Correlation properties of continuous-time autoregressive processes delayed by the inverse of the stable subordinator

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N.N. Leonenko[a,1], **I. Papić**[b]

[a] *School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF244AG, UK*
[b] *Department of Mathematics, J.J. Strossmayer University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia*

Abstract. We define the delayed Lévy-driven continuous-time autoregressive process via the inverse of the stable subordinator. We derive correlation structure for the observed non-stationary delayed Lévy-driven continuous-time autoregressive processes of order p, emphasising low orders, and we show they exhibit long-range dependence property. Distributional properties are discussed as well.

Key words. Continuous-time autoregressive process, Lévy noise, Delayed stochastic process, Inverse of the stable subordinator, Mittag-Leffler function, Correlation structure

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1 Introduction

In the last few decades stochastic processes delayed via the inverse of the subordinators gained popularity due to the fact they can model "trapping events", i.e. time periods when the process rests. In finance, such models can describe delays between trades (Scalas (2006)) or interest rate data for developing countries (Janczura et al. (2011)). One way to capture this kind of behaviour are time-fractional models. In hydrology, time-fractional models can capture behaviour such as sticking and trapping of contaminant particles in a porous medium or river flow (Chakraborty et al. (2009), Schumer et al. (2003)). In statistical physics, modeling random waiting times between particle jumps in continuous-time random walks is done via fractional models (Meerschaert & Scheffler (2004)). For systematic read on stochastic models for fractional calculus see Meerschaert & Sikorskii (2012). On the other hand, classical models such as autoregressive moving average (ARMA) processes, their continuous counterparts, i.e. continuous-time autoregressive moving average

[1]Corresponding author; e-mail: LeonenkoN@cardiff.ac.uk; Telephone: +44(0)29 208 75521; Fax: +44(0)29 208 74199

(CARMA) processes as well as diffusions cannot capture such behaviour and therefore aren't adequate for modeling "trapping events". Motivated by this issue their delayed counterparts via inverse of subordinators have been studied. Recently, fractional Pearson diffusions have been studied in detail (Leonenko et al. (2013*a*,*b*, 2017)). Pearson diffusions include the well known Ornstein-Uhlenbeck process (Uhlenbeck & Ornstein (1930)), as well as Cox-Ingersoll-Ross(CIR) diffusion (Cox et al. (2005)). Fractional Pearson diffusions, i.e. time-changed (delayed) Pearson diffusions via the inverse of the stable subordinator have the long-range dependence property, as well as applicable class of stationary distributions. On the other hand, continuous-time autoregressive process delayed via the inverse of the stable subordinator (dCAR process) have been recently analyzed in Gajda et al. (2016), Wyłomańska & Gajda (2016). In particular, in Wyłomańska & Gajda (2016) codifference structure for dCAR(1) process is examined, as well as the simulation and estimation procedures. In Gajda et al. (2016) dCAR(1) process is used to model technical data, i.e. it is used to model behavior of particular mechanical system.

Motivated by the work in Gajda et al. (2016), Wyłomańska & Gajda (2016) we study correlation properties of the dCAR(p) process, i.e. the second-order Lévy-driven continuous-time autoregressive process delayed by the inverse of the stable subordinator and examine the long-range dependence of the process. Therefore, we do not observe Lévy-driven continuous-time autoregressive processes with infinite driving variance (for the full description of the process see Section 2).

Continuous-time autoregressive process of order p, CAR(p) process, can be symbolically represented in analogy to discrete case with equation:

$$dX^{p-1}(t) + \alpha_1 X^{p-1}(t)dt + \ldots + \alpha_p X(t)dt = \sigma dW(t),\ t \geq 0,$$

where driving process $\{W(t),\, t \geq 0\}$ is the standard Brownian motion.

In this paper, we focus on Lévy-driven CAR(p) process, i.e. CAR(p) process with Lévy process as the driving process. Reason for the usage of such processes is the rich class of non-Gaussian and heavy-tailed marginal distributions of underlying process, due to usage of Lévy process instead of Brownian motion as the driving process. Brockwell made such extensions for the Lévy-driven CARMA(p, q) processes, giving necessary and sufficient conditions for such process to be weakly and strictly stationary process, as well as the explicit form of the corresponding cumulant generating function with several examples (see Brockwell (2001*b*), Brockwell & Marquardt (2005)).

Moreover, we focus on dCAR(p) process, i.e. Lévy-driven CAR(p) process delayed by the inverse of the stable subordinator for low degrees of p since we will be able to give explicit calculations and formulas for the correlation structure and distributional properties, which makes it a more trackable process then the general case.

The paper is organized as follows. Section 2 contains preliminary facts regarding Lévy-driven CAR(p) process, while in Section 3 we define the corresponding dCAR(p) process, i.e. delayed Lévy-driven CAR(p) process. In Section 4 we explicitly derive the correlation structure for the dCAR(p) processes, emphasizing low orders. Next, in Section 5 we propose the definition for the long-range dependence of the non-stationary stochastic process and show that dCAR(p) processes are long-range dependent, emphasizing low orders, while in Section 6 we examine their distributional properties.

2 Lévy-driven CAR(p) processes

Formal definition of the Lévy-driven CAR(p) process is as follows. Let us introduce p-variate process

$$\boldsymbol{S}(t) := [X(t), X^1(t), \ldots, X^{p-1}(t)]^T,\ \ p \in \mathbb{N} \tag{2.1}$$

which satisfies SDE

$$d\boldsymbol{S}(t) - A\boldsymbol{S}(t)dt = \boldsymbol{e}dL(t),\ \ t \geq 0, \tag{2.2}$$

where $\{L(t),\, t \geq 0\}$ is the Lévy process (e.g. see Sato (1999)) such that $EL(1)^2 < \infty$,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{bmatrix},\ \boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{S}(0) \text{ is independent of the driving Lévy process } \{L(t),\, t \geq 0\} \tag{2.3}$$

(if $p = 1$ then $A = -\alpha_1$).
Solution of SDE (2.2) satisfies

$$\boldsymbol{S}(t) = e^{A(t-s)}\boldsymbol{S}(s) + \int_s^t e^{A(t-u)}\boldsymbol{e}dL(u),\ \ t > s \geq 0. \tag{2.4}$$

Definition 2.1. Lévy-driven CAR(p) process is defined as $\{X(t),\, t \geq 0\}$, the first component of the process (2.1), where the process $\{\boldsymbol{S}(t),\, t \geq 0\}$ is the strictly stationary solution of (2.4) which satisfies (2.3).

Additionally, we will assume that the driving process $\{L(t),\, t \geq 0\}$ in SDE (2.2) is the second-order Lévy processes which satisfy

$$EL(t) = \mu t,\ \ Var(L(t)) = \sigma^2 t,\ \ t \geq 0,\ \text{ for some real constants } \mu,\, \sigma^2. \tag{2.5}$$

Lévy-driven CAR(p) processes with this additional assumption are referred to as *second-order Lévy-driven CAR(p) processes* by Brockwell (see Brockwell (2001*b*)). Since we will only consider such processes, we will simply refer to them as Lévy-driven CAR(p) processes.
Necessary and sufficient conditions for weak stationarity of Lévy-driven CAR(p) process are given via Proposition 1. in Brockwell & Marquardt (2005). For process $\{\boldsymbol{S}(t),\, t \geq 0\}$ to be weakly stationary it is both necessary and sufficient that all eigenvalues of matrix A have strictly negative real parts and

$$\boldsymbol{S}(0)\ \text{ has mean and covariance matrix of }\ \int_0^\infty e^{Au}\boldsymbol{e}dL(u).$$

