled version of the Floquet-Bloch transform,³ which is given as the continuous extension of the following action on *e.g.* continuous functions with compact support

$$(\mathcal{U}_\varepsilon f)(\theta, y) = \sqrt{\frac{\varepsilon}{2\pi}} \sum_{z \in \mathbb{Z}} f(\varepsilon(y - z))e^{i\theta z}, \quad y \in Y, \theta \in [0, 2\pi), \quad (6.82)$$

we see that \mathcal{U}_ε unitarily maps $L^2_{\rho^\varepsilon}(\mathbb{R})$ to the Bochner space $L^2(0, 2\pi; L^2_\rho(Y))$ and $\mathcal{U}_\varepsilon A^\varepsilon f(\theta, \cdot) = A_\theta^\varepsilon \mathcal{U}_\varepsilon f(\theta, \cdot)$. Here, A_θ^ε is the operator defined in $L^2_\rho(Y)$ and associated with the form

$$\beta_\theta^\varepsilon(u, v) := \int_{Y_0} a_0 u'\overline{v'} + \varepsilon^{-2} \int_{Y_1} a_1 u'\overline{v'}, \quad u, v \in H_\theta^1(Y).$$

We recall that $H_\theta^1(Y)$ is the complex Hilbert space of $H^1(Y)$ -functions that are θ -quasiperiodic. We equip the space $H_\theta^1(Y)$ with the graph norm

$$\| \| u \| \| := \sqrt{\int_{Y_0} a_0 |u'|^2 + \int_{Y_1} a_1 |u'|^2 + \int_Y \rho |u|^2}, \quad (6.83)$$

and consider the subspace

$$V_\theta := \{v \in H_\theta^1(Y) : v' \equiv 0 \text{ in } Y_1\}$$

and its orthogonal complement V_θ^\perp in H_θ^1 with respect to the inner product associated with $\| \| \cdot \| \|$. The following result, established in [8], is of fundamental importance in studying the asymptotics of A^ε , equivalently A_θ^ε .

Lemma 6.1. *There exists a constant $C_P > 0$, independent of θ , such that*

$$\| \| P_\theta^\perp u \| \| \leq C_P \| \sqrt{a_1} u' \|_{L^2(Y_1)}, \quad \forall u \in H_\theta^1(Y), \quad (6.84)$$

where P_θ^\perp is the orthogonal projection of $H_\theta^1(Y)$ onto V_θ^\perp .

For $\theta \in [0, 2\pi)$ and all $f \in L^2_\rho(Y)$, we consider the resolvent problem

$$-((\varepsilon^{-2} a_1 + a_0)u_\theta^\varepsilon)' + \rho u_\theta^\varepsilon = \rho f \quad \text{on } (0, 1). \quad (6.85)$$

We look for an asymptotic expansion of u_θ^ε in the form

$$u_\theta^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^{2n} u_\theta^{(2n)}, \quad u_\theta^{(2n)} \in H_\theta^1(Y) \quad \forall n \in \mathbb{N}. \quad (6.86)$$

The following result holds.

³See Appendix A below for further information on the Floquet-Bloch transform.

Theorem 6.2. For each $\theta \in [0, 2\pi)$ and $f \in L^2_\rho(Y)$, consider the unique solution $u_\theta^{(0)} \in V_\theta$ to the problem

$$\int_{Y_0} a_0(u_\theta^{(0)})' \overline{\varphi}' + \int_Y \rho u_\theta^{(0)} \overline{\varphi} = \int_Y \rho f \overline{\varphi} \quad \forall \varphi \in V_\theta,$$

and for all $n \in \mathbb{N}$ consider the unique solution $u_\theta^{(2n)} \in V_\theta^\perp$ to

$$-\left(a_1(u_\theta^{(2n)})'\right)' = \left(a_0(u_\theta^{(2(n-1))})'\right)' - \rho u_\theta^{(2(n-1))} + \delta_{1n} \rho f,$$

where δ_{1n} is the Kronecker delta function. Then, for each $N \in \mathbb{N}$ the sum

$$U_\theta^{(N)} := \sum_{n=0}^N \varepsilon^{2n} u_\theta^{(2n)}$$

approximates the solution u_θ^ε to (6.85) in the following sense:

$$\| \|u_\theta^\varepsilon - U_\theta^{(N)}\| \| \leq C_P^{2(N+1)} \varepsilon^{2(N+1)} \|f\|_{L^2_\rho(Y)}.$$

