From Iterated Revision to Iterated Contraction: Extending the Harper Identity*

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Abstract

The study of iterated belief change has principally focused on *revision*, with the other main operator of AGM belief change theory, namely *contraction*, receiving comparatively little attention. In this paper we show how principles of iterated revision can be carried over to iterated contraction by generalising a principle known as the 'Harper Identity'. The Harper Identity provides a recipe for defining the *belief set* resulting from contraction by a sentence A in terms of (i) the initial belief set and (ii) the belief set resulting from revision by $\neg A$. Here, we look at ways to similarly define the *conditional belief set* resulting from contraction by A. After noting that the most straightforward proposal of this kind leads to triviality, we characterise a promising family of alternative suggestions that avoid such a result. One member of that family, which involves the operation of rational closure, is noted to be particularly theoretically fruitful and normatively appealing.

Keywords: belief revision, iterated belief change, belief contraction, Harper Identity, rational closure

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1. Introduction

Since the publication of Darwiche and Pearl's seminal paper on the problem of iterated belief revision—the problem of how to adjust one's corpus of beliefs in response to a sequence of successive additions to its members—in the mid 1990's [2], a substantial body of research has accumulated on the topic [3, 4, 5, 6, 7, 8]. In contrast, work on the parallel problem of iterated *contraction*—the problem of how to adjust that corpus in response to a sequence of successive retractions— was only initiated far more recently and remains comparatively underdeveloped [9, 10, 11, 12, 13, 14, 15].

One obvious way to level out this discrepancy would be to introduce a principle that enables us to derive, from constraints on iterated revision, corresponding constraints on iterated contraction. But while there exists a well known and widely accepted identity connecting *single-shot* revision and contraction, the 'Harper Identity' [16], there has been no discussion to date of how to extend this principle to the iterated case.[2]

The Harper Identity states that the posterior *belief set* resulting from contracting sentence A should be formed by combining (i) the prior belief set and (ii) the posterior belief set resulting from revision by $\neg A$. Moving to the iterated case, we want to consider whether or not an analogous treatment can be given of an agent's *conditional belief set* after contraction by A, a set that captures the beliefs that he or she would have after any subsequent revision or contraction. In other words, we are looking for ways to define the conditional belief set resulting from contracting A in terms of (iii) the prior conditional belief set, and (iv) the posterior conditional belief set resulting from revision by $\neg A$. It will prove convenient to carry out this exercise by exploiting the well-known representability of conditional belief sets by total preorders over the set of possible

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[2]It should be noted that [12] and [14] do propose a principle that they call the 'New Harper Identity'. But while this may be suggestive of an attempted extension of the Harper Identity to the iterated case, the New Harper Identity simply appears to be a representation, in terms of plausibility orderings, of a particular set of postulates for iterated contraction.

worlds. Seen through this lens, our task will amount to finding an appropriate procedure for combining a pair of total preorders.

The plan of the remainder of this paper is as follows. We first introduce some basic terminology and definitions (Section 2). We then show that the simplest extension of the Harper Identity to iterated belief change is too strong, being inconsistent, on pains of triviality, with basic principles of belief dynamics (Section 3). Next, we sketch out an alternative proposal (Section 4). We propose a pair of weak upper and lower bound principles, which we ultimately show to characterise, in our functional domain of interest, a family of binary combination operators for plausibility orderings that we call *TeamQueue (TQ) combinators* (Subsection 4.1). We show that the addition of a further principle of 'Parity', singles out a particularly noteworthy TQ combinator that we call the *Synchronous TeamQueue (STQ) combinator*. Indeed, extending the Harper Identity via the STQ method turns out to yield the 'flattest' plausibility ordering that is consistent with the strict plausibility inequalities common to both inputs. Equivalently: the conditional belief set obtained by this method corresponds to the *rational closure* [17] of the set of conditional beliefs common to both inputs (Subsection 4.2). Finally, we illustrate the usefulness of the TQ approach by establishing some correspondences between principles of iterated revision and iterated contraction (Section 5). First, we show that any TQ combinator whatsoever can be used to recover a number of recently discussed postulates for iterated contraction from the Darwiche-Pearl postulates for iterated revision (Subsection 5.1). Second, we show that the *STQ* combinator allows us to derive counterparts for contraction of the best known revision operators in the literature (Subsection 5.1). We then briefly conclude and mention some ideas for future work (Section 6). In order to improve readability, proofs of the fairly significant number of propositions and theorems have been relegated to the appendix.

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2. Preliminaries

We represent the beliefs of an agent by a so-called *belief state* Ψ, which, for the purposes of the present paper, can be treated as a primitive. The state Ψ determines a *belief set* $[\Psi]$, a deductively closed set of sentences, drawn from a finitely generated propositional, truth-functional language L. The set of classical logical consequences of a sentence $A \in L$ is denoted by $\mathrm{Cn}(A)$. The set of propositional worlds is denoted by W, and the set of models of a given sentence A is denoted by $[\![A]\!]$. We occasionally abuse notation and use x to denote not a world but a sentence. In particular, whenever a world x appears within the scope of a logical connective, it should be understood as referring to some sentence whose set of models is exactly $\{x\}$. So, for example, given $x, y \in W$, the sentence $x \vee y$ is such that $[\![x \vee y]\!] = \{x, y\}$.

The dynamics of belief states are modelled by two operations—contraction and revision, which respectively return the posterior belief states $\Psi * A$ and $\Psi \div A$ resulting from an adjustment of the prior belief state Ψ to accommodate, respectively, the inclusion and the exclusion of A. We assume that these operations satisfy the so-called AGM postulates [18], which enforce a principle of 'minimal mutilation' of the prior belief set in meeting the relevant exclusion or inclusion constraint. Regarding revision, we have:

(K1*) $\mathrm{Cn}([\Psi * A]) \subseteq [\Psi * A]$

(K2*) $A \in [\Psi * A]$

(K3*) $[\Psi * A] \subseteq \mathrm{Cn}([\Psi] \cup \{A\})$

(K4*)   If $\neg A \notin [\Psi]$, then $\mathrm{Cn}([\Psi] \cup \{A\}) \subseteq [\Psi * A]$

(K5*)   If A is consistent, then so too is $[\Psi * A]$

(K6*)   If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$, then $[\Psi * A] = [\Psi * B]$

(K7*) $[\Psi * A \wedge B] \subseteq \mathrm{Cn}([\Psi * A] \cup \{B\})$

(K8*)   If $\neg B \notin [\Psi * A]$, then $\mathrm{Cn}([\Psi * A] \cup \{B\}) \subseteq [\Psi * A \wedge B]$

Regarding contraction:

$(K1^{\div})$ $\mathrm{Cn}([\Psi \div A]) \subseteq [\Psi \div A]$

$(K2^{\div})$ $[\Psi \div A] \subseteq [\Psi]$

$(K3^{\div})$ If $A \notin [\Psi]$, then $[\Psi \div A] = [\Psi]$

$(K4^{\div})$ If $A \notin \mathrm{Cn}(\varnothing)$, then $A \notin [\Psi \div A]$

$(K5^{\div})$ If $A \in [\Psi]$, then $[\Psi] \subseteq \mathrm{Cn}([\Psi \div A] \cup \{A\})$

$(K6^{\div})$ If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$, then $[\Psi \div A] = [\Psi \div B]$

$(K7^{\div})$ $[\Psi \div A] \cap [\Psi \div B] \subseteq [\Psi \div A \wedge B]$

$(K8^{\div})$ If $A \notin [\Psi \div A \wedge B]$, then $[\Psi \div A \wedge B] \subseteq [\Psi \div A]$

We also assume that they are linked in the single-step case by the Harper Identity:

(HI) $[\Psi \div A] = [\Psi] \cap [\Psi * \neg A]$

This principle incorporates the compelling directive according to which, in giving up A, and hence leaving open the possibility that $\neg A$, one ought to thereby also retract anything that one would not have endorsed had one come to believe that $\neg A$. It adds to this the 'minimal change' requirements that (i) nothing further is retracted and (ii) nothing is introduced either (i.e. the postulate $(K2^{\div})$).

A classic result tells us that (HI) enables us to recover the AGM postulates for contraction from the AGM postulates for revision. More precisely, it has been shown that: if $*$ and $\div$ satisfy (HI), then (1) if $*$ satisfies $(K1^*)$–$(K6^*)$, then $\div$ satisfies $(K1^{\div})$–$(K6^{\div})$ and (2) if $*$ additionally satisfies $(K7^*)$ (resp. $(K8^*)$), then $\div$ satisfies $(K7^{\div})$ (resp. $(K8^{\div})$) [19, Thms. 3.4 & 3.5].

The single shot revision and contraction behaviour of an agent can be usefully represented by a set of 'Ramsey Test conditionals', whose members are drawn from an extension L_{c} of L to include all sentences of the form $A \Rightarrow B$, where $A, B \in L$, and which satisfies the so called Ramsey Test ([20], [21]):

(RT) $A \Rightarrow B \in [\Psi]_{\mathrm{c}}$ iff $B \in [\Psi * A]$

where $[\Psi]_{\mathrm{c}}$ denotes the 'conditional belief set' associated with Ψ, that subsumes $[\Psi]$ and includes the relevant Ramsey Test conditionals. While this principle

explicitly interprets the content of $[\Psi]_c$ in terms of the beliefs held after possible single *revisions*, note that, in view of (HI), this conditional belief set also implicitly determines the beliefs held after possible single *contractions*. The AGM postulates for revision ensure that conditional belief sets are 'rational' in the technical sense of Lehmann & Magidor [17].[3]

We follow a number of authors in making use of a 'semantic' representation of the 'syntactic' single-step revision and contraction behaviour associated with a particular belief state Ψ in terms of a total preorder (*TPO*) $\preceq_\Psi$, i.e., a complete and transitive binary relation, over W. This TPO is sometimes interpreted as ordering the worlds according to *plausibility*, with more plausible worlds situated lower down the ordering. In this representation, the set $\min(\preceq_\Psi, [\![A]\!]) := \{x \in [\![A]\!] \mid \forall y \in [\![A]\!], x \preceq_\Psi y\}$ of minimal A-worlds corresponds to the set of worlds in which all and only the sentences in $[\Psi * A]$ are true, with $[\![[\Psi]]\!] = \min(\preceq_\Psi, W)$ for any Ψ (see, for instance, the representation results in [23, 24]). In terms of conditional belief sets, the condition $A \Rightarrow B \in [\Psi]_c$ thus corresponds to $\min(\preceq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$: a conditional belief is held iff the consequent holds in all the minimal worlds in which the antecedent holds. We will exploit this correspondence throughout the paper. See Figure 1 for illustration.