On the other hand, necessary and sufficient conditions for strict stationarity of Lévy-driven CAR(p) process are given via Proposition 2. in Brockwell & Marquardt (2005). For process $\{\boldsymbol{S}(t),\, t \geq 0\}$ to be strictly stationary it is both necessary and sufficient that all eigenvalues of matrix A have strictly negative real parts and

$$\boldsymbol{S}(0) \overset{d}{=} \int_0^\infty e^{Au}\boldsymbol{e}dL(u).$$

Now it follows that the same conditions are necessary and sufficient for Lévy-driven CAR(p) processes $\{X(t),\, t \geq 0\}$ to be weakly and strictly stationary.
Eigenvalues of matrix A are the roots of the characteristic equation (see e.g. Brockwell (2001a))

$$C(\lambda) = \lambda^p + \alpha_1 \lambda^{p-1} + \alpha_2 \lambda^{p-2} + \ldots \alpha_{p-1}\lambda + \alpha_p = 0. \tag{2.6}$$

When $p = 1$ or $p = 2$, characteristic roots of equation (2.6) have negative real parts if and only if all coefficients in the same equation are positive. From here we assume that conditions for strict stationarity of process $\{X(t),\, t \geq 0\}$ are fulfilled.
Since stationary Lévy-driven CAR(p) process has the same autocovariance structure as stationary CAR(p) process (driven by Brownian motion), it follows that autocovariance function (ACF) of the stationary Lévy-driven CAR(p) process is of the form

$$Cov(X(t), X(s)) = \sum_{\lambda\colon C(\lambda)=0} \frac{\sigma^2}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \frac{(z-\lambda)^m e^{z|t-s|}}{C(z)C(-z)} \right] \Bigg|_{z=\lambda}$$

where m is the multiplicity of the root λ of the equation (2.6) (see Brockwell (2001a)). If the roots are distinct, last formula simplifies to

$$Cov(X(t), X(s)) = \sum_{\lambda\colon C(\lambda)=0} \frac{\sigma^2 e^{\lambda|t-s|}}{C'(\lambda)C(-\lambda)}. \tag{2.7}$$

We also use some ideas from Scalas & Viles (2014).
Therefore, stationary Lévy-driven CAR(1) process has autocorrelation function (ACF) of the form

$$Corr(X(t), X(s)) = e^{-\alpha_1|t-s|}, \tag{2.8}$$

where $\alpha_1 > 0$ in order to have the stationarity of the process.
In the case of stationary Lévy-driven CAR(2) process (2.6) becomes

$$z^2 + \alpha_1 z + \alpha_2 = 0, \tag{2.9}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ again for the stationarity of the process. Depending on the sign of the discriminant $D = \alpha_1^2 - 4\alpha_2$ of the equation (2.9) we will have three cases for the ACF of the stationary Lévy-driven CAR(2) process:

- $D > 0$ - **the over-damped case**

$$Corr(X(t), X(s)) = \frac{-\lambda_2 e^{\lambda_1|t-s|} + \lambda_1 e^{\lambda_2|t-s|}}{\lambda_1 - \lambda_2}, \tag{2.10}$$

  where λ_1, λ_2 are two distinct real roots of the equation (2.9).

- $D < 0$ - **the under-damped case**

$$Corr(X(t), X(s)) = \frac{-\bar{\lambda} e^{\lambda|t-s|} + \lambda e^{\bar{\lambda}|t-s|}}{\lambda - \bar{\lambda}}, \tag{2.11}$$

  where $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$ are two distinct complex roots of the equation (2.9) and $\alpha < 0$, $\beta > 0$. Notice that

$$\alpha = -\frac{\alpha_1}{2}, \quad \beta = \sqrt{\alpha_2 - \frac{\alpha_1^2}{4}}.$$

  ACF (2.11) can also be written in the following form

$$Corr(X(t), X(s)) = \left(\cos\left(\beta(t-s)\right) - \frac{\alpha}{\beta} \sin\left(\beta|t-s|\right) \right) e^{\alpha|t-s|}. \tag{2.12}$$

- $D = 0$ - **the critically-damped case**

$$Corr(X(t), X(s)) = \left(1 + \frac{\alpha_1}{2}|t-s|\right) e^{-\frac{\alpha_1}{2}|t-s|}, \tag{2.13}$$

where $\lambda_1 = \lambda_2 = -\frac{\alpha_1}{2}$ is the double real root of the equation (2.9).

3 dCAR(p) processes

Let $\{X(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(p) process defined in the last section. The delayed Lévy-driven CAR(p) process (dCAR(p) process) $\{X_\gamma(t),\, t \geq 0\}$ is defined via a non-Markovian time-change E_t independent of $X(t)$:

$$X_\gamma(t) := X(E_t), \quad t \geq 0,$$

where $E_t = \inf\{x > 0 : D_x > t\}$ is the inverse of the standard γ-stable Lévy subordinator $\{D_t,\, t \geq 0\}$, $0 < \gamma < 1$, with the Laplace transform $E[e^{-sD_t}] = \exp\{-ts^\gamma\}$, $s > 0$. Since E_t rests for periods of time with non-exponential distribution, the process $\{X_\gamma(t),\, t \geq 0\}$ is non-Markovian and non-stationary process.

Let us denote by $f_t(\cdot)$, the density of the random variable E_t for a fixed t and by $g_\gamma(\cdot)$ density of the random variable D_1. Then it follows

$$f_t(x) = \frac{t}{\gamma} x^{-1-\frac{1}{\gamma}} g_\gamma(tx^{-1/\gamma}). \tag{3.1}$$

The Laplace transform of this density is

$$\mathbb{E}[e^{-sE_t}] = \int_0^\infty e^{-sx} f_t(x)dx = \mathcal{E}_\gamma(-st^\gamma),\ s \in \mathbb{C}, \tag{3.2}$$

where

$$\mathcal{E}_\gamma(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+\gamma j)},\ z \in \mathbb{C}$$

is the Mittag-Leffler function (see Gorenflo et al. (2016)).

To see that (3.2) is valid for any complex s, notice that for any $0 < \gamma < 1$ and any complex s

$$\gamma \mathcal{E}_\gamma(-s) = \int\limits_0^\infty e^{-sx} x^{-1-1/\gamma} g_\gamma(x^{-1/\gamma})dx \tag{3.3}$$

(see Theorem 2.10.2, Zolotarev (1986)). Now combining (3.1) and (3.3) we directly obtain (3.2).

On the other hand, two-parametric Mittag-Leffler function is defined as

$$\mathcal{E}_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)},$$

where $\alpha,\, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0,\ Re(\beta) > 0$.

This function was first studied by Wiman in 1905 (see Wiman (1905)). Notice when $\beta = 1$, two-parametric Mittag-Leffler function reduces to classical Mittag-Leffler function $\mathcal{E}_\alpha(z)$. For details regarding Mittag-Leffler function we refer to Gorenflo et al. (2016), Popov & Sedletskii (2011).