Remark 6.3. By an application of the min-max principle, Theorem 6.2 implies that the n -th eigenvalue $\lambda_n^\varepsilon(\theta)$ of the operator A_θ^ε is ε^2 -close, uniformly in θ , to the n -th eigenvalue $\lambda_n(\theta)$ of A_θ , i.e. for each $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that

$$|\lambda_n^\varepsilon(\theta) - \lambda_n(\theta)| \leq c_n \varepsilon^2 \quad \forall \theta \in [0, 2\pi).$$

In particular, this indirectly implies, since λ_n is the uniform limit of continuous functions, that λ_n is continuous in θ . A direct proof of this fact can be arrived at by the definition of the operators A_θ and the continuity properties (in the Hausdorff sense) of their domains $D(A_\theta)$, see [8, Appendix B].

Proof. Substituting (6.86) into (6.85) and equating powers of ε yields a system of recurrence relations for the functions $u_\theta^{(2n)}$, $n \in \mathbb{N}$. The first equation in this system, which corresponds to ε^{-2} , is

$$-\left(a_1(u_\theta^{(0)})'\right)' = 0 \quad \text{on } (0, 1), \tag{6.87}$$

which implies that $u_\theta^{(0)} \in V_\theta = \{v \in H_\theta^1(Y) : v' \equiv 0 \text{ on } Y_1\}$ (recall that $a_1 \equiv 0$ on Y_0). The remaining equations, obtained by considering the terms of order ε^{2j} , $j = 0, 1, 2, \dots$ are

$$-\left(a_1(u_\theta^{(2n)})'\right)' = \left(a_0(u_\theta^{(2(n-1))})'\right)' - \rho u_\theta^{(2(n-1))} + \delta_{1n} \rho f, \quad \text{on } (0, 1), \quad n \in \mathbb{N}, \tag{6.88}$$

where, as before, δ_{in} denotes the Kronecker delta function. The existence of solutions to differential equations with degenerate coefficients such as (6.88) was first studied in [11] for the case $\theta = 0$, and it was shown therein that existence is guaranteed by inequalities of the type (6.84). By following this general framework, and it can be readily shown that (6.84) implies the following result.

Lemma 6.4. For a given $F \in H_\theta^{-1}(Y)$, the dual space of $H_\theta^1(Y)$, there exist (infinitely many) solutions $u \in H_\theta^1(Y)$ to the problem

$$\int_{Y_1} a_1 u' \overline{\varphi}' =_{H_\theta^{-1}(Y)} \langle F, \varphi \rangle_{H_\theta^1(Y)} \quad \forall \varphi \in H_\theta^1(Y),$$

if and only if F satisfies the condition

$${}_{H_\theta^{-1}(Y)} \langle F, v \rangle_{H_\theta^1(Y)} = 0 \quad \forall v \in V_\theta.$$

Such solutions are unique in V_θ^\perp , i.e. for any two solutions u_1, u_2 one has $u_1 - u_2 \in V_\theta$.