Viewed in this way, the question of iterated belief change becomes intimately bound with the dynamics of $\preceq_\Psi$ under contraction and revision. This is because, following one operation of contraction or revision by a sentence A, the beliefs following a *subsequent* belief change step will be determined by the TPOs $\preceq_{\Psi \div A}$ or $\preceq_{\Psi * A}$, associated with $\Psi \div A$ and $\Psi * A$ respectively. (HI) then translates into the constraint that $\min(\preceq_{\Psi \div A}, W) = \min(\preceq_\Psi, W) \cup \min(\preceq_{\Psi * \neg A}, W)$.[4]

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[3]More precisely, the postulates are collectively equivalent to the proposition that $\mathrel{\mid\!\sim}_\Psi = \{\langle A, B \rangle \mid A \Rightarrow B \in [\Psi]_c\}$ is a rational consequence relation—a relation that satisfies the properties of Reflexivity, Left Classical Equivalence, Right Weakening, Left Disjunction, Right Conjunction and Rational Monotony—that also satisfies a condition known as 'consistency preservation'; see [22].

[4] Indeed, (HI) amounts to: $C \in [\Psi \div A]$ iff $C \in [\![\Psi]\!]$ and $C \in [\![(\Psi * \neg A) * B]\!]$. So, translating into the language of minimal sets, it can be presented as: $\min(\preccurlyeq_{\Psi \div A}, W) \subseteq [\![C]\!]$

$A \wedge B$	$A \wedge \neg B$	$\neg A \wedge B$	$\neg A \wedge \neg B$
			y
	w	z	
x			

Figure 1: TPO represention $\preceq_\Psi$ of a conditional belief set $[\Psi]_c$ drawn from a language with atomic sentences $\{A, B\}$. The relation $\preceq_\Psi$ orders the worlds from bottom to top, with the minimal world on the lowest level. The columns group worlds according to the sentences that they validate. The set $[\Psi]_c$ includes, for example, the conditionals $\top \Rightarrow A \wedge B$ and $\neg A \Rightarrow B$ but not the conditionals $(A \leftrightarrow \neg B) \Rightarrow A$ or $(A \leftrightarrow \neg B) \Rightarrow B$.

We will denote the set of all TPOs over W by $T(W)$. The strict part of $\preceq_\Psi$ will be denoted by $\prec_\Psi$ and its symmetric part by $\sim_\Psi$. We also shall occasionally abuse notation and use $[\preceq]_c$ to denote the conditional belief set represented by a TPO $\preceq$. We shall say that $\preceq$ satisfies a particular sentence or conditional sentence iff that sentence or conditional sentence is in $[\preceq]_c$ and that it satisfies a particular set of sentences or conditional sentences iff it satisfies all the sentences or conditional sentences in that set. Finally, a TPO $\preceq_\Psi$ will sometimes also be represented by an ordered partition $\langle S_1, S_2, \ldots S_m \rangle$ of W, with $x \preceq_\Psi y$ iff $r(x, \preceq_\Psi) \leq r(y, \preceq_\Psi)$, where $r(x, \preceq_\Psi)$ denotes the 'rank' of x with respect to $\preceq_\Psi$ and is defined by taking $S_{r(x,\preceq_\Psi)}$ to be the cell in the partition that contains x.

3. A naïve proposal and some triviality results

What should an agent believe after performing a contraction followed by a revision? We would like to extend the Harper Identity to cover this case.[5] The

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iff both $\min(\preccurlyeq_\Psi, W) \subseteq \llbracket C \rrbracket$ and $\min(\preccurlyeq_{\Psi * \neg A}, W) \subseteq \llbracket C \rrbracket$. This is equivalent to: $\min(\preccurlyeq_{\Psi \div A}, W) \subseteq \llbracket C \rrbracket$ iff $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_{\Psi * \neg A}, W) \subseteq \llbracket C \rrbracket$, which will hold for arbitrary C iff $\min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_{\Psi * \neg A}, W)$.

[5]Note that, by (HI), if one knows the outcome, in terms of resulting belief sets, of a contraction followed by a revision, one thereby also knows the outcome of two successive

most straightforward suggestion would be to simply replace, in (HI), all terms denoting belief sets by terms denoting the corresponding conditional belief sets. For reasons that will become clear shortly, we shall call this proposal the 'Naïve Iterated Harper Identity':

(NiHI) $\qquad [\Psi \div A]_{\mathrm{c}} = [\Psi]_{\mathrm{c}} \cap [\Psi * \neg A]_{\mathrm{c}}$

Equivalently, the suggestion can be presented as follows:

$$[(\Psi \div A) * B] = [\Psi * B] \cap [(\Psi * \neg A) * B]$$

If $B \equiv \top$ then we obtain (HI) as a special case. Under weak assumptions, (NiHI) can equivalently be restated in terms of contraction only:

Proposition 1. (NiHI) *entails*

(NiHI$^{\div}$) $\qquad [(\Psi \div A) \div B] = [\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B]$

and is equivalent to it in the presence of (K3*) *and the Levi Identity:*

(LI) $\qquad [\Psi * A] = \mathrm{Cn}([\Psi \div \neg A] \cup \{A\}).$

However, as Gärdenfors' classic triviality result and its subsequent refinements [25, 26, 27] have taught us, the unqualified extension of principles of belief dynamics to cover conditional beliefs is a risky business. And indeed, it turns out that the above proposal is too strong: it can be shown that, under mild constraints on single shot revision and contraction, it places unacceptable restrictions on the space of permissible belief sets or conditional belief sets. Indeed, we first note the following:

Proposition 2. *In the presence of* (K5*), (K6*) *and* (K3$^{\div}$), (HI) *and the left-to-right half of* (NiHI) *jointly entail the following "Restricted Right Euclidean" principle:*

(RRE) $\qquad$ *If A is consistent and $B \in [\Psi * A]$, then $[\Psi] \cap [\Psi * A] \subseteq [\Psi * B]$*

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contractions.

But (RRE)[6] is a very strong principle indeed. To obtain a sense of just how implausible it is, consider the following result:

Proposition 3. (RRE) *entails that there does not exist a belief state* Ψ *such that: (i)* $[\Psi] = \mathrm{Cn}(A \wedge B)$*, (ii)* $[\Psi * \neg A] = \mathrm{Cn}(\neg A \wedge B)$ *and (iii)* $[\Psi * A \leftrightarrow \neg B] = \mathrm{Cn}(A \leftrightarrow \neg B)$*, where* A *and* B *are propositional atoms.*

From this, we can see that (RRE) notably entails that there is no epistemic state associated with the TPO depicted in Figure 1. Furthermore, if we supplement (RRE) with just a few mild assumptions, matters get even worse:

Proposition 4. *In the presence of* $(\mathrm{K1}^*)$*,* $(\mathrm{K2}^*)$ *and* $(\mathrm{K5}^*)$*,* (RRE) *entails that there do not exist a state* Ψ *and non-tautologous sentences* A *and* B *in* L *whose disjunction* $A \vee B$ *is tautologous and are such that* $A, B \in [\Psi]$*.*

For a concrete illustration of this last implication, let 'A' stand for 'I have no more than one dollar in my pocket' and let 'B' stand for 'I have at least one dollar in my pocket'. (RRE) would prohibit my holding the belief that $A \wedge B$, i.e. that I have exactly one dollar in my pocket, a constraint that is clearly unacceptable.

(NiHI) and its shortcomings can equivalently be recast in semantic terms. A key concept, which will play a central role throughout the remainder of the paper, is the following:

Definition 1. *A binary* combinator $\oplus$ *is a function that takes pairs of TPOs as inputs and yields a TPO as an output. For convenience,* $\preceq_1 \oplus \preceq_2$ *will sometimes be denoted by* $\preceq_{1 \oplus 2}$*.*

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[6]The name for this principle was chosen for the following reason. Framed in terms of conditionals, (RRE) amounts to: If A is consistent and $A \Rightarrow B, \top \Rightarrow C, A \Rightarrow C \in [\Psi]_c$, then $B \Rightarrow C \in [\Psi]_c$. This is a restricted form of: If A is consistent and $A \Rightarrow B, A \Rightarrow C \in [\Psi]_c$, then $B \Rightarrow C \in [\Psi]_c$. A relation R is said to have the Right Euclidean property just in case it satisfies: If aRb and aRc, then bRc.

Indeed, in view of the equivalence between conditional belief sets and TPOs, we are, in seeking to extend the Harper Identity to the iterated case, plausibly looking for an appropriate combinator $\oplus$ such that:

(Combi) $\preceq_{\Psi \div A} = \preceq_{\Psi} \oplus \preceq_{\Psi * \neg A}$

Now, just as (HI) corresponds, given (Combi), to the following combination constraint:

$(\text{HI}_{\oplus})$ $\min(\preceq_{1 \oplus 2}, W) = \min(\preceq_1, W) \cup \min(\preceq_2, W)$

(NiHI) amounts to:

$(\text{NiHI}_{\oplus})$ For all $S \subseteq W$, $\min(\preceq_{1 \oplus 2}, S) = \min(\preceq_1, S) \cup \min(\preceq_2, S)$[7]

What our result above effectively demonstrates is that no combinator $\oplus$ satisfies $(\text{NiHI}_{\oplus})$ unless we place undesirable restrictions on its domain: $(\text{NiHI}_{\oplus})$ is too much to ask for.[8]

We will continue approaching our issue of interest from a predominantly semantic perspective for the remainder of the paper.