4 Correlation structure of dCAR(p) processes

In this section we compute formulas for the correlation structure of dCAR(p) processes, emphasizing low orders. The correlation function (CF) of the dCAR(p) process $\{X_\gamma(t),\ t \geq 0\}$ where $0 < \gamma < 1$ is of the form

$$Corr[X_\gamma(t), X_\gamma(s)] = Corr[X(E_t), X(E_s)] = \int_0^\infty \int_0^\infty Corr[X(u), X(v)] H(du, dv), \quad (4.1)$$

where the last integral is the Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u, v) := P[E_t \leq u, E_s \leq v]$ of the process $\{E_t,\ t \geq 0\}$.
In order to compute the last integral we use the idea of bivariate integration by parts (see Lemma 2.2, Gill et al. (1995))

$$\begin{aligned}\int_0^\infty \int_0^\infty F(u,v)H(du,dv) &= \int_0^\infty \int_0^\infty H([u,\infty] \times [v,\infty])F(du,dv) + \int_0^\infty H([u,\infty] \times (0,\infty])F(du,0) \\ &+ \int_0^\infty H((0,\infty] \times [v,\infty])F(0,dv) + F(0,0)H((0,\infty] \times (0,\infty]).\end{aligned} \quad (4.2)$$

This approach was exploited for calculating the correlation structure of the fractional Pearson diffusions, i.e. time changed (delayed) stationary Pearson diffusions via the inverse of the γ-stable subordinator (see Leonenko et al. (2013a)). Difference between dCAR(p) processes and fractional diffusions is in the outer process, which is stationary Lévy-driven CAR(p) process in dCAR(p) case, and stationary diffusion in the fractional diffusion case.
Since we will use results from Leonenko et al. (2013a), for clarity we define fractional Pearson diffusion. Let $\{Y(t),\ t \geq 0\}$ be the stationary Pearson diffusion, i.e. a stationary solution of SDE

$$dY(t) = \mu(Y(t))dt + \sigma(Y(t))dW(t),\ t \geq 0,$$

where

$$\mu(x) = -\theta(x - \mu),\ \sigma^2(x) = 2\theta k(b_2 x^2 + b_1 x + b_0),\ \theta > 0,\ k > 0,\ \mu \in \mathbb{R},$$

b_0, b_1, b_2 reals and not all simultaneously equal to 0, and $\{W(t),\ t \geq 0\}$ is the standard Brownian motion. Pearson diffusion has the ACF of the form

$$Corr(Y(s), Y(t)) = e^{-\theta|t-s|},$$

while fractional Pearson diffusion, i.e. the process $\{Y_\gamma(t),\ t \geq 0\}$ where

$$Y_\gamma(t) := Y(E_t),\ t \geq 0,$$

has correlation structure of the form (see Theorem 3.1., Leonenko et al. (2013a))

$$Corr(Y_\gamma(t), Y_\gamma(s)) = \mathcal{E}_\gamma(-\theta t^\gamma) + \frac{\gamma \theta t^\gamma}{\Gamma(1+\gamma)} \int_0^{s/t} \frac{\mathcal{E}_\gamma(-\theta t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz. \quad (4.3)$$

Remark 4.1. Notice that the integral representation (4.1) for CF of the general delayed stochastic process depends only on the CF of the non-delayed process $\{X(t),\ t \geq 0\}$ (i.e. the outer stationary process) and the bivariate distribution $H(u, v)$ of the process $\{E_t,\ t \geq 0\}$. So if two non-delayed processes have the same CF, their delayed counterparts will have the same CF as well.

The next theorem provides a general formula for correlation structure of the dCAR(p) process for which the corresponding characteristic equation (2.6) has distinct roots. In the case of non-distinct roots, extended techniques must be used (see Theorem 4.8).

Theorem 4.2. *Let $\{X_p(t),\ t \geq 0\}$ be the stationary Lévy-driven CAR(p) process defined in section 2 with the autocovariance function given by* (2.7)*. Then the correlation function of the corresponding dCAR(p) process $\{X_\gamma(t),\ t \geq 0\}$ is given by*

$$Corr(X_\gamma(t), X_\gamma(s)) = \frac{\sum\limits_{\lambda: C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[\mathcal{E}_\gamma(\lambda t^\gamma) - \frac{\gamma \lambda t^\gamma}{\Gamma(1+\gamma)} \int\limits_0^{s/t} \frac{\mathcal{E}_\gamma(\lambda t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz \right]}{\sum\limits_{\lambda: C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \tag{4.4}$$

where $t \geq s > 0$.

Proof.

$$\begin{aligned}
Corr(X_\gamma(t), X_\gamma(s)) &= Corr(X_p(E_t), X_p(E_s)) \\
&= \int\limits_0^\infty \int\limits_0^\infty Corr(X_p(t), X_p(s)) H(du,\ dv) \\
&= \frac{1}{\sum\limits_{\lambda:\, C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \int\limits_0^\infty \int\limits_0^\infty \sum_{\lambda:\, C(\lambda)=0} \frac{e^{\lambda |t-s|}}{C'(\lambda)C(-\lambda)} H(du,\ dv) \\
&= \frac{1}{\sum\limits_{\lambda:\, C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \sum_{\lambda:\, C(\lambda)=0} \frac{1}{C'(\lambda)C(-\lambda)} \int\limits_0^\infty \int\limits_0^\infty e^{\lambda |u-v|} H(du,\ dv),
\end{aligned} \tag{4.5}$$

where the integral after the first equality is a Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u, v) = \mathbb{P}(E_t \leq u, E_s \leq v)$ of the process $\{E_t,\ t \geq 0\}$. Since integrands in (4.5) have the same form as the ACF of the stationary Pearson diffusion, from (4.3) (i.e. Theorem 3.1., Leonenko et al. (2013*a*)) the result immediately follows. □

Remark 4.3. The last theorem is also valid for complex eigenvalues. To see this, notice that the Laplace transform of the density of random variable E_t (3.2) is valid for any complex number s and procedure from Leonenko et al. (2013*a*) is valid for complex eigenvalues as well.

Corollary 4.4 (dCAR(1) process)**.** *Let $\{X_1(t),\ t \geq 0\}$ be the stationary Lévy-driven CAR(1) process defined in section 2 with the correlation function given by* (2.8)*. Then the correlation function of the corresponding dCAR(1) process $\{X_\gamma(t),\ t \geq 0\}$ is given by*

$$Corr(X_\gamma(t), X_\gamma(s)) = \mathcal{E}_\gamma(-\alpha_1 t^\gamma) + \frac{\gamma \alpha_1 t^\gamma}{\Gamma(1+\gamma)} \int\limits_0^{s/t} \frac{\mathcal{E}_\gamma(-\alpha_1 t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz \tag{4.6}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z + \alpha_1 = 0,$$

where $\alpha_1 > 0$. Now simply apply Theorem 4.2 and the result follows. $\square$

Remark 4.5. If in the outer process, stationary Lévy-driven CAR(1) process is replaced with the stationary CAR(1) process driven by Brownian motion, the outer process becomes the Ornstein-Uhlenbeck process (one of the six Pearson diffusions), while the corresponding dCAR(1) process becomes the fractional Ornstein-Uhlenbeck process. For details regarding fractional Ornstein-Uhlenbeck process and other fractional Pearson diffusions we refer to Leonenko et al. (2013*a*,*b*, 2017).

Corollary 4.6 (the over-damped case). *Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 2 with the correlation function given by* (2.10)*. Then the correlation function of the corresponding dCAR(2) process $\{X_\gamma(t),\, t \geq 0\}$ is given by*

$$\begin{aligned} Corr(X_\gamma(t), X_\gamma(s)) =& \frac{\lambda_1 \mathcal{E}_\gamma(\lambda_2 t^\gamma) - \lambda_2 \mathcal{E}_\gamma(\lambda_1 t^\gamma)}{\lambda_1 - \lambda_2} + \\ &+ \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \frac{\gamma t^\gamma}{\Gamma(1+\gamma)} \int_0^{s/t} \frac{\mathcal{E}_\gamma(\lambda_1 t^\gamma (1-z)^\gamma) - \mathcal{E}_\gamma(\lambda_2 t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz \end{aligned} \tag{4.7}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 > 0$, while the corresponding roots are λ_1 and λ_2. Now simply apply Theorem 4.2 and the result follows. $\square$