Consequently, the system (6.88) is solvable if and only if the conditions

$$\int_{Y_0} a_0(u_\theta^{(2n)})' \overline{\varphi'} + \int_Y \rho u_\theta^{(2n)} \overline{\varphi} = \delta_{0n} \int_Y \rho f \overline{\varphi} \quad \forall \varphi \in V_\theta, \quad n+1 \in \mathbb{N}, \quad (6.89)$$

hold. The equation for $n=0$ uniquely determines $u_\theta^{(0)}$ and for $n \geq 1$, due to the choice (6.83) of the norm on $H_\theta^1(Y)$, demonstrates that $u_\theta^{(2n)} \in V_\theta^\perp$. Substituting $\varphi = u_\theta^{(0)}$ into the identity (6.89) for $n=0$, recalling (6.83), the fact that $a_1(u_\theta^{(0)})' \equiv 0$ and using the Cauchy-Schwarz inequality, we obtain

$$|||u_\theta^{(0)}|||^2 \leq \|f\|_{L_\rho^2(Y)} \|u_\theta^{(0)}\|_{L_\rho^2(Y)}.$$

Hence, $u_\theta^{(0)}$ satisfies the bound

$$|||u_\theta^{(0)}||| \leq \|f\|_{L_\rho^2(Y)} \quad \forall \theta \in [0, 2\pi). \quad (6.90)$$

By Lemmas 6.1 and 6.4, the solution $u_\theta^{(2n)} \in V_\theta^\perp$ to (6.88) is unique and

$$|||u_\theta^{(2n)}||| \leq C_P \|\sqrt{a_1}(u_\theta^{(2n)})'\|_{L^2(Y_1)}. \quad (6.91)$$

Equations (6.88) and the orthogonality of V_θ and V_θ^\perp with respect to the inner product associated with the norm (6.83), in particular, the orthogonality of $u_\theta^{(0)}$ and $u_\theta^{(2)}$, imply that

$$\int_{Y_1} a_1 |(u_\theta^{(2n)})'|^2 = \delta_{1n} \int_Y \rho f \overline{u_\theta^{(2n)}} - (1 - \delta_{1n}) \left(\int_{Y_0} a_0 (u_\theta^{(2(n-1))})' \overline{(u_\theta^{(2n)})'} + \int_Y \rho u_\theta^{(2(n-1))} \overline{u_\theta^{(2n)}} \right), \quad n \geq 1,$$

and (6.91) yields

$$|||u_\theta^{(2)}||| \leq C_P^2 \|f\|_{L_\rho^2(Y_1)}, \quad |||u_\theta^{(2n)}||| \leq C_P^2 |||u_\theta^{(2(n-1))}|||, \quad n \geq 2.$$

By iterating the above inequalities we establish that

$$|||u_\theta^{(2n)}||| \leq C_P^{2n} \|f\|_{L_\rho^2(Y_1)}, \quad n \geq 1. \quad (6.92)$$

Having identified each term in the expansion, for each $n \in \mathbb{N}$ we define the remainder R_θ^ε (dropping the index N for brevity), according to the formula

$$u_\theta^\varepsilon = \sum_{n=0}^N \varepsilon^{2n} u_\theta^{(2n)} + \varepsilon^{2N} R_\theta^\varepsilon, \quad (6.93)$$

and find, via (6.87) and (6.88), that $R_\theta^\varepsilon \in H_\theta^1(Y)$ solves the problem

$$-((\varepsilon^{-2} a_1 + a_0)(R_\theta^\varepsilon)')' + \rho R_\theta^\varepsilon = \delta_{0N} \rho f + (a_0(u_\theta^{(2N)})')' - \rho u_\theta^{(2N)} \quad \text{on } (0, 1),$$

that is

$$\int_{Y_1} \varepsilon^{-2} a_1 (R_\theta^\varepsilon)' \overline{v'} + \int_{Y_0} a_0 (R_\theta^\varepsilon)' \overline{v'} + \int_Y \rho R_\theta^\varepsilon \overline{v} = \delta_{0N} \int_Y \rho f \overline{v} - \int_{Y_0} a_0 (u_\theta^{(2N)})' \overline{v'} - \int_Y \rho u_\theta^{(2N)} \overline{v} \quad \forall v \in H_\theta^1(Y).$$