4. An alternative approach

4.1. TeamQueue combinators

In this section, we retreat from the naïve proposal above to an altogether weaker set of minimal postulates for $\oplus$, before taking a look at a concrete family of 'Team Queuing' combinators that satisfy them. We first establish a general

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[7] From the brief proof in Footnote 4, it is easy to establish, substituting $[\![B]\!]$ for W, that (NiHI) amounts to the claim that $\min(\preccurlyeq_{\Psi \div A}, [\![B]\!]) = \min(\preccurlyeq_{\Psi}, [\![B]\!]) \cup \min(\preccurlyeq_{\Psi * \neg A}, [\![B]\!])$. The correspondence between this and $(\text{NiHI}_{\oplus})$, given (Combi), should be clear.

[8] The problem that we have just noted for (NiHI) is closely related to the observation that an intersection of two rational sets of conditionals need not itself be rational, which is familiar from the literature on nonmonotonic reasoning [17]. We return to this connection below, in Subsection 4.2.

characterisation of this family before showing that our set of minimal postulates suffices to characterise it in our restricted functional domain of interest.

Since we are in the business of extending the Harper Identity, we will begin by requiring satisfaction of (HI). We call combinators that satisfy this property '*basic*' combinators. In addition, even though (NiHI) is too strong, certain weakenings of it do seem to be compelling. Specifically, it seems appropriate to require that our combination method leads to the following weak lower and upper bound principles:

(LB) $\qquad [\Psi \div A]_c \supseteq [\Psi]_c \cap [\Psi * \neg A]_c$

(UB) $\qquad [\Psi \div A]_c \subseteq [\Psi]_c \cup [\Psi * \neg A]_c$

Equivalently:

$$[(\Psi \div A) * B] \supseteq [\Psi * B] \cap [(\Psi * \neg A) * B]$$

$$[(\Psi \div A) * B] \subseteq [\Psi * B] \cup [(\Psi * \neg A) * B]$$

We note that (LB) corresponds to the direction of (NiHI) that was *not* implicated in our earlier triviality results. Given (Combi), these will be ensured by requiring, respectively, the following upper and lower bounds on $\min(\preceq_{1\oplus 2}, S)$ for any $S \subseteq W$:[9]

(UB$_\oplus$) $\qquad \min(\preceq_{1\oplus 2}, S) \subseteq \min(\preceq_1, S) \cup \min(\preceq_2, S)$

(LB$_\oplus$) $\qquad$ Either $\min(\preceq_{1\oplus 2}, S) \supseteq \min(\preceq_1, S)$
$\qquad\qquad$ or $\min(\preceq_{1\oplus 2}, S) \supseteq \min(\preceq_2, S)$

These last principles can also be formulated using only binary comparisons:

Proposition 5. (UB$_\oplus$) *and* (LB$_\oplus$) *are respectively equivalent to the following:*

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[9]Note that an upper (respectively lower) bound on sets of *worlds* yields a lower (respectively upper) bound on sets of *beliefs*. We omit the proofs of the correspondences here, since they are similar in flavour to those of the correspondences between (HI) and (HI$_\oplus$), or again (NiHI) and (NiHI$_\oplus$). See Footnote 4 and Footnote 7.

$(\text{SPU}_\oplus^+)$ *If $x \prec_1 y$ and $z \prec_2 y$ then either $x \prec_{1\oplus 2} y$ or $z \prec_{1\oplus 2} y$*

$(\text{WPU}_\oplus^+)$ *If $x \preceq_1 y$ and $z \preceq_2 y$ then either $x \preceq_{1\oplus 2} y$ or $z \preceq_{1\oplus 2} y$*

$(\text{SPU}_\oplus^+)$ and $(\text{WPU}_\oplus^+)$ owe their names to their being respective strengthenings of the following principles of Strict and Weak Preference Unanimity, which are analogues of the 'Weak Pareto' and 'Pareto Weak Preference' principles found in the social choice literature:[10]

$(\text{SPU}_\oplus)$ If $x \prec_1 y$ and $x \prec_2 y$ then $x \prec_{1\oplus 2} y$

$(\text{WPU}_\oplus)$ If $x \preceq_1 y$ and $x \preceq_2 y$ then $x \preceq_{1\oplus 2} y$

With this set of basic principles in place, we now consider a concrete family of basic combinators that satisfy both $(\text{SPU}_\oplus^+)$ and $(\text{WPU}_\oplus^+)$, and, indeed, can be shown to be *characterised* by precisely these constraints in our functional domain of interest. We call these 'TeamQueue' combinators.

The basic idea behind this family—and the motivation behind the name given to it—can be grasped by means of the following analogy: A number of couples go shopping for groceries. The supermarket that they frequent is equipped with two tills. For each till, we find a sequence of groups of people queueing to pay for their items. In order to minimise the time spent in the store, each couple operates by "team queueing": each member of the pair joins a group in a different queue and leaves their place to join their partner's group in case this second group arrives at the till first. After synchronously processing their first group of customers, the tills may then operate at different and variable speeds. We consider the temporal sequence of sets of couples leaving the store. In our setting, the queues are the two TPOs (with lower elements towards the head of the queue) and the couples are pairs of copies of each world.

More formally, we assume, for each ordered pair $\langle \preceq_1, \preceq_2 \rangle$ of TPOs, a sequence $\langle a_{\preceq_1, \preceq_2}(i) \rangle_{i \in \mathbb{N}}$ such that:

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[10]For an accessible book-length introduction to social choice, see for instance [28]. For an overview of various issues at the intersection of social choice and computer science, see [29].

(a1) $\emptyset \neq a_{\preceq_1,\preceq_2}(i) \subseteq \{1,2\}$ for each i

(a2) $a_{\preceq_1,\preceq_2}(1) = \{1,2\}$

$a_{\preceq_1,\preceq_2}(i)$ specifies which queue is to be processed at each step. Condition (a1) ensures either one or both are processed, while condition (a2) states that both are processed at the initial stage, ensuring that $(\mathrm{HI}_\oplus)$ holds for the resulting combinators. The ordered partition $\langle T_1, T_2, \ldots, T_m \rangle$ corresponding to $\preceq_{1\oplus2}$ is then constructed inductively as follows:

$$T_i = \bigcup_{j \in a_{\preceq_1,\preceq_2}(i)} \min(\preceq_j, \bigcap_{k<i} T_k^c)$$

where 'T^c' denotes the complement of set T and m is minimal such that $\bigcup_{i \leq m} T_i = W$. With this in hand, we can now offer:

Definition 2. $\oplus$ *is a* TeamQueue (TQ) *combinator iff, for each ordered pair* $\langle \preceq_1, \preceq_2 \rangle$ *of TPOs there exists a sequence* $\langle a_{\preceq_1,\preceq_2}(i) \rangle_{i \in \mathbb{N}}$ *satisfying (a1) and (a2) and such that* $\preceq_{1\oplus2}$ *is obtained as above.*

It is easily verified that TeamQueue combinators are indeed basic combinators. The following example provides an elementary illustration of the combinator at work:

Example 1. *Suppose that* $W = \{w, x, y, z\}$*, that* $\preceq_1$ *is the TPO represented by the ordered partition* $\langle \{z\}, \{w\}, \{x,y\} \rangle$*, and that* $\preceq_2$ *is represented by* $\langle \{x,z\}, \{y\},$ $\{w\} \rangle$*. Let* $\oplus$ *be a TeamQueue combinator such that* $\langle a_{\preceq_1,\preceq_2}(i) \rangle_{i \in \mathbb{N}} = \langle \{1,2\}, \{2\},$ $\{1\}, \ldots \rangle$*. Then the ordered partition corresponding to* $\preceq_{1\oplus2}$ *is* $\langle T_1, T_2, T_3 \rangle =$ $\langle \{x,z\}, \{y\}, \{w\} \rangle$*, since*

$$
\begin{aligned}
T_1 &= \bigcup_{j \in \{1,2\}} \min(\preceq_j, W) = \{x, z\} \\
T_2 &= \min(\preceq_2, T_1^c) = \{y\} \\
T_3 &= \min(\preceq_1, T_1^c \cap T_2^c) = \{w\}
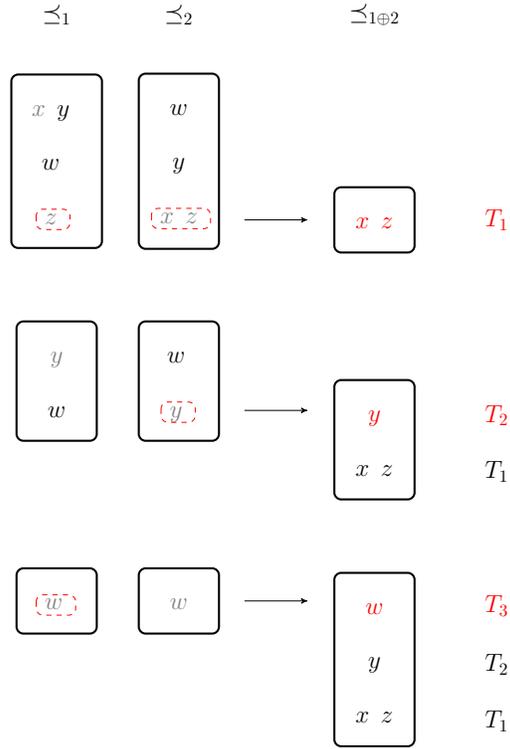\end{aligned}
$$

See Figure 2 for illustration.