Corollary 4.7 (the under-damped case). *Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 2 with the correlation function given by* (2.11)*. Then the correlation function of the corresponding dCAR(2) process $\{X_\gamma(t),\, t \geq 0\}$ is given by*

$$\begin{aligned} Corr(X_\gamma(t), X_\gamma(s)) =& \frac{\lambda \mathcal{E}_\gamma(\bar{\lambda} t^\gamma) - \bar{\lambda} \mathcal{E}_\gamma(\lambda t^\gamma)}{\lambda - \bar{\lambda}} \\ &+ \frac{\lambda \bar{\lambda}}{\lambda - \bar{\lambda}} \frac{\gamma t^\gamma}{\Gamma(1+\gamma)} \int_0^{s/t} \frac{\mathcal{E}_\gamma(\lambda t^\gamma (1-z)^\gamma) - \mathcal{E}_\gamma(\bar{\lambda} t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz \end{aligned} \tag{4.8}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 < 0$, while the corresponding roots are λ and $\bar{\lambda}$. Now simply apply Theorem 4.2 and the result follows. $\square$

4.1 The critically-damped case

Theorem 4.8. *Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 2 with the correlation function given by* (2.13). *Then the correlation function of the corresponding dCAR(2) process $\{X_\gamma(t),\, t \geq 0\}$ is given by*

$$Corr(X_\gamma(t), X_\gamma(s)) = \frac{\alpha_1}{2\gamma} t^\gamma \mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2}t^\gamma\right) + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}t^\gamma\right) + \frac{\alpha_1^2\, t^{2\gamma}}{4\Gamma(1+\gamma)} \int_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2}t^\gamma(1-z)^\gamma\right) dz \tag{4.9}$$

where $t \geq s > 0$.

Proof.

$$\begin{aligned} Corr(X_\gamma(t), X_\gamma(s)) = Corr(X_2(E_t), X_2(E_s)) &= \int_0^\infty\int_0^\infty Corr(X(t), X(s)) H(du,\, dv) \\ &= \int_0^\infty\int_0^\infty \left(1 + \frac{\alpha_1}{2}|u-v|\right) e^{-\frac{\alpha_1}{2}|u-v|} H(du,\, dv), \end{aligned} \tag{4.10}$$

where the last integral is a Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u,v) = \mathbb{P}(E_t \leq u, E_s \leq v)$ of the process $\{E_t,\, t \geq 0\}$.
Let $F(u,v) = \left(1 + \frac{\alpha_1}{2}|u-v|\right) e^{-\frac{\alpha_1}{2}|u-v|}$. Following bivariate integration by parts approach as in Leonenko et al. (2013*a*), i.e. bivariate integration by parts formula (4.2) we obtain

$$\begin{aligned} \int_0^\infty\int_0^\infty F(u,\, v) H(du, dv) &= \int_0^\infty\int_0^\infty \mathbb{P}(E_t \geq u, E_s \geq v) F(du, dv) + \int_0^\infty \mathbb{P}(E_t \geq u) F(du, 0) + \\ &\quad + \int_0^\infty \mathbb{P}(E_s \geq v) F(0, dv) + 1 \\ &= I_1 + I_2 + I_3 + 1. \end{aligned} \tag{4.11}$$

Since $F(du, v) = f_v(u)du$ for $v \geq 0$ where

$$f_v(u) = -\frac{\alpha_1^2}{4}(u-v)e^{-\frac{\alpha_1}{2}(u-v)} I(u > v) - \frac{\alpha_1^2}{4}(u-v)e^{-\frac{\alpha_1}{2}(v-u)} I(u \leq v),$$

using (3.2), it follows

$$\begin{aligned} I_2 = \int_0^\infty \mathbb{P}(E_t \geq u) F(du, 0) &= \int_0^\infty \mathbb{P}(E_t \geq u) \left(-\frac{\alpha_1^2}{4} u e^{-\frac{\alpha_1}{2}u}\right) du \\ &= \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2}u} u \mathbb{P}(E_t \geq u)\Big|_0^\infty - \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2}u} \left(\mathbb{P}(E_t \geq u) - u f_t(u)\right) du \\ &= \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2}u} u f_t(u) du - \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2}u} \mathbb{P}(E_t \geq u) du \\ &= \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2}u} u f_t(u) du + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}t^\gamma\right) - 1. \end{aligned} \tag{4.12}$$

Similarly,

$$I_3 = \int_0^\infty \mathbb{P}(E_s \geq v) F(0, dv) = \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2} v} v f_s(v) dv + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} s^\gamma\right) - 1. \tag{4.13}$$

Now, (4.11) reduces to

$$\int_0^\infty \int_0^\infty F(u,\, v) H(du, dv) = I_1 + \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2} u} u \left(f_s(u) + f_t(u)\right) du + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} s^\gamma\right) - 1.$$

Since $F(du, dv) = h(u, v) du dv$, where

$$\begin{aligned} h(u,v) = &\left(\frac{\alpha_1^2}{4} e^{-\frac{\alpha_1}{2}(u-v)} - \frac{\alpha_1^3}{8}(u-v) e^{-\frac{\alpha_1}{2}(u-v)}\right) I(u > v) + \\ &+ \left(\frac{\alpha_1^2}{4} e^{-\frac{\alpha_1}{2}(v-u)} - \frac{\alpha_1^3}{8}(v-u) e^{-\frac{\alpha_1}{2}(v-u)}\right) I(u \leq v) \end{aligned}$$

and the process $\{E_t,\, t \geq 0\}$ is nondecreasing, it follows that for $u \leq v$

$$\mathbb{P}(E_t \geq u, E_s \geq v) = P(E_s \geq v).$$

Write

$$I_1 = I_1^{(a)} + I_2^{(b)} + I_3^{(c)},$$

where

$$\begin{aligned} I_1^{(a)} &= \int_{u<v} \mathbb{P}(E_t \geq u, E_s \geq v) F(du, dv) = \int_{u<v} \mathbb{P}(E_s \geq v) F(du, dv) \\ I_1^{(b)} &= \int_{u=v} \mathbb{P}(E_t \geq u, E_s \geq v) F(du, dv) = \int_{u=v} \mathbb{P}(E_s \geq v) F(du, dv) \\ I_1^{(c)} &= \int_{u>v} \mathbb{P}(E_t \geq u, E_s \geq v) F(du, dv). \end{aligned}$$

Once again, using integration by parts and (3.2) we obtain

$$\begin{aligned} I_1^{(a)} &= \frac{\alpha_1^2}{4} \int_{v=0}^\infty \int_{u=0}^v \mathbb{P}(E_s \geq v) e^{-\frac{\alpha_1}{2}(v-u)} du dv - \frac{\alpha_1^3}{8} \int_{v=0}^\infty \int_{u=0}^v \mathbb{P}(E_s \geq v)(v-u) e^{-\frac{\alpha_1}{2}(v-u)} du dv \\ &= \frac{\alpha_1^2}{4} \int_0^\infty \mathbb{P}(E_s \geq v) v e^{-\frac{\alpha_1}{2} v} dv \\ &= 1 - \int_0^\infty \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2} v} v f_s(v) dv - \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} s^\gamma\right). \end{aligned} \tag{4.14}$$

Notice that $I_1^{(a)} = -I_3$.