Setting $v \in V_\theta$, recalling the norm (6.83) and (6.89), demonstrates that $R_\theta^\varepsilon \in V_\theta^\perp$. Additionally, setting $v = R_\theta^\varepsilon$ above implies that

$$\varepsilon^{-2} \int_{Y_1} a_1 |(R_\theta^\varepsilon)'|^2 \leq \delta_{0N} \int_Y \rho f \overline{R_\theta^\varepsilon} - \int_{Y_0} a_0 (u_\theta^{(2N)})' \overline{(R_\theta^\varepsilon)'} - \int_Y \rho u_\theta^{(2N)} \overline{R_\theta^\varepsilon},$$

and inequalities (6.84), (6.92), along with another application of the Cauchy-Schwarz inequality yields

$$\|R_\theta^\varepsilon\| \leq C_P^{2(N+1)} \varepsilon^2 \|f\|_{L_\rho^2(Y)}.$$

Finally, by combining this inequality with (6.93) we deduce that

$$\left\| u_\theta^\varepsilon - \sum_{n=0}^N \varepsilon^{2n} u_\theta^{(2n)} \right\| \leq C^{2(N+1)} \varepsilon^{2(N+1)} \|f\|_{L_\rho^2(Y)},$$

as required. □

A Appendix: Norm-resolvent asymptotics of A^ε and the approximating operator A^0

We consider the space H to be the closure in $L_\rho^2(\mathbb{R})$ of

$$H^+ := \left\{ v \in H^1(\mathbb{R}) : v' \equiv 0 \text{ on } \Omega_1 := \bigcup_{z \in \mathbb{Z}} (Y_1 + z) \right\}.$$

Both H and H^+ are Hilbert spaces when equipped with the inner products inherited from $L_\rho^2(\mathbb{R})$ and $H^1(\mathbb{R})$ respectively, and clearly H^+ is dense in H with continuous embedding (recall ρ is taken to be uniformly positive and bounded). The norm of H^+ , which is the standard H^1 -norm, is equivalent to the graph norm

$$\|\cdot\|_{H^+} := \left(\|\cdot\|_{L_\rho^2(\mathbb{R})}^2 + \beta^0(\cdot, \cdot) \right)^{1/2}, \quad (1.94)$$

where β^0 is the sesquilinear form

$$\beta^0(u, v) := \int_{\Omega_0} a_0 u' \overline{v'}, \quad u, v \in H^+.$$

We shall henceforth consider H^+ to be equipped with the graph norm (1.94), and denote by H^- the dual space consisting of bounded linear functionals on H^+ . As β^0 is a non-negative closed symmetric sesquilinear form, it generates a densely defined non-negative self-adjoint linear operator A^0 in H . The domain $D(A^0)$ is the dense subset of H^+ consisting of the solutions to the problem: for each $f \in H^+$ we consider $u \in H^+$ the unique solution to the problem

$$\beta^0(u, v) + \int_{\mathbb{R}} \rho u \overline{v} = \int_{\mathbb{R}} \rho f \overline{v} \quad \forall v \in H^+,$$

and set $A^0 u = f - u$ for $u \in D(A^0)$. The operator A^0 is unitarily equivalent to the fibre integral operator $\int^\oplus A_\theta$, cf. Remark 2.1, and the unitary map is given by the continuous extension of the Floquet-Bloch transform \mathcal{U} , cf. [15, Section 2.2], which acts on smooth functions f with compact support as

$$\mathcal{U}f(\theta, y) := \frac{1}{\sqrt{2\pi}} \sum_{z \in \mathbb{Z}} f(y - z) e^{i\theta z}, \quad \theta \in [0, 2\pi), \quad y \in Y.$$

Indeed, \mathcal{U} is well known to be a unitary operator between $L_\rho^2(\mathbb{R})$ and the Bochner space $L^2(0, 2\pi; L_\rho^2(Y))$ and it is straightforward to see that

$$\mathcal{U}A^0 f(\theta; \cdot) = A_\theta \mathcal{U}f(\theta; \cdot), \quad \forall f \in L_\rho^2(\mathbb{R}), \quad \theta \in [0, 2\pi).$$