Figure 2: Illustration of the construction of $\preceq_{1\oplus2}$ in Example 1, with the successive formations of T_1, then T_2, and finally T_3. The leftmost two boxes at each stage correspond to the restrictions of the relevant TPOs to W, T_1^c and T_2^c, respectively, with the lesser worlds in the ordering being placed below the greater ones. Worlds to be deleted from the input at the next step are represented in grey. Dashed boxes depict the sets of worlds whose union is to be selected for T_i.

As noted above, TeamQueue combinators satisfy both $(\mathrm{SPU}_\oplus^+)$ and $(\mathrm{WPU}_\oplus^+)$. In fact, one can show that this family can actually be characterised by these two conditions, in the presence of a third:

Theorem 1. $\oplus$ *is a TeamQueue combinator iff it is a basic combinator that satisfies* $(\mathrm{SPU}_\oplus^+)$, $(\mathrm{WPU}_\oplus^+)$ *and the following 'no overtaking' property;*

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$(\text{NO}_{\oplus})$ *For $i \neq j$, if $x \prec_i y$ and $z \preceq_j y$, then either $x \prec_{1\oplus 2} y$ or*

             $z \preceq_{1\oplus 2} y$

Taken together, the three postulates $(\text{SPU}_{\oplus}^{+})$, $(\text{WPU}_{\oplus}^{+})$ and $(\text{NO}_{\oplus})$ say that in $\preceq_{1\oplus 2}$, no world x is allowed to improve its position with respect to *both* input orderings $\preceq_1$ and $\preceq_2$. Indeed each postulate blocks one of the three possible ways in which this 'no double improvement' condition could be violated. We note that this last condition can be shown to correspond to a rather remarkable property:

Proposition 6. $\oplus$ *is a TeamQueue combinator iff it is a basic combinator that satisfies the following 'factoring' property, for all $S \subseteq W$:*

$(\text{F}_{\oplus})$ $\min(\preceq_{1\oplus 2}, S)$ *is equal to either* $\min(\preceq_1, S)$, $\min(\preceq_2, S)$ *or*

             $\min(\preceq_1, S) \cup \min(\preceq_2, S)$

Given (Combi), $(\text{F}_{\oplus})$ yields the claim that $[(\Psi \div A) * B]$ is equal to either $[\Psi * B]$, $[(\Psi * \neg A) * B]$ or $[\Psi * B] \cap [(\Psi * \neg A) * B]$.

To wrap up this section, it should be noted that the results so far have been perfectly domain-independent, in the sense that they hold for combinators whose domain corresponded to the entire space $T(W) \times T(W)$ of pairs of TPOs defined over W. Our problem of interest is somewhat narrower in scope, however, since we are interested in the special case in which one of the TPOs is obtained from the other by means of a revision. In particular, we assume the first two semantic postulates of [2]:

$(\text{C1}_{\preceq}^{*})$ If $x, y \in [\![A]\!]$ then $x \preceq_{\Psi * A} y$ iff $x \preceq_{\Psi} y$

$(\text{C2}_{\preceq}^{*})$ If $x, y \in [\![\neg A]\!]$ then $x \preceq_{\Psi * A} y$ iff $x \preceq_{\Psi} y$

In other words, $\preceq_1$ and $\preceq_2$ will always be $[\![A]\!]$-variants for some sentence A, in the following sense:

Definition 3. *Given $\preceq_1, \preceq_2 \in T(W)$ and $S \subseteq W$, we say $\preceq_1$ and $\preceq_2$ are S-variants iff $(x \preceq_1 y$ iff $x \preceq_2 y)$ holds for all $x, y \in (S \times S) \cup (S^c \times S^c)$. We*

let $V(W)$ denote the set of all $\langle \preceq_1, \preceq_2 \rangle \in T(W) \times T(W)$ such that $\preceq_1, \preceq_2$ are S-variants for some $S \subseteq W$.

Example 2. *Suppose that $W = \{w, x, y, z\}$, that $\preceq_1$ is the TPO represented by the ordered partition $\langle \{w\}, \{x\}, \{y\}, \{z\} \rangle$, and that $\preceq_2$ is represented by $\langle \{w\}, \{x, y\}, \{z\} \rangle$. Then $\preceq_1$ and $\preceq_2$ are $\{y, z\}$-variants, since (i) $w \prec_1 x$ and $w \prec_2 x$, as well as (ii) $y \prec_1 z$ and $y \prec_2 z$. They are not, however, $\{x, y\}$-variants, since $x \prec_1 y$ but $y \preceq_2 x$.*

This leads to the following domain restriction on $\oplus$:

$$(\text{DOM}_\oplus) \quad Domain(\oplus) \subseteq V(W)$$

As it turns out, this constraint allows for a noteworthy simplification of the characterisation of TeamQueue combinators, allowing us to do away with $(\text{NO}_\oplus)$ entirely:

Proposition 7. *Given $(\text{DOM}_\oplus)$, $\oplus$ is a TeamQueue combinator iff it is a basic combinator that satisfies $(\text{SPU}_\oplus^+)$ and $(\text{WPU}_\oplus^+)$.*

In fact, we can show that the characterisation under our domain restriction only requires the weaker versions of our preference unanimity conditions:

Proposition 8. *Given $(\text{DOM}_\oplus)$, $\oplus$ satisfies $(\text{SPU}_\oplus^+)$ and $(\text{WPU}_\oplus^+)$ iff it satisfies $(\text{SPU}_\oplus)$ and $(\text{WPU}_\oplus)$, respectively.*

Given Proposition 5 and Proposition 6, the potentially surprising upshot of Proposition 7 is that, in our domain of interest, satisfaction of (LB) and (UB) entails satisfaction of $(\text{F}_\oplus)$.

4.2. The Synchronous TeamQueue Combinator

A special case of TeamQueue combinators takes $a_{\preceq_1, \preceq_2}(i) = \{1, 2\}$ for all ordered pairs $\langle \preceq_1, \preceq_2 \rangle$ and all i. This represents a particularly *even handed* way of combining TPOs. In terms of our supermarket analogy, it corresponds to the situation in which the tills process groups of customers at the same speed.

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Definition 4. *The* Synchronous TeamQueue (STQ) *combinator is the Team-Queue combinator for which* $a_{\preceq_1, \preceq_2}(i) = \{1, 2\}$ *for all ordered pairs* $\langle \preceq_1, \preceq_2 \rangle$ *and all i. We will denote the STQ combinator by* $\oplus_{\mathrm{STQ}}$.

Example 3. *Suppose that $W = \{x, y, z, w\}$, that $\preceq_1$ is the TPO represented by the ordered partition $\langle \{z\}, \{w\}, \{x, y\} \rangle$ and $\preceq_2$ is represented by $\langle \{x, z\}, \{y\}, \{w\} \rangle$. Then the ordered partition corresponding to $\preceq_{1 \oplus_{\mathrm{STQ}} 2}$ is $\langle T_1, T_2 \rangle = \langle \{x, z\}, \{w, y\} \rangle$. See Figure 3 for illustration.*[11]

We have a number of characterisations of this combinator. It can first of all be characterised semantically, in the absence of domain restrictions, as follows:

Theorem 2. $\oplus_{\mathrm{STQ}}$ *is the only basic combinator that satisfies both* $(\mathrm{SPU}_{\oplus}^+)$ *and the following 'Parity' constraint:*

$(\mathrm{PAR}_{\oplus})$ *If $x \prec_{1 \oplus 2} y$ then for each $i \in \{1, 2\}$ there exists z such that $x \sim_{1 \oplus 2} z$ and $z \prec_i y$*

Note that $(\mathrm{WPU}_{\oplus}^+)$ is *not* listed among the characteristic principles. Indeed, it is entailed by the conjunction of $(\mathrm{SPU}_{\oplus}^+)$ and $(\mathrm{PAR}_{\oplus})$. Figure 2, which illustrates

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[11]The STQ combinator is to be distinguished from the 'min' combinator $\oplus_{\min}$ that ranks worlds according to the minimum rank they receive in the inputs: $x \preceq_{1 \oplus_{\min} 2} y$ iff $\min(r(x, \preceq_1), r(x, \preceq_2)) \leq \min(r(y, \preceq_1), r(y, \preceq_2))$.

$\oplus_{\mathrm{STQ}}$ and $\oplus_{\min}$ are indeed distinct, even in our restricted functional domain of interest. To see this, let $x \prec_{\Psi} y \prec_{\Psi} \{w, z\}$. Let $*$ be the lexicographic revision operator (see Subsection 5.2 for definition and references). Then $y \prec_{\Psi * \neg(x \vee z)} w \prec_{\Psi * \neg(x \vee z)} x \prec_{\Psi * \neg(x \vee z)} z$. Defining $\div$ via (Combi) and $\oplus_{\mathrm{STQ}}$, we get: $\{x, y\} \prec_{\Psi \div x \vee z} \{w, z\}$. Using $\oplus_{\min}$, however $\{x, y\} \prec_{\Psi \div x \vee z} w \prec_{\Psi \div x \vee z} z$.