Since function $f_v(u) du = F(du, v)$ does not have a jump at $u = v$ it follows

$$I_1^{(b)} = \int_{u=v} \mathbb{P}(E_s \geq v) F(du, dv) = 0. \tag{4.15}$$

Next,

$$I_1^{(c)} = \frac{\alpha_1^2}{4}\int\limits_{v=0}^{\infty}\mathbb{P}(E_t \geq u, E_s \geq v)\int\limits_{u=v}^{\infty} e^{-\frac{\alpha_1}{2}(u-v)}dudv - \frac{\alpha_1^3}{8}\int\limits_{v=0}^{\infty}\mathbb{P}(E_t \geq u, E_s \geq v)\int\limits_{u=v}^{\infty}(u-v)e^{-\frac{\alpha_1}{2}(u-v)}dudv. \tag{4.16}$$

From Leonenko et al. (2013*a*), p. 741 we have

$$\mathbb{P}(E_t \geq u, E_s \geq v) = \int\limits_{y=0}^{s}\frac{\gamma}{y}vf_y(v)\int\limits_{x=0}^{t-y}\frac{\gamma}{x}(u-v)f_x(u-v)dxdy.$$

Now using this expression together with Fubini theorem in (4.16) we obtain

$$\begin{aligned}
I_1^{(c)} = & \frac{\alpha_1^2}{4}\int\limits_{y=0}^{s}\frac{\gamma}{y}\int\limits_{x=0}^{t-y}\frac{\gamma}{x}\int\limits_{v=0}^{\infty}vf_y(v)\int\limits_{u=v}^{\infty}(u-v)f_x(u-v)e^{-\frac{\alpha_1}{2}(u-v)}du\,dv\,dx\,dy \\
& - \frac{\alpha_1^3}{8}\int\limits_{0}^{s}\frac{\gamma}{y}\int\limits_{x=0}^{t-y}\frac{\gamma}{x}\int\limits_{v=0}^{\infty}vf_y(v)\int\limits_{u=v}^{\infty}(u-v)^2f_x(u-v)e^{-\frac{\alpha_1}{2}(u-v)}du\,dv\,dx\,dy.
\end{aligned}$$

Since

$$\int\limits_{u=v}^{\infty}(u-v)f_x(u-v)e^{-\frac{\alpha_1}{2}(u-v)}du = \int\limits_{0}^{\infty}zf_x(z)e^{-\frac{\alpha_1}{2}z}dz, \tag{4.17}$$

$$\int\limits_{u=v}^{\infty}(u-v)^2f_x(u-v)e^{-\frac{\alpha_1}{2}(u-v)}du = \int\limits_{0}^{\infty}z^2f_x(z)e^{-\frac{\alpha_1}{2}z}dz \tag{4.18}$$

and

$$\int\limits_{v=0}^{\infty}vf_y(v)dv = \mathbb{E}[E_s] = \frac{y^\gamma}{\Gamma(1+\gamma)},$$

(see Baeumer & Meerschaert (2007), Eq. (9)) it follows

$$\begin{aligned}
I_1^{(c)} = & \frac{\alpha_1^2\gamma^2}{4\Gamma(1+\gamma)}\int\limits_{y=0}^{s}\frac{1}{y^{1-\gamma}}\int\limits_{x=0}^{t-y}\frac{1}{x}\int\limits_{0}^{\infty}zf_x(z)e^{-\frac{\alpha_1}{2}z}dz\,dx\,dy \\
& - \frac{\alpha_1^3\gamma^2}{8\Gamma(1+\gamma)}\int\limits_{y=0}^{s}\frac{1}{y^{1-\gamma}}\int\limits_{x=0}^{t-y}\frac{1}{x}\int\limits_{0}^{\infty}z^2f_x(z)e^{-\frac{\alpha_1}{2}z}dz\,dx\,dy.
\end{aligned} \tag{4.19}$$

As in Leonenko et al. (2013*a*), we proceed by expanding $e^{-\frac{\alpha_1}{2}z}$ in (4.17) and (4.18) to obtain

$$\int\limits_{0}^{\infty}zf_x(z)e^{-\frac{\alpha_1}{2}z}dz = -\frac{2}{\alpha_1}\sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j}{\Gamma(1+\gamma j)}$$

and

$$\int\limits_{0}^{\infty}z^2f_x(z)e^{-\frac{\alpha_1}{2}z}dz = \frac{4}{\alpha_1^2}\left(\sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j^2}{\Gamma(1+\gamma j)} - \sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j}{\Gamma(1+\gamma j)}\right).$$

On the other hand,

$$\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) = \frac{\gamma}{x}\sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j}{\Gamma(1+\gamma j)} \tag{4.20}$$

and

$$\frac{d^2}{dx^2}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) = -\frac{\gamma}{x^2}\sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j}{\Gamma(1+\gamma j)} + \frac{\gamma^2}{x^2}\sum_{j=0}^{\infty}\frac{\left(-\frac{\alpha_1}{2}x^\gamma\right)^j j^2}{\Gamma(1+\gamma j)},$$

which implies

$$\int_0^\infty z f_x(z) e^{-\frac{\alpha_1}{2}z}dz = -\frac{2x}{\gamma\alpha_1}\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) \tag{4.21}$$

and

$$\int_0^\infty z^2 f_x(z) e^{-\frac{\alpha_1}{2}z}dz = \frac{4}{\alpha_1^2}\left[\frac{x^2}{\gamma^2}\frac{d^2}{dx^2}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) + \left(\frac{x}{\gamma^2}-\frac{x}{\gamma}\right)\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\right].$$

Using these expressions in (4.19) we obtain

$$\begin{aligned}
I_1^{(c)} &= -\frac{\alpha_1\gamma}{2\Gamma(1+\gamma)}\int_{y=0}^{s}\frac{1}{y^{1-\gamma}}\int_{x=0}^{t-y}\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)dx\,dy \\
&\quad -\frac{\alpha_1\gamma^2}{2\Gamma(1+\gamma)}\int_{y=0}^{s}\frac{1}{y^{1-\gamma}}\int_{x=0}^{t-y}\left[\frac{x}{\gamma^2}\frac{d^2}{dx^2}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) + \left(\frac{1}{\gamma^2}-\frac{1}{\gamma}\right)\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\right]dx\,dy \\
&= -\frac{\alpha_1}{2\Gamma(1+\gamma)}\int_{y=0}^{s}\frac{1}{y^{1-\gamma}}\int_{x=0}^{t-y}\left[x\frac{d^2}{dx^2}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) + \frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\right]dx\,dy
\end{aligned}$$

Since

$$x\frac{d^2}{dx^2}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) + \frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right) = \left(x\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\right)'$$

it follows

$$I_1^{(c)} = -\frac{\alpha_1}{2\Gamma(1+\gamma)}\int_{y=0}^{s}\frac{1}{y^{1-\gamma}}\left[x\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\Bigg|_{x=t-y}\right]dy.$$

Using the definition of the two-parametric Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}(\cdot)$ together with (4.20), after straightforward calculations we obtain

$$\left[x\frac{d}{dx}\mathcal{E}_\gamma\left(-\frac{\alpha_1}{2}x^\gamma\right)\right]\Bigg|_{x=z} = -\frac{\alpha_1}{2}z^\gamma\mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2}z^\gamma\right) \tag{4.22}$$

so that

$$I_1^{(c)} = \frac{\alpha_1^2}{4\Gamma(1+\gamma)}\int_{y=0}^{s} y^{\gamma-1}(t-y)^\gamma\mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2}(t-y)^\gamma\right)dy.$$

Substituting $y = t\,z$ in the last integral it follows

$$I_1^{(c)} = \frac{\alpha_1^2\,t^{2\gamma}}{4\Gamma(1+\gamma)}\int_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma\mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2}t^\gamma(1-z)^\gamma\right)dz. \tag{4.23}$$

On the other hand, (4.21) and (4.22) imply that (4.12) reduces to

$$I_2 = \frac{\alpha_1}{2\gamma} t^\gamma \mathcal{E}_{\gamma,\,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma\right) + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) - 1. \tag{4.24}$$

Finally, combining together (4.13), (4.14), (4.15), (4.23) and (4.24) we obtain

$$\begin{aligned} Corr(X_\gamma(t), X_\gamma(s)) &= I_1 + I_2 + I_3 + 1 = I_1^{(a)} + I_1^{(b)} + I_1^{(c)} + I_2 + I_3 + 1 \\ &= \frac{\alpha_1}{2\gamma} t^\gamma \mathcal{E}_{\gamma,\,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma\right) + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) \\ &+ \frac{\alpha_1^2\, t^{2\gamma}}{4\Gamma(1+\gamma)} \int_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma (1-z)^\gamma\right) dz. \end{aligned} \tag{4.25}$$