Furthermore, it is clear that \mathcal{U} unitarily maps H^+ to the space $L^2(0, 2\pi; V_\theta)$ (we recall that $V_\theta = \{v \in H_\theta^1(Y) : v' \equiv 0 \text{ on } Y_1\}$). It is easy to verify that the spectrum of A^0 coincides with the union of the spectra of A_θ over all $\theta \in [0, 2\pi)$, *i.e.*

$$\sigma(A^0) = \bigcup_{\theta} \sigma(A_\theta) = \bigcup_{n \in \mathbb{N}} [\min_{\theta} \lambda_n(\theta), \max_{\theta} \lambda_n(\theta)].$$

Theorem 2.2 implies in particular that A^ε is order- $O(\varepsilon^2)$ close in the norm-resolvent sense to A^0 (up to unitary equivalence), *i.e.* there exists a constant $C > 0$ such that

$$\|\mathcal{R}_\varepsilon(A^\varepsilon + 1)^{-1}\mathcal{R}_\varepsilon^{-1} - (A^0 + 1)^{-1}\|_{L^2_\rho(\mathbb{R}) \rightarrow L^2_\rho(\mathbb{R})} \leq C\varepsilon^2$$

for all $\varepsilon \in (0, 1)$, where $\mathcal{R}_\varepsilon : L^2_{\rho^\varepsilon}(\mathbb{R}) \rightarrow L^2_\rho(\mathbb{R})$ is the unitary transformation $\mathcal{R}_\varepsilon(f)(y) = \varepsilon^{1/2}f(\varepsilon y)$.

B Appendix: Spectral decomposition of A^0

As the operator A^0 is self-adjoint, it has a spectral decomposition and we shall now characterise the space H^+ and its dual H^- in terms of a realisation of this spectral decomposition. For each θ , the self-adjoint operator A_θ has compact resolvent and for each of its eigenvalues $\lambda_n(\theta)$, $n \in \mathbb{N}$, we denote by $\psi_n(\theta; \cdot)$ the corresponding $L^2_\rho(Y)$ -normalised eigenfunction. Then the mapping Ψ given by

$$\Psi f(\theta; \cdot) = \{c_n(\theta)\}_{n \in \mathbb{N}}, \quad c_n(\theta) := \int_Y \rho(y) f(\theta, y) \overline{\psi_n(\theta; y)} dy,$$

unitarily maps $L^2(0, 2\pi; L^2_\rho(Y))$ to $\mathfrak{h} := L^2(0, 2\pi; \ell^2)$ so that

$$\Psi(\mathcal{U}A^0 f)(\theta, n) = \lambda_n(\theta)\Psi(\mathcal{U}f)(\theta, n),$$

where for $u \in \mathfrak{h}$, we denote by $u(\theta, n)$ is the n -th element of the sequence $u(\theta)$. It is easy to verify that $\Psi \circ \mathcal{U}$ unitarily maps H^+ to

$$\mathfrak{h}^+ := \{u(\theta, n) \in \mathfrak{h} : (\lambda_n(\theta) + 1)^{1/2}u(\theta, n) \in \mathfrak{h}\}.$$

By standard duality arguments, see for example [16, Chapter 1, Section 6.2], we show that $\Psi \circ \mathcal{U}$ unitarily maps H^- , the dual space of bounded linear functionals on H^+ , to

$$\mathfrak{h}^- := \{f : (0, 2\pi) \rightarrow \ell^2 \text{ measurable} : (\lambda_n(\theta) + 1)^{-1/2}f(\theta, n) \in \mathfrak{h}\},$$

in the sense that $F \in H^-$ if and only if there exists $f \in \mathfrak{h}^-$ such that

$${}_{H^-} \langle F, v \rangle_{H^+} = \sum_{n \in \mathbb{N}} \int_0^{2\pi} f(\theta, n) \overline{(\Psi \mathcal{U})v(\theta, n)} d\theta \quad \forall v \in H^+,$$

and we have

$$\|F\|_{H^-} = \sqrt{\sum_{n \in \mathbb{N}} \int_0^{2\pi} \frac{|f(\theta, n)|^2}{\lambda_n(\theta) + 1} d\theta}.$$