Note that $\oplus_{\min}$ does however belong to the TQ family. To demonstrate this, it suffices to show that $\oplus_{\min}$ satisfies $(\mathrm{WPU}_{\oplus}^+)$, $(\mathrm{SPU}_{\oplus}^+)$ and $(\mathrm{NO}_{\oplus})$. We simply prove satisfaction of $(\mathrm{WPU}_{\oplus}^+)$, since the other principles are handled in a similar manner. Let θ denote $\min(r(y, \preceq_1), r(y, \preceq_2))$. Now suppose $y \prec_{1 \oplus_{\min} 2} x$ and $y \prec_{1 \oplus_{\min} 2} z$. Then (i) $\theta < \min(r(x, \preceq_1), r(x, \preceq_2))$ and (ii) $\theta < \min(r(z, \preceq_1), r(z, \preceq_2))$. We know that either $\theta = r(y, \preceq_1)$ or $\theta = r(y, \preceq_2)$. If $\theta = r(y, \preceq_1)$, then (i) gives us $r(y, \preceq_1) < r(x, \preceq_1)$, i.e. $y \prec_1 x$. If $\theta = r(y, \preceq_2)$, we similarly recover $y \prec_2 z$ from (ii).
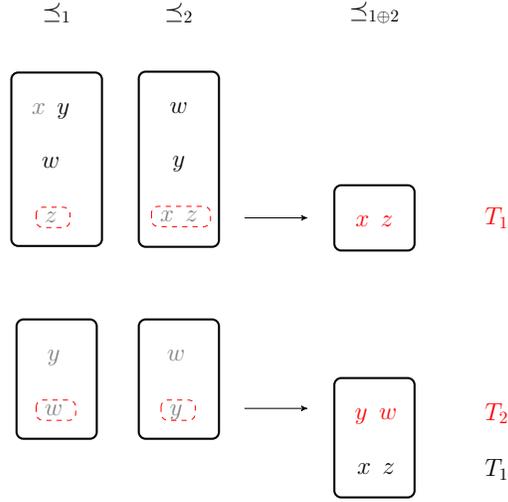
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$\preceq_1$ $\preceq_2$ $\preceq_{1\oplus2}$

x y w

w y

z x z $\longrightarrow$ x z T_1

y w

w y $\longrightarrow$ y w T_2

                                 x z T_1

Figure 3: Illustration of the construction of $\preceq_{1\oplus2}$ in Example 3.

a TQ combinator that is distinct from $\oplus_{\mathrm{STQ}}$, provides an example of a failure of ($\mathrm{PAR}_\oplus$). Specifically, while we have $y \prec_{1\oplus2} w$, it is *not* the case that, for each $i \in \{1, 2\}$, there exists z' such that $y \sim_{1\oplus2} z'$ and $z' \prec_i w$. Indeed, the only z' such that $y \sim_{1\oplus2} z'$ is y itself and so ($\mathrm{PAR}_\oplus$) would require $y \prec_1 w$. But instead, we have $w \prec_1 y$.

Whilst ($\mathrm{PAR}_\oplus$) may not be immediately easy to grasp, it can be given a helpful reformulation in terms of the notion of *strong belief* [30, 31]. A sentence $A \in [\Psi]$ is strongly believed in Ψ in case the only way it can be dislodged by the next revision input B is if B is logically *inconsistent* with A. In other words:

Definition 5. *A is* strongly believed *in Ψ iff (i) $A \in [\Psi]$, and (ii) $A \in [\Psi * B]$ for all sentences B such that $A \wedge B$ is consistent.*

Semantically, a consistent sentence A is strongly believed in Ψ iff $x \prec_\Psi y$ for every $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$. With this in hand, one can show:

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Proposition 9. $(\mathrm{PAR}_\oplus)$ *is equivalent to:*

$(\mathrm{SB}_\oplus)$ *If $x \prec_{1\oplus2} y$ for every $x \in S^c$, $y \in S$, then $\min(\preceq_1, S) \cup$*
    $\min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$

Given (Combi), $(\mathrm{SB}_\oplus)$ yields: If $\neg B$ is strongly believed in $\Psi \div A$ then $[(\Psi \div A) * B] \subseteq [\Psi * B] \cap [(\Psi * \neg A) * B]$. Thus, although we cannot have (NiHI) for all A, B, the STQ combinator *does* guarantee the principle to hold for a certain restricted class of pairs of sentences, namely those A, B such that $\neg B$ is strongly believed after contracting by A.

Having said all this, there is however another, possibly more perspicuous, way of characterising $\oplus_{\mathrm{STQ}}$. In order to present it, however, we need to introduce one further item of notation:

Definition 6. *Let $\sqsupseteq$ be a binary relation on $T(W)$ such that $\langle S_1, S_2, \ldots, S_m \rangle \sqsupseteq \langle T_1, T_2, \ldots, T_m \rangle$ iff either (i) $S_i = T_i$ for all $i = 1, \ldots, m$, or (ii) $S_i \supset T_i$ for the first i such that $S_i \neq T_i$.*

Note that $\sqsupseteq$ compares TPOs via their representing partitions. We may assume that both partitions are of equal length m, since we can always fill up the tail of the shorter one with instances of the empty set, if necessary. The relation $\sqsupseteq$ is clearly reflexive, transitive and antisymmetric. It partially orders $T(W)$ according to what one might intuitively call comparative '*flatness*', with the flatter TPOs appearing 'greater' in the ordering. With this interpretation in mind, we can show that $\oplus_{\mathrm{STQ}}$ yields the unique flattest TPO satisfying $(\mathrm{SPU}_\oplus^+)$:

Theorem 3. $\preceq_{1 \oplus_{\mathrm{STQ}} 2} \quad \sqsupseteq \quad \preceq_{1\oplus2}$, *for any TPOs $\preceq_1$ and $\preceq_2$, and any combinator $\oplus$ satisfying* $(\mathrm{SPU}_\oplus^+)$.

In view of Proposition 5, this can be restated as the observation that $\oplus_{\mathrm{STQ}}$ yields the flattest TPO that satisfies the *set of conditionals* satisfied by both inputs. With the result presented in this manner, readers familiar with the literature on nonmonotonic reasoning will anticipate the following corollary:

Corollary 1. $[\preceq_{1 \oplus_{\mathrm{STQ}} 2}]_{\mathrm{c}} = \mathrm{C_R}([\preceq_1]_{\mathrm{c}} \cap [\preceq_2]_{\mathrm{c}})$, *where* $\mathrm{C_R}$ *is the rational closure function* [17, *Definitions 20, 21*].

with the Synchronous TeamQueue approach to our problem of interest consequently amounting to:

(iHI) $[\Psi \div A]_{\mathrm{c}} = \mathrm{C_R}([\Psi]_{\mathrm{c}} \cap [\Psi * \neg A]_{\mathrm{c}})$

Indeed, Booth & Paris [32], for instance, establish, as a corollary of a further result, that the rational closure of a set of conditionals Γ corresponds to the flattest TPO that satisfies it.[12] Similar results can be found in Rott [33], in relation to his E-minimal entailment, and Pearl [34], in relation to his 1-entailment, notions that are essentially identical to rational closure (regarding rational closure's relation to 1-entailment, see [34]; regarding its relation to E-minimal entailment, see [35]).

The rational closure of a set of conditionals Γ has been argued to provide the most conservative or again least opinionated rational conditional belief set that subsumes Γ.[13] Lehmann & Magidor offer a dialogical justification of this claim, grounded in their syntactic definition of rational closure. On the basis of the aforementioned semantic result in [33], Rott argues that the conditional belief set given by the rational closure of Γ minimises degrees of confidence in the sentences of L among those conditional belief sets that include Γ. Finally, Hill

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[12]Indeed, they establish that a particular kind of construction (essentially an n-ary STQ combination of the TPOs satisfying Γ) yields a TPO that corresponds to the rational closure of Γ. In the course of doing this, they establish that this TPO is also the flattest TPO satisfying Γ. See the proof of their Theorem 2.

[13]Here, 'most conservative' or 'least opinionated' cannot be taken to mean 'set-theoretically smallest', since the set of rational consequence relations satisfying a given set of conditionals is not closed under intersection. In the literature, these notions are understood informally, although it would probably be desirable to sharpen these into something more precise. One could indeed seek to (1) define a formal measure of 'opinionation for a set of conditionals that would extend a plausible measure for sets of propositional formulae and (2) show that the rational closure of a set of conditionals Γ is the superset of Γ that minimises this quantity. We do not currently have such a proposal or result to offer, unfortunately.

& Paris [36] show that, on their preferred probabilistic semantics for rational conditionals, the rational closure of Γ corresponds to the maximally entropic (i.e. least informative) probability function that satisfies Γ. In view of all this, we suggest that (iHI) constitutes a prima facie very attractive generalisation of (HI).

5. Putting the proposal to work

As we stated earlier, a central result of AGM theory says that, under assumption of HI, if $*$ satisfies the AGM revision postulates, then $\div$ automatically satisfies the AGM contraction postulates. In this section, we turn our attention to some of the postulates for both iterated revision and contraction that have been proposed in the literature. We show that, if $\preceq_{\Psi \div A}$ is defined from $\preceq_{\Psi}$ and $\preceq_{\Psi * \neg A}$ using (Combi) via a TeamQueue combinator, then satisfaction of some well known postulates for iterated revision leads to satisfaction of well known postulates for iterated contraction.