□

Remark 4.9. When $t = s$, it must be true that $Corr(X_\gamma(t), X_\gamma(s)) = 1$.
For $t = s$ (4.23) becomes

$$\begin{aligned} I_1^{(c)} &= \frac{\alpha_1^2\, t^{2\gamma}}{4\Gamma(1+\gamma)} \int_{z=0}^{1} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma (1-z)^\gamma\right) dz \\ &= -\frac{\alpha_1 t^\gamma}{2\Gamma(1+\gamma)} \int_{z=0}^{1} z^{\gamma-1} \left[x \frac{d}{dx} \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} x^\gamma\right)\right]\Bigg|_{x=t(1-z)} dz \\ &= -\frac{\alpha_1 t^\gamma \gamma}{2\Gamma(1+\gamma)} \int_{z=0}^{1} z^{\gamma-1} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma (1-z)^\gamma\right)^j j}{\Gamma(1+\gamma j)} dz \\ &= -\frac{\alpha_1 t^\gamma \gamma}{2\Gamma(1+\gamma)} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma\right)^j j}{\Gamma(1+\gamma j)} \int_{z=0}^{1} z^{\gamma-1}(1-z)^{\gamma j} dz. \end{aligned} \tag{4.26}$$

Since formula for the beta density yields

$$\int_0^x y^{a-1}(x-y)^{b-1} dy = B(a,\, b) x^{a+b-1}$$

where $B(a,\, b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a > 0$, $b > 0$, (4.26) reduces to

$$\begin{aligned} I_1^{(c)} &= -\frac{\alpha_1 t^\gamma}{2} \frac{\gamma\Gamma(\gamma)}{\Gamma(1+\gamma)} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma\right)^j j}{\Gamma(1+\gamma(j+1))} \\ &= \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma\right)^{j+1} j}{\Gamma(1+\gamma(j+1))} \\ &= \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma\right)^{j+1} (j+1)}{\Gamma(1+\gamma(j+1))} - \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\gamma\right)^{j+1}}{\Gamma(1+\gamma(j+1))} \\ &= \frac{t}{\gamma} \frac{d}{dt} \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) - \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) + 1 \\ &= -\frac{\alpha_1}{2\gamma} t^\gamma \mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma\right) - \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) + 1. \end{aligned}$$

Now, from (4.25) it follows $Corr(X_\gamma(t), X_\gamma(s)) = 1$.

Remark 4.10. For dCAR(p) process of order $p > 2$ such that the non-distinct roots of the corresponding characteristic equation (2.6) have the highest multiplicity $m = 2$, one can expect similar correlation structure as in Theorem 4.8. On the other hand, with higher multiplicities ($m > 2$), higher derivatives of Mittag-Leffler function should appear in the correlation structure, and case-by-case analysis is expected.

5 Long-range dependence for dCAR(p) process

In this section we propose definition for the long-range dependence of the non-stationary stochastic process and apply it for the dCAR(p) processes, emphasizing low orders.

Definition 5.1. Let $\{X(t), t \geq 0\}$ be the non-stationary stochastic process with the correlation function $Corr(X(t), X(s))$ which satisfies

$$Corr(X(t), X(s)) \sim c(s)\, t^{-d},\ t \to \infty,$$

i.e.,

$$\lim_{t\to\infty} \frac{Corr(X(t), X(s))}{t^{-d}} = c(s),$$

for a fixed $s > 0$, some constant $c(s) > 0$ and $d > 0$.
We say that $\{X(t), t \geq 0\}$ has the long-range dependence if $d \in \langle 0, 1\rangle$ and the short-range dependence if $d \in \langle 1, 2\rangle$.

Remark 5.2. Long-range dependence is usually observed for second-order stationary processes in terms of covariance function. In case of non-stationary second-order processes natural extension is given via Definition 5.1. For brief discussion and other possible definitions of long-range dependence in various settings see Heyde & Yang (1997). Long-range dependence property in the form of Definition 5.1 was first used in Maheshwari & Vellaisamy (2016) and Maheshwari & Vellaisamy (2017). Moreover, equivalent form was used in Leonenko et al. (2013*a*).

Before we proceed, we need some technical results regarding Mittag-Leffler functions. First notice that

$$\mathcal{E}_\gamma(\theta t^\gamma) \sim -\frac{1}{\theta\Gamma(1-\gamma)t^\gamma},\ t \to \infty, \tag{5.1}$$

where θ is a complex number such that $\operatorname{Re}\theta < 0$ and $0 < \gamma < 1$ (see Theorem 1.4., Podlubny (1998)). Since

$$\mathcal{E}_{\gamma,\gamma}(\theta t^\gamma) \sim O(|\theta t^\gamma|^{-2}),\ t \to \infty$$

(again, see Theorem 1.4., Podlubny (1998)), from (4.22) and (5.1) immediately after applying L'Hospital's rule for complex valued functions (see Carter (1958)) it follows

$$\mathcal{E}_{\gamma,\gamma}(\theta t^\gamma) \sim \frac{\gamma}{\theta^2\Gamma(1-\gamma)t^{2\gamma}},\ t \to \infty. \tag{5.2}$$

On the other hand, if θ is a complex number such that $\operatorname{Re}\theta < 0$, $0 < \gamma < 1$ and C a real constant, then (see Theorem 1.6., Podlubny (1998))

$$|\mathcal{E}_\gamma(\theta t^\gamma)| \leq \frac{C}{1+|\theta|t^\gamma},\ t > 0, \tag{5.3}$$

$$|\mathcal{E}_{\gamma,\gamma}(\theta t^\gamma)| \leq \frac{C}{1+|\theta|t^\gamma},\ t > 0. \tag{5.4}$$

Next, we prove two lemmas needed for the proof of the long-range dependence for dCAR(p) processes.

Lemma 5.3. *Let θ be a complex number such that* $\operatorname{Re}\theta < 0$. *If* $0 < \gamma < 1$ *and* $t \geq s > 0$ *then*

$$\int_0^{s/t} \frac{\mathcal{E}_\gamma(\theta t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz \sim -\frac{1}{\theta\gamma\Gamma(1-\gamma)} \frac{s^\gamma}{t^{2\gamma}}, \; t \to \infty.$$

Proof. By change of variable $z = s/ty$ we have

$$\int_0^{s/t} \frac{\mathcal{E}_\gamma(\theta t^\gamma (1-z)^\gamma)}{z^{1-\gamma}} dz = \left(\frac{s}{t}\right)^\gamma \int_0^1 \frac{\mathcal{E}_\gamma(\theta t^\gamma (1-sy/t)^\gamma)}{y^{1-\gamma}} dy.$$

From (5.3) we see that last integrand is bounded with $g(y) = C/y^{1-\gamma}$ and $\int_0^1 g(y) < \infty$, so by using Lebesgue's dominated convergence theorem together with (5.1) we obtain

$$\begin{aligned}\left(\frac{s}{t}\right)^\gamma \int_0^1 \frac{\mathcal{E}_\gamma(\theta t^\gamma (1-sy/t)^\gamma)}{y^{1-\gamma}} dy &\sim -\left(\frac{s}{t}\right)^\gamma \frac{1}{\theta\Gamma(1-\gamma)t^\gamma} \int_0^1 y^{\gamma-1} dy, \; t \to \infty \\ &= -\frac{1}{\theta\gamma\Gamma(1-\gamma)} \frac{s^\gamma}{t^{2\gamma}}, \; t \to \infty.\end{aligned}$$