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References

- [1] Agmon, Sh. *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations: Bounds on Eigenfunctions of N -body Schrödinger Operators* (Mathematical Notes **29**, Princeton University Press (1982).
- [2] Alama, S., Avellaneda, M., Deift, P. A., Hempel, R. On the existence of eigenvalues of a divergence-form operator $A + \lambda B$ in a gap of $\sigma(A)$. *Asymptotic Anal.*, **8**(4), 311–344 (1994).
- [3] Birman, M. Sh. On the spectrum of singular boundary-value problems (Russian). *Math. Sb.* **55** (**97**), 125–174 (1961). English transl. in *Eleven Papers in Analysis*, AMS Transl. 2 **53**, AMS, Providence, RI, 23–60 (1966).
- [4] Birman, M. S., Solomjak, M. Z. *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Springer Netherlands (1987).
- [5] Cherdantsev, M. Asymptotic analysis of some spectral problems in high contrast homogenisation and in thin domains. *PhD Thesis*, University of Bath (2008).
- [6] Cherdantsev, M. Spectral convergence for high-contrast elliptic periodic problems with a defect via homogenization. *Mathematika* **55** (1–2), 29–57 (2009).
- [7] Cherednichenko, K. D., Cooper, S. Resolvent estimates for high-contrast elliptic problems with periodic coefficients. *Archive for Rational Mechanics and Analysis* **219** (3), 1061–1086 (2016).
- [8] Cherednichenko, K. D., Cooper, S., Guenneau, S. Spectral analysis of one-dimensional high-contrast elliptic problems with periodic coefficients. *Multiscale Modeling and Simulation* **13** (1), 72–98 (2015).
- [9] Eastham, M. S. P. *The spectral theory of periodic differential equations*, Scottish Academic Press (1973).
- [10] Figotin, A., Klein, A. Localized classical waves created by defects, *J. Statist. Phys.* **86**(1–2), 165–177 (1997).
- [11] Kamotski, I. V., Smyshlyaev, V. P.: Two-scale homogenization for a class of partially degenerating PDE systems. *arXiv:1309.4579* (2011).
- [12] Jikov, V. V., Kozlov, S. M., Oleinik, O. A. *Homogenization of Differential Operators and Integral Functionals*, Springer, Berlin (1994).
- [13] Kamotski, I. V., Smyshlyaev, V. P. Localised modes due to defects in high contrast periodic media via homogenisation. *Bath Institute for Complex Systems, preprint 3/06* (2011)
- [14] Knight, J. C., Photonic crystal fibres. *Nature* **424**, 847–851 (2003).
- [15] Kuchment, P. *Floquet Theory for Partial Differential Equations*, Birkhauser Verlag, Basel (1993).
- [16] Lions, J. L., Magenes, E. *Non-homogeneous Boundary Value Problems, Vol. 1*, Springer, Berlin, Heidelberg (1972).
- [17] Simon, B. Reed, M. *Modern Methods of Mathematical Physics, Volume 4: Analysis of Operators*. Academic Press (1978).
- [18] Russell, P. St J. Photonic crystal fibers. *Science* **299** 385–362 (2003).
- [19] Zettl, A., *Sturm-Liouville Theory (Mathematical Surveys and Monographs volume 121)*, The American Mathematical Society, USA (2005).
- [20] Zhikov, V. V. On an extension of the method of two-scale convergence and its applications, *Sbornik: Mathematics* **191**(7), 973–1014 (2000).

- [21] Zhikov, V. V., Pastukhova, S. E. On the Trotter-Kato theorem in a variable space, *Funct. Anal. Appl.* **41**(4), 264–270 (2007).
- [22] Vishik, M. I., Lyusternik, L. A. Regular degeneration and boundary layer for linear differential equations with small parameter. (Russian) *Uspekhi Mat. Nauk (N. S.)* (1957).
- [23] Weyl, H. Über beschränkte quadratische Formen, deren Differenz vollstetig ist, *Rend. Circ. Mat. Palermo* **27**, 373–392 (1909).