5.1. The Darwiche-Pearl (DP) postulates

The most widely cited postulates for iterated revision are the four DP postulates of [2]. These, like most of the postulates for iterated belief change, come in two flavours: a *semantic* one in terms of requirements on the TPO $\preceq_{\Psi * A}$ associated to the revised state $\Psi * A$, and a *syntactic* one in terms of requirements on the belief set $[(\Psi * A) * B]$ following a sequence of two revisions. Turning first to the semantic versions, we have already encountered the first two of these postulates—$(\text{C1}^*_{\preceq})$ and $(\text{C2}^*_{\preceq})$—in the previous section. The other two are

$(\text{C3}^*_{\preceq})$ If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \prec_{\Psi} y$ then $x \prec_{\Psi * A} y$

$(\text{C4}^*_{\preceq})$ If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preceq_{\Psi} y$ then $x \preceq_{\Psi * A} y$

Each of these has an equivalent corresponding syntactic version as follows:

(C1^*) If $A \in \text{Cn}(B)$ then $[(\Psi * A) * B] = [\Psi * B]$

(C2^*) If $\neg A \in \text{Cn}(B)$ then $[(\Psi * A) * B] = [\Psi * B]$

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(C3*) If $A \in [\Psi * B]$ then $A \in [(\Psi * A) * B]$

(C4*) If $\neg A \notin [\Psi * B]$ then $\neg A \notin [(\Psi * A) * B]$

Chopra *et al* [9] proposed a list of 'counterparts' to the DP postulates for contraction. The semantic versions of these were:

(C1$\dot{\preceq}$) If $x, y \in [\![\neg A]\!]$ then $x \preceq_{\Psi \div A} y$ iff $x \preceq_{\Psi} y$

(C2$\dot{\preceq}$) If $x, y \in [\![A]\!]$ then $x \preceq_{\Psi \div A} y$ iff $x \preceq_{\Psi} y$

(C3$\dot{\preceq}$) If $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and $x \prec_{\Psi} y$ then $x \prec_{\Psi \div A} y$

(C4$\dot{\preceq}$) If $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and $x \preceq_{\Psi} y$ then $x \preceq_{\Psi \div A} y$

Chopra *et al* [9] showed (their Theorem 2) that, in the presence of the AGM postulates (reformulated as in our setting to apply to belief *states* rather than just belief *sets*) each of these postulates has an equivalent syntactic version as follows:

(C1$\dot{}$) If $\neg A \in \mathrm{Cn}(B)$ then $[(\Psi \div A) * B] = [\Psi * B]$

(C2$\dot{}$) If $A \in \mathrm{Cn}(B)$ then $[(\Psi \div A) * B] = [\Psi * B]$

(C3$\dot{}$) If $\neg A \in [\Psi * B]$ then $\neg A \in [(\Psi \div A) * B]$

(C4$\dot{}$) If $A \notin [\Psi * B]$ then $A \notin [(\Psi \div A) * B]$[14]

However, while Chopra *et al* remarked that their postulates are 'Darwiche-Pearl-like', the precise nature of the connection was not elucidated. As it turns out, the definition of $\preceq_{\Psi \div A}$ from $\preceq_{\Psi}$ and $\preceq_{\Psi * \neg A}$ using (Combi) via a TeamQueue combinator affords us a clearer view on the matter:

Proposition 10. *Let $\oplus$ be an arbitrary TeamQueue combinator, let $*$ be an AGM revision operator and let $\div$ be such that $\preceq_{\Psi \div A}$ is defined from $*$ via (Combi) using $\oplus$. Then, for each $i = 1, 2, 3, 4$, if $*$ satisfies (C$i_{\preceq}^{*}$) then $\div$ satisfies (C$i_{\preceq}^{\dot{}}$).*

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[14](C1$\dot{\preceq}$)–(C4$\dot{\preceq}$) have been recently rediscovered in [37], in which a set of somewhat different-looking syntactic counterparts is also proposed.

As a corollary, given the AGM postulates, we obviously recover an analogous result for the syntactic versions.

To round off this subsection, we now turn our attention to the postulate of 'Principled Factored Intersection' endorsed by Nayak *et al* [13], which they show to be satisfied by a number of proposals for iterated contraction:

(PFI)   Given $B \in [\Psi \div A]$

    (a) If $\neg B \rightarrow \neg A \in [(\Psi \div A) \div B]$, then

$$[(\Psi \div A) \div B] = [\Psi \div A] \cap [\Psi \div \neg A \rightarrow B]$$

    (b) If $\neg B \rightarrow \neg A, \neg B \rightarrow A \notin [(\Psi \div A) \div B]$, then

$$[(\Psi \div A) \div B] = [\Psi \div A] \cap [\Psi \div \neg A \rightarrow B] \cap [\Psi \div A \rightarrow B]$$

    (c) If $\neg B \rightarrow A \in [(\Psi \div A) \div B]$, then

$$[(\Psi \div A) \div B] = [\Psi \div A] \cap [\Psi \div A \rightarrow B]$$

The rationale for (PFI) remains rather unclear to date, however. Indeed, the only justifications provided appear to be (a) that (PFI) avoids a particular difficulty faced by another constraint that has been proposed in the literature—namely Rott's 'Qualified Intersection' principle [38]—and which can be reformulated in a manner that is superficially rather similar to (PFI) and (b) that (PFI) entails a pair of prima facie appealing principles. Neither of these considerations strike us as being particularly compelling. For one, Rott's Qualified Intersection principle remains itself unclearly motivated. Secondly, plenty of ill-advised principles can be shown to have certain plausible consequences.

The TeamQueue approach, however, allows us to rest (PFI) on a far firmer foundation. Indeed:

Proposition 11. *Let $\oplus$ be a TeamQueue combinator, let $*$ be an AGM revision operator and let $\div$ be such that $\preceq_{\Psi \div A}$ is defined from $*$ via* (Combi) *using $\oplus$. If $*$ satisfies* $(C1^{*}_{\preceq})$ *and* $(C2^{*}_{\preceq})$ *then $\div$ satisfies* (PFI).

5.2. Popular strengthenings of the DP postulates

Three popular approaches to supplementing the AGM and DP postulates for revision can be found in the literature: the 'natural' [5], 'restrained' [3], and

'lexicographic' [39] approaches. All of these have the semantic consequence that the prior TPO $\preceq_\Psi$ determines the posterior TPO $\preceq_{\Psi*A}$. All three promote the lowest A-worlds in $\preceq_\Psi$ to become the lowest overall in $\preceq_{\Psi*A}$, but differ on what to do with the rest of the ordering. Natural revision leaves everything else unchanged, restrained revision preserves the strict ordering $\prec_\Psi$ while additionally making every A-world x strictly lower than every $\neg A$-world y for which $x \preceq_\Psi y$, and lexicographic revision just makes every A-world lower than every $\neg A$-world, while preserving the ordering within each of $[\![A]\!]$ and $[\![\neg A]\!]$.

The obvious question to ask is then the following: Which principles of iterated contraction does one recover from the natural, restrained and lexicographic revision operators, respectively, if one defines $\div$ from $*$, via (Combi), using $\oplus_{\mathrm{STQ}}$? As it turns out, both the natural and the restrained revision operator yield the very same iterated contraction operator, which has been discussed in the literature under the name of 'natural contraction' [13], and which sets $\min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$ to be the lowest ranked equivalence class in $\preceq_{\Psi \div A}$, while leaving $\preceq_{\Psi \div A}$ otherwise unchanged from $\preceq_\Psi$. Indeed, more generally:

Proposition 12. *Let $*$ be any revision operator—such as the natural or restrained revision operator—satisfying* (C1$^*_\preceq$)*,* (C2$^*_\preceq$)*,* (C4$^*_\preceq$) *and the following property:*

$$\text{If } x, y \notin \min(\preceq_\Psi, [\![A]\!]) \text{ and } x \prec_\Psi y, \text{ then } x \prec_{\Psi*A} y$$

Let $\div$ be the contraction operator defined from $*$ via (Combi) *using $\oplus_{\mathrm{STQ}}$. Then $\div$ is the natural contraction operator.*

We do not have a characterisation of the operator that is recovered from lexicographic revision in this manner—that is, the operator $\div$ such that $\preceq_{\Psi \div A} = \preceq_\Psi$ $\oplus_{\mathrm{STQ}} \preceq_{\Psi*_L \neg A}$, where $*_L$ is lexicographic revision—which we call the *STQ-lex contraction* operator ($\div_{\mathrm{STQL}}$). We can report, however, that it is *distinct from both lexicographic contraction ($\div_L$) and priority contraction ($\div_P$)*, the other two iterated contraction operators discussed in the literature alongside natural contraction [13].

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Roughly, lexicographic contraction works by setting the ith-ranked equivalence class S_i of $\preceq_{\Psi \dot{-} A}$ to be the union of the ith-lowest A-worlds with the ith-lowest $\neg A$-worlds. The difference between this approach and STQ-lex contraction can be illustrated by the following example:

Example 4. *Suppose that $W = \{v, w, x, y, z\}$ and $\preceq_{\Psi}$ is the TPO represented by $\langle \{v\}, \{w\}, \{x\}, \{y\}, \{z\} \rangle$. Let $\llbracket A \rrbracket = \{v, z\}$. We then have $\preceq_{\Psi *_L \neg A} = \langle \{w\}, \{x\}, \{y\}, \{v\}, \{z\} \rangle$. Lexicographic contraction yields $\preceq_{\Psi \dot{-}_L A} = \langle \{v, w\}, \{z, x\}, \{y\} \rangle$ while STQ-lex contraction gives us $\preceq_{\Psi \dot{-}_{\text{STQL}} A} = \langle \{v, w\}, \{x\}, \{y\}, \{z\} \rangle$. See Figure [4] for an illustration.*

In fact, somewhat more generally, lexicographic contraction cannot be recovered by *any* kind of TeamQueue combination of $\preceq_{\Psi}$ and $\preceq_{\Psi *_L \neg A}$, for *any* revision operator $*$ satisfying either $(\text{C3}^*_{\preceq})$ or $(\text{C4}^*_{\preceq})$. Indeed, as we know from Proposition [10], if it could, then it would have to satisfy either $(\text{C3}^{\dot{-}}_{\preceq})$ or $(\text{C4}^{\dot{-}}_{\preceq})$. But clearly it does not, as Example [4] shows: there, we have $z \in \llbracket \neg A \rrbracket$, $y \in \llbracket A \rrbracket$ and $y \prec_{\Psi} z$ but $z \prec_{\Psi \dot{-}_L A} y$, in contradiction with both principles.