□

Lemma 5.4. *Let θ be a complex number such that* $\operatorname{Re}\theta < 0$. *If* $0 < \gamma < 1$ *and* $t \geq s > 0$ *then*

$$\int_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\gamma}\left(\theta t^\gamma (1-z)^\gamma\right) dz \sim \frac{1}{\theta^2\Gamma(1-\gamma)} \frac{s^\gamma}{t^{3\gamma}}, \; t \to \infty.$$

Proof. Once again, by change of variable $z = s/ty$ we have

$$\int_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\gamma}\left(\theta t^\gamma (1-z)^\gamma\right) dz = \left(\frac{s}{t}\right)^\gamma \int_0^1 y^{\gamma-1} \left(1-sy/t\right)^\gamma \mathcal{E}_{\gamma,\gamma}\left(\theta t^\gamma (1-sy/t)^\gamma\right) dy.$$

From (5.4) we see that last integrand is bounded with $g(y) = Cy^{\gamma-1}$ and $\int_0^1 g(y) < \infty$, so by using Lebesgue's dominated convergence theorem together with (5.2) we obtain

$$\begin{aligned}\left(\frac{s}{t}\right)^\gamma \int_0^1 y^{\gamma-1} \left(1-sy/t\right)^\gamma \mathcal{E}_{\gamma,\gamma}\left(\theta t^\gamma (1-sy/t)^\gamma\right) dy &\sim \left(\frac{s}{t}\right)^\gamma \frac{\gamma}{\theta^2\Gamma(1-\gamma)t^{2\gamma}} \int_0^1 y^{\gamma-1}\left(1-sy/t\right)^{-\gamma} dy \\ &\sim \left(\frac{s}{t}\right)^\gamma \frac{\gamma}{\theta^2\Gamma(1-\gamma)t^{2\gamma}} \int_0^1 y^{\gamma-1} dy \\ &= \frac{1}{\theta^2\Gamma(1-\gamma)} \frac{s^\gamma}{t^{3\gamma}}, \; t \to \infty.\end{aligned}$$

□

Theorem 5.5. *Let $\{X_\gamma(t), \, t \geq 0\}$ be the dCAR(p) process as in Theorem 4.2 with corresponding correlation function (4.4). Then stochastic process $\{X_\gamma(t), \, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$*

$$Corr(X_\gamma(t), X_\gamma(s)) \sim \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(-\frac{\sum\limits_{\lambda: C(\lambda)=0} \left(\lambda C'(\lambda)C(-\lambda)\right)^{-1}}{\sum\limits_{\lambda: C(\lambda)=0} \left(C'(\lambda)C(-\lambda)\right)^{-1}} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right), \; t \to \infty.$$

Proof. Since distinct roots λ of equation (2.6) have negative real parts, using (5.1) together with Lemma 5.3 it follows

$$
\begin{aligned}
Corr(X_\gamma(t), X_\gamma(s)) &= \frac{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[\mathcal{E}_\gamma(\lambda t^\gamma) - \frac{\gamma\lambda t^\gamma}{\Gamma(1+\gamma)} \int\limits_0^{s/t} \frac{\mathcal{E}_\gamma(\lambda t^\gamma(1-z)^\gamma)}{z^{1-\gamma}} dz\right]}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \\
&\sim \frac{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[-\frac{1}{\lambda\Gamma(1-\gamma)t^\gamma} + \frac{\gamma\lambda t^\gamma}{\Gamma(1+\gamma)} \cdot \frac{1}{\lambda\gamma\Gamma(1-\gamma)} \frac{s^\gamma}{t^{2\gamma}}\right]}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}}, \ t \to \infty \\
&= \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(-\frac{\sum\limits_{\lambda:C(\lambda)=0} (\lambda C'(\lambda)C(-\lambda))^{-1}}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right), \ t \to \infty.
\end{aligned}
$$

□

Corollary 5.6 (dCAR(1) process). *Let $\{X_\gamma(t),\, t \geq 0\}$ be the dCAR(1) process as in Corollary 4.4 with corresponding correlation function* (4.6). *Then stochastic process $\{X_\gamma(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$*

$$
Corr(X_\gamma(t), X_\gamma(s)) \sim \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(\frac{1}{\alpha_1} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right), \ t \to \infty.
$$

Proof. In this case, characteristic equation is of the form

$$
C(z) = z + \alpha_1 = 0,
$$

where $\alpha_1 > 0$. Now simply apply Theorem 5.5 and the result follows. □

Corollary 5.7 (the over-damped case). *Let $\{X_\gamma(t),\, t \geq 0\}$ be the dCAR(2) process in the over-damped case, i.e. as in Corollary 4.6 with corresponding correlation function* (4.7). *Then stochastic process $\{X_\gamma(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$*

$$
Corr(X_\gamma(t), X_\gamma(s)) \sim \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(-\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right), \ t \to \infty.
$$

Proof. In this case, characteristic equation is of the form

$$
C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,
$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 > 0$, while the corresponding roots are λ_1 and λ_2. Now simply apply Theorem 5.5 and the result follows. □

Corollary 5.8 (the under-damped case). *Let $\{X_\gamma(t),\, t \geq 0\}$ be the dCAR(2) process in the under-damped case, i.e. as in Corollary 4.7 with corresponding correlation function* (4.8). *Then stochastic process $\{X_\gamma(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$*

$$
Corr(X_\gamma(t), X_\gamma(s)) \sim \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(-\frac{\lambda + \bar{\lambda}}{\lambda\bar{\lambda}} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right), \ t \to \infty.
$$

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 < 0$, while the corresponding roots are λ and $\bar{\lambda}$. Now simply apply Theorem 5.5 and the result follows. $\square$

Theorem 5.9 (the critically-damped case). *Let $\{X_\gamma(t),\, t \geq 0\}$ be the dCAR(2) process in the critically-damped case, i.e. as in Theorem 4.8 with corresponding correlation function* (4.9)*. Then stochastic process $\{X_\gamma(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$*

$$Corr(X_\gamma(t), X_\gamma(s)) \sim \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(\frac{4}{\alpha_1} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right),\ t \to \infty.$$

Proof. Since $\alpha_1 > 0$, using (5.1), (5.2) together with Lemma 5.4 for $\theta = -\alpha_1/2$ it follows

$$\begin{aligned} Corr(X_\gamma(t), X_\gamma(s)) &= \frac{\alpha_1}{2\gamma} t^\gamma \mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma\right) + \mathcal{E}_\gamma\left(-\frac{\alpha_1}{2} t^\gamma\right) \\ &+ \frac{\alpha_1^2\, t^{2\gamma}}{4\Gamma(1+\gamma)} \int\limits_{z=0}^{s/t} z^{\gamma-1}(1-z)^\gamma \mathcal{E}_{\gamma,\gamma}\left(-\frac{\alpha_1}{2} t^\gamma (1-z)^\gamma\right) dz \\ &\sim \frac{\alpha_1}{2\gamma} t^\gamma \cdot \frac{4\gamma}{\alpha_1^2 \Gamma(1-\gamma) t^{2\gamma}} + \frac{2}{\alpha_1 \Gamma(1-\gamma) t^\gamma} + \frac{\alpha_1^2\, t^{2\gamma}}{4\Gamma(1+\gamma)} \cdot \frac{4}{\alpha_1^2 \Gamma(1-\gamma)} \frac{s^\gamma}{t^{3\gamma}} \\ &= \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \left(\frac{4}{\alpha_1} + \frac{s^\gamma}{\Gamma(1+\gamma)} \right),\ t \to \infty. \end{aligned}$$