The posterior TPO $\preceq_{\Psi \dot{-}_P A}$ obtained by priority contraction is defined piecewise. If $A \in [\Psi]$ or A is a tautology, then set $\preceq_{\Psi \dot{-}_P A} = \preceq_{\Psi}$. Otherwise, define $\preceq_{\Psi \dot{-}_P A}$ by taking the TPO $\preceq_{\Psi *_L \neg A}$, resulting from the lexicographic revision of Ψ by $\neg A$ and then shifting any A-worlds that were initially in the minimal set of $\preceq_{\Psi}$ into its minimal set, leaving all else unchanged. The non-equivalence of this procedure to STQ-lex contraction can be seen in the following example:

Example 5. *Suppose that $W = \{w, x, y, z\}$ and $\preceq_{\Psi}$ is the TPO represented by $\langle \{x\}, \{y, z\}, \{w\} \rangle$. Let $\llbracket A \rrbracket = \{x, y\}$, so that $\preceq_{\Psi *_L \neg A} = \langle \{z\}, \{w\}, \{x\}, \{y\} \rangle$. Then STQ-lex contraction yields $\preceq_{\Psi \dot{-}_{\text{STQL}} A} = \langle \{x, z\}, \{y, w\} \rangle$ while priority contraction gives us $\preceq_{\Psi \dot{-}_P A} = \langle \{x, z\}, \{w\}, \{y\} \rangle$. See Figure [5] for an illustration.*

More generally, priority contraction cannot be recovered by combination of $\preceq_{\Psi}$ with *any* other ordering, by *any* combination method that satisfies $(\text{PAR}_{\oplus})$. Indeed, we know from Proposition [9] that $(\text{PAR}_{\oplus})$ is equivalent to $(\text{SB}_{\oplus})$, which in turn, given (Combi) entails: If $x \prec_{\Psi \dot{-} A} y$ for every $x \in S^c$, $y \in S$, then
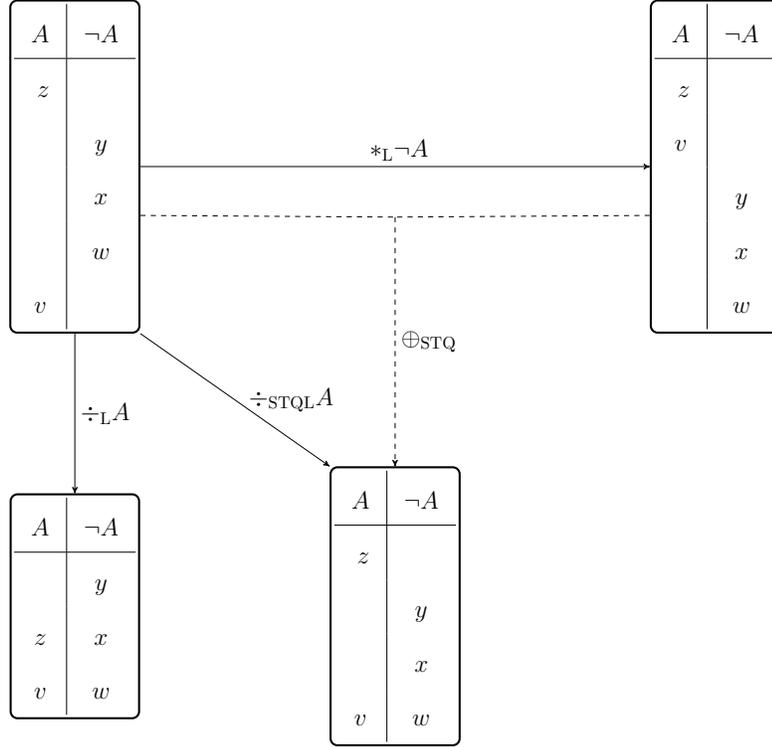
A	$\neg A$
z	
	y
	x
	w
v	

A	$\neg A$
z	
v	
	y
	x
	w

A	$\neg A$
	y
z	x
v	w

A	$\neg A$
z	
	y
	x
v	w

$*_{\mathrm{L}}\neg A$ $\oplus_{\mathrm{STQ}}$ $\div_{\mathrm{L}}A$ $\div_{\mathrm{STQL}}A$

Figure 4: Illustration of Example 4.

$\min(\preceq_\Psi, S) \subseteq \min(\preceq_{\Psi \div A}, S)$. However, as Example 5 demonstrates, this last condition is violated by priority contraction. Indeed, let $S = \{y, w\}$. Then $u \prec_{\Psi \div A} v$ for every $u \in S^c$, $v \in S$, $y \in \min(\preceq_\Psi, S)$ but $y \notin \min(\preceq_{\Psi \div A}, S)$.

Both lexicographic and priority contraction can, however, still be recovered via the TeamQueue approach, broadly construed. Lexicographic contraction can be recovered from lexicographic revision by combining, not $\preceq_\Psi$ and $\preceq_{\Psi *_L \neg A}$, but rather $\preceq_{\Psi *_L A}$ and $\preceq_{\Psi *_L \neg A}$ using $\oplus_{\mathrm{STQ}}$. Priority contraction can be recovered from lexicographic revision by combining $\preceq_\Psi$ and $\preceq_{\Psi * \neg A}$ using a TeamQueue combinator. However, the combinator involved is not $\oplus_{\mathrm{STQ}}$ but rather the
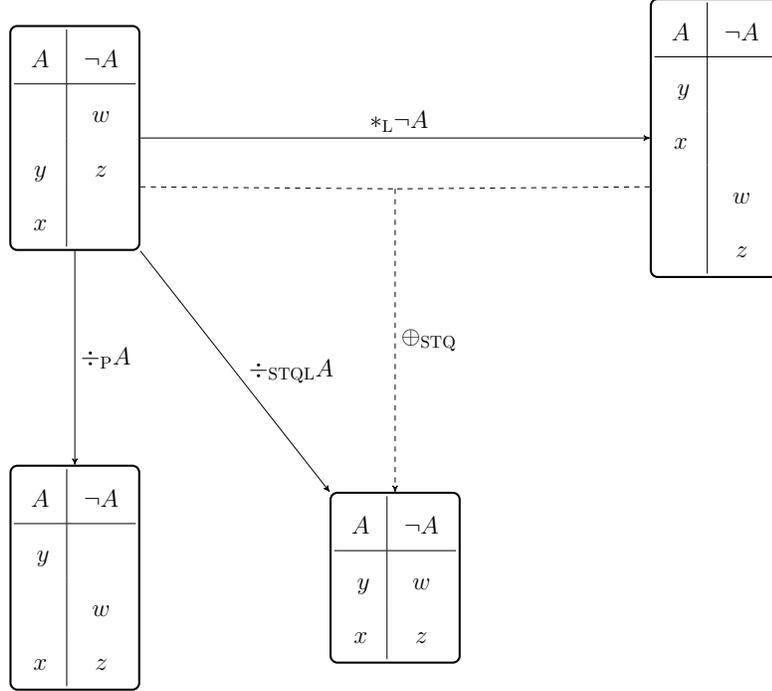
Figure 5: Illustration of Example 5.

TeamQueue combinator that is most 'biased' towards $\preceq_2$: the combinator $\oplus_2$ for which, for all ordered pairs $\langle \preceq_1, \preceq_2 \rangle$, $a_{\preceq_1, \preceq_2}(1) = \{1, 2\}$, then $a_{\preceq_1, \preceq_2}(j) = \{2\}$ for all $j > 1$. We then set: $\preceq_{\Psi \div A} = \preceq_\Psi$, if $A \in [\Psi]$ or A is a tautology, otherwise $\preceq_{\Psi \div A} = \preceq_\Psi \oplus_2 \preceq_{\Psi * \neg A}$.

Before closing this paper, we should briefly mention a property that is shared by both natural and restrained revision and which was introduced in [3] and then, independently, in [6].[15] This is the property known as (P*) or 'Independence', which strengthens both (C3*) and (C4*):

(P*)   If $\neg A \notin [\Psi * B]$, then $A \in [(\Psi * A) * B]$

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[15]We thank an anonymous referee for pressing us on this.

Its semantic counterpart is given by:

$(P_{\preceq}^{*})$ If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preccurlyeq_{\Psi} y$, then $x \prec_{\Psi * A} y$

One might wonder whether these principles have counterparts for contraction, in the sense of principles that are common and exclusive to those contraction operators that are definable, via a suitable extension of (HI) to the iterated case, from a revision operator that satisfies $(P^{*})/(P_{\preceq}^{*})$.

This question is most likely answerable in the negative. Indeed, if we assume that the appropriate extension of (HI) is given by the STQ method, then, as we have shown, both natural revision (which lacks property (P^{*})) and restrained revision (which has property (P^{*})) are mapped onto the same contraction operator, namely natural contraction.

Of course, if one does not commit to the STQ method, the question remains open. But whatever the fact of the matter may be in this alternative case, we note that the following properties—which some may be tempted to consider— will certainly *not* constitute respective counterparts for (P^{*}) and $(P_{\preceq}^{*})$:

$(P^{\div})$ If $A \notin [\Psi * B]$, then $\neg A \in [(\Psi \div A) * B]$

$(P_{\preceq}^{\div})$ If $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and $x \preccurlyeq_{\Psi} y$, then $x \prec_{\Psi \div A} y$

Indeed, if we set $B = \top$, $(P^{\div})$ gives us: If $A \notin [\Psi]$, then $\neg A \in [\Psi \div A]$. But this conflicts with the AGM postulate of Inclusion for contraction, i.e. $(K2^{\div})$, since, in a situation in which $A, \neg A \notin [\Psi]$, we wind up with $[\Psi \div A] \nsubseteq [\Psi]$.