$\square$

6 Distribution of dCAR(p) processes

Let

$$p(x,t) := \frac{d}{dx} \mathbb{P}\left(X(t) \leq x\right)$$

denote the density of the Lévy-driven CAR(p) process $\{X(t),\ t \geq 0\}$, and like in previous sections, let

$$f_t(x) = \frac{d}{dx} \mathbb{P}\left(E(t) \leq x\right)$$

denote the density of the inverse of the γ-stable subordinator $\{E(t),\ t \geq 0\}$. Then for the density of the dCAR(p) process $\{X_\gamma(t),\ t \geq 0\}$

$$q(x,t) := \frac{d}{dx} \mathbb{P}\left(X_\gamma(t) \leq x\right) = \frac{d}{dx} \mathbb{P}\left(X(E(t)) \leq x\right)$$

the following representation is valid

$$q(x,t) = \int\limits_0^\infty p(x,s) f_t(s) ds. \tag{6.1}$$

To see this, since $X(t)$ and $E(t)$ are independent, using conditional argument yields

$$\begin{aligned}\mathbb{P}\left(X(E(t)) \leq x\right) &= \mathbb{E}\left[\mathbb{P}\left(X(E(t)) \leq x\right) | E(t)\right] \\ &= \int_0^\infty \mathbb{P}(X(u) \leq x | E(t) = u) f_t(u) du \\ &= \int_0^\infty \mathbb{P}(X(u) \leq x) f_t(u) du.\end{aligned}$$

After differentiating (which can be justified by the dominated convergence theorem) we arrive at (6.1).
Since the density $f_t(x)$ of the process $\{E(t),\ t \geq 0\}$ is given via (3.1), it is clear that once we know the density $p(x,t)$ of the process $\{X(t),\ t \geq 0\}$, we can calculate the density $q(x,t)$ of the dCAR(p) process $\{X_\gamma(t),\ t \geq 0\}$ via (6.1).

Example 6.1. Let us consider the non-stationary CAR(1) process with $L(t) = W(t)$, i.e. driven by the standard Brownian motion. SDE (2.2) reduces to

$$dX(t) + \alpha_1 X(t) dt = dW(t).$$

Therefore CAR(1) process reduces to the well known Ornstein-Uhlenbeck diffusion with transition density (cf. Karlin & Taylor (1981) page 332)

$$p(x; x_0, t) = \frac{1}{\sqrt{2\pi (2\alpha_1)^{-1} (1 - e^{1-2\alpha_1 t})}} \exp\left\{ - \frac{x - x_0 e^{-\alpha_1 t}}{(\alpha_1)^{-1}(1 - e^{-2\alpha_1 t})} \right\}. \tag{6.2}$$

If we denote probability density of the initial distribution of CAR(1) process with p_0, then the density of CAR(1) process is given by

$$p(x,t) = \int_{\mathbb{R}} p_0(x_0) p(x; x_0, t) dx_0,$$

where the transition density $p(x; x_0, t)$ is given by (6.2). Now, density of the corresponding dCAR(1) process, i.e. expression (6.1) becomes

$$q(x,t) = \int_{\mathbb{R}} p_0(x_0) \left(\int_0^\infty p(x; x_0, s) f_t(s) ds \right) dx_0,$$

where p_0 is the initial distribution of the non-stationary CAR(1) process, f_t is the probability density of the inverse of the stable subordinator (3.1) and the transition density of CAR(1) process $p(x; x_0, s)$ is given by (6.2).

However, in this paper, we consider only stationary Lévy-driven CAR(p) process $\{X(t),\ t \geq 0\}$. If $m(x)$ denotes its probability density, then from (6.1) it is clear that the density of corresponding dCAR(p) process stays the same over all time, i.e. it has the probability density $m(x)$. Therefore, density of the dCAR(p) process is the same as the density of the corresponding stationary Lévy-driven CAR(p) process.

Stationary Lévy-driven CAR(p) process $\{X(t),\ t \geq 0\}$ has cumulant generating function (cgf) for $\{X(t_1),\, X(t_2),\, \dots, X(t_n),\ 0 < t_1 < t_2 < \dots < t_n\}$ (see Brockwell (2001*b*), Brockwell & Marquardt (2005))

$$\begin{aligned}\ln \mathbb{E}\left[exp(i\theta_1 X(t_1) + \dots + i\theta_n X(t_n))\right] = &\int_0^\infty \xi\left(\sum_{i=1}^n \theta_i \boldsymbol{b}^T e^{A(t_i+u)}\boldsymbol{e}\right) du + \int_0^{t_1} \xi\left(\sum_{i=1}^n \theta_i \boldsymbol{b}^T e^{A(t_i-u)}\boldsymbol{e}\right) du \\ &+ \int_{t_1}^{t_2} \xi\left(\sum_{i=2}^n \theta_i \boldsymbol{b}^T e^{A(t_i-u)}\boldsymbol{e}\right) + \dots + \int_{t_{n-1}}^{t_n} \xi\left(\theta_n \boldsymbol{b}^T e^{A(t_n-u)}\boldsymbol{e}\right) du,\end{aligned} \tag{6.3}$$

where $\boldsymbol{b} = [1, 0, \dots 0]^T,\ \boldsymbol{b} \in \mathbb{R}^p$, the characteristic function of the driving Lévy process $\{L(t),\ t \geq 0\}$ of the CAR(p) process (see section 2)

$$\phi_t(\theta) := \mathbb{E}\left[e^{i\theta L(t)}\right] = e^{t\xi(\theta)},$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2 s^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\theta x} - 1 - i\theta x\, \mathbb{I}_{|x|<1}\right) \nu(dx),$$

for some $m \in \mathbb{R}$, $s \geq 0$ and Lévy measure ν. In particular, marginal distribution of $X(t)$ (and therefore of $X_\gamma(t)$ as well) has cgf

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta \boldsymbol{b}^T e^{Au}\boldsymbol{e}\right) du. \tag{6.4}$$

In our setting, (6.4) reduces to cases:

- stationary Lévy-driven CAR(1) process with the correlation function given by (2.8)

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, e^{-\alpha_0 u}\right) du$$

- stationary Lévy-driven CAR(2) process with the correlation function given by (2.10)

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, \frac{e^{\lambda_1 u} - e^{\lambda_2 u}}{\lambda_1 - \lambda_2}\right) du$$

- stationary Lévy-driven CAR(2) process with the correlation function given by (2.11)

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, \frac{e^{\lambda u} - e^{\bar{\lambda} u}}{\lambda - \bar{\lambda}}\right) du$$

- stationary Lévy-driven CAR(2) process with the correlation function given by (2.13)

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, u e^{-\frac{\alpha_1}{2}u}\right) du.$$

Example 6.2. Let us consider the stationary Lévy-driven CAR(1) process with the driving process being compound Poisson process with finite jump-rate λ and bilateral exponential jump size distribution with probability density $f(x) = \beta/2e^{-\beta|x|}$, while corresponding characteristic exponent is of the form

$$\xi(\theta) = -\frac{\lambda\theta^2}{\beta^2 + \theta^2}.$$

Then, marginal distribution of the corresponding dCAR(1) process has cumulant generating function of the form

$$\ln \mathbb{E}\left[exp(i\theta X_\gamma(t))\right] = \int_0^\infty \xi\left(\theta\, e^{-\alpha_0 u}\right) du = -\frac{\lambda}{2\alpha_0} \ln\left(1 + \frac{\theta^2}{\beta^2}\right),$$

which shows that corresponding dCAR(1) process has marginals distributed as the difference between two independent gamma distributed random variables with exponent $\lambda/(2\alpha_0)$ and scale parameter β.

Many examples regarding distribution of stationary Lévy-driven CAR(p) process can be found in Barndorff-Nielsen & Shephard (2001), Brockwell (2001*b*) and Brockwell & Marquardt (2005).

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