6. Conclusions

We have shown that the issue of extending the Harper Identity to iterated belief change is not a straightforward affair but that it is one that can be fruitfully approached by combining a pair of total preorders by means of a TeamQueue combinator. In the course of the discussion, one particular combinator has

stood out as being of particular interest: the Synchronous TeamQueue combinator $\oplus_{\mathrm{STQ}}$. We have notably proven that the conditional belief set obtained by Synchronous TeamQueue combination corresponds to the rational closure of the set of conditional beliefs common to both inputs. As a number of authors have noted, the rational closure of a set of conditionals Γ corresponds to what is arguably the least opinionated conditional belief set that includes Γ. We have also shown that $\oplus_{\mathrm{STQ}}$ can be put to work to derive various counterparts for contraction of the three best known iterated revision operators.

One issue that we would like to explore in future work is the question of whether or not it is possible to show that the Darwiche-Pearl postulates are *equivalent* to the ones proposed by Chopra *et al*, given a suitable further bridge principle taking us from iterated *contraction* to iterated *revision*. Such a task would first involve providing a compelling generalisation of the Levi Identity mentioned in Proposition 1 above.

Also of interest would be the derivation of analogues for contraction of yet *further* principles of iterated revision, such as the ones recently identified in [40].

Acknowledgments

Appendix

Proposition 1. (NiHI) *entails*

$(\mathrm{NiHI}^{\dot{-}})$ $[(\Psi \div A) \div B] = [\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B]$

and is equivalent to it in the presence of $(\mathrm{K3}^{*})$ *and the Levi Identity:*

(LI) $[\Psi * A] = \mathrm{Cn}([\Psi \div \neg A] \cup \{A\}).$

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Proof:

(a) **From** (NiHI) **to** (NiHI$^{\div}$)**:** By (HI), which (NiHI) entails, $[(\Psi \div A) \div B] = [\Psi \div A] \cap [(\Psi \div A) * \neg B] = [\Psi] \cap [\Psi * \neg A] \cap [(\Psi \div A) * \neg B]$. Furthermore, by (NiHI), we have $[(\Psi \div A) * \neg B] = [\Psi * \neg B] \cap [(\Psi * \neg A) * \neg B]$. Hence $[(\Psi \div A) \div B] = [\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B]$, as required.

(b) **From** (NiHI$^{\div}$) **to** (NiHI)**:** By (LI), we have $[(\Psi \div A) * \neg B] = \mathrm{Cn}([(\Psi \div A) \div B] \cup \{\neg B\})$. By (NiHI$^{\div}$), we have $[(\Psi \div A) \div B] = [\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B]$. So to recover (NiHI), we need to show that $\mathrm{Cn}([\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B] \cup \{\neg B\}) = [\Psi * \neg B] \cap [(\Psi * \neg A) * \neg B]$. The left-to-right direction, i.e. $\mathrm{Cn}([\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B] \cup \{\neg B\}) \subseteq [\Psi * \neg B] \cap [(\Psi * \neg A) * \neg B]$, is immediate. Regarding the right-to-left, assume, for some arbitrary C, that $C \in [\Psi * \neg B] \cap [(\Psi * \neg A) * \neg B]$. Firstly, it follows by (K3*) and the deduction theorem that $\neg B \to C \in [\Psi]$ and $\neg B \to C \in [\Psi * \neg A]$. Secondly, it follows by deductive closure of belief sets that $\neg B \to C \in [\Psi * \neg B] \cap [(\Psi * \neg A) * \neg B]$. Therefore $\neg B \to C \in [\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B]$ and hence $C \in \mathrm{Cn}([\Psi] \cap [\Psi * \neg B] \cap [\Psi * \neg A] \cap [(\Psi * \neg A) * \neg B] \cup \{\neg B\})$, as required. $\qquad\square$

Proposition 2. *In the presence of* (K5*), (K6*) *and* (K3$^{\div}$), (HI) *and the left-to-right half of* (NiHI) *jointly entail the following "Restricted Right Euclidean" principle:*

(RRE)   *If A is consistent and $B \in [\Psi * A]$, then $[\Psi] \cap [\Psi * A] \subseteq [\Psi * B]$*

Proof: Assume (K5*), (K6*) and (K3$^{\div}$), (HI) and the left-to-right half of (NiHI). Assume that A is consistent and that $B \in [\Psi * A]$. Since A is consistent, so too is $[\Psi * A]$, by (K5*), and hence $\neg B \notin [\Psi * A]$. Since, by (HI), we

have $[\Psi \div \neg A] = [\Psi] \cap [\Psi * A]$ (with help from (K6*)), it follows that $\neg B \notin$ $[\Psi \div \neg A]$. Given (K3$\div$), we then have $[(\Psi \div \neg A) \div \neg B] = [\Psi \div \neg A]$, and, by (HI), $[(\Psi \div \neg A) \div \neg B] = [\Psi] \cap [\Psi * A]$.

By (HI), $[(\Psi \div \neg A) \div \neg B] = [\Psi \div \neg A] \cap [(\Psi \div \neg A) * B] \subseteq [(\Psi \div \neg A) * B]$. By the left-to-right half of (NiHI), we have $[(\Psi \div \neg A) * B] \subseteq [\Psi * B] \cap [(\Psi * \neg \neg A) * B] \subseteq [\Psi * B]$. Hence $[(\Psi \div \neg A) \div \neg B] \subseteq [\Psi * B]$.

From $[(\Psi \div \neg A) \div \neg B] \subseteq [\Psi * B]$ and $[(\Psi \div \neg A) \div \neg B] = [\Psi] \cap [\Psi * A]$, it then follows that $[\Psi] \cap [\Psi * A] \subseteq [\Psi * B]$, as required. $\qquad\square$

Proposition 3. (RRE) *entails that there does not exist a belief state Ψ such that: (i) $[\Psi] = \mathrm{Cn}(A \wedge B)$, (ii) $[\Psi * \neg A] = \mathrm{Cn}(\neg A \wedge B)$ and (iii) $[\Psi * A \leftrightarrow \neg B] = \mathrm{Cn}(A \leftrightarrow \neg B)$, where A and B are propositional atoms.*

Proof: Assume (RRE) and, for reductio, that there exists a belief set satisfying (i) to (iii). It follows from (ii) that $A \leftrightarrow \neg B \in [\Psi * \neg A]$. Given the latter, it then follows from (RRE) that $[\Psi] \cap [\Psi * \neg A] \subseteq [\Psi * A \leftrightarrow \neg B]$. But by (i) and (ii), $[\Psi] \cap [\Psi * \neg A] = \mathrm{Cn}(A \wedge B) \cap \mathrm{Cn}(\neg A \wedge B) = \mathrm{Cn}(B)$. Hence, by $[\Psi] \cap [\Psi * \neg A] \subseteq [\Psi * A \leftrightarrow \neg B]$, we have $B \in [\Psi * A \leftrightarrow \neg B]$. But (iii) tells us that $[\Psi * A \leftrightarrow \neg B] = \mathrm{Cn}(A \leftrightarrow \neg B)$. Contradiction. $\qquad\square$

Proposition 4. *In the presence of* (K1*), (K2*) *and* (K5*), (RRE) *entails that there do not exist a state Ψ and non-tautologous sentences A and B in L whose disjunction $A \vee B$ is tautologous and are such that $A, B \in [\Psi]$.*

Proof: Assume, for contradiction, that there exists a state Ψ and non-tautologous sentences A and B in L whose disjunction $A \vee B$ is tautologous and are such that $A, B \in [\Psi]$. By (K1*) and (K2*), $\neg A \vee \neg B \in [\Psi * \neg A]$. Since A is non-tautologous, and hence $\neg A$ is consistent, it follows by (RRE) that $[\Psi] \cap [\Psi * \neg A] \subseteq [\Psi * \neg A \vee \neg B]$. Since $A \vee B$ is tautologous, by (K1*) and (K2*), we have $B \in [\Psi * \neg A]$. Hence, since we have assumed $B \in [\Psi]$, it follows by the above inclusion that $B \in [\Psi * \neg A \vee \neg B]$. By the same reasoning (swapping the roles of A

and B), we obtain $A \in [\Psi * \neg A \vee \neg B]$. So we have $A, B \in [\Psi * \neg A \vee \neg B]$ but also, by (K2*), $\neg A \vee \neg B \in [\Psi * \neg A \vee \neg B]$. Hence $[\Psi * \neg A \vee \neg B]$ is inconsistent. Since both A and B have been assumed to be non-tautologous, $\neg A \vee \neg B$ is consistent. It then follows, by (K5*), that $[\Psi * \neg A \vee \neg B]$ is consistent. Contradiction. $\square$

Proposition 5. (UB$_\oplus$) *and* (LB$_\oplus$) *are respectively equivalent to the following:*

(SPU$_\oplus^+$)   *If $x \prec_1 y$ and $z \prec_2 y$ then either $x \prec_{1 \oplus 2} y$ or $z \prec_{1 \oplus 2} y$*

(WPU$_\oplus^+$)   *If $x \preceq_1 y$ and $z \preceq_2 y$ then either $x \preceq_{1 \oplus 2} y$ or $z \preceq_{1 \oplus 2} y$*

Proof:

(a) **From** (UB$_\oplus$) **to** (SPU$_\oplus^+$)**:** Suppose that $x \prec_1 y$ and $z \prec_2 y$. From the former, we know that $\min(\preceq_1, \{x, y, z\}) \subseteq \{x, z\}$ and from the latter we know that $\min(\preceq_2, \{x, y, z\}) \subseteq \{x, z\}$. Thus, by (UB$_\oplus$), $\min(\preceq_{1 \oplus 2}, \{x, y, z\}) \subseteq \{x, z\}$. From this, it must the case that $y \notin \min(\preceq_{1 \oplus 2}, \{x, y, z\})$, so either $x \prec_{1 \oplus 2} y$ or $z \prec_{1 \oplus 2} y$, as required.

(b) **From** (SPU$_\oplus^+$) **to** (UB$_\oplus$)**:** Assume for contradiction that there exists an x, such that $x \in \min(\preceq_{1 \oplus 2}, S)$ but $x \notin \min(\preceq_1, S) \cup \min(\preceq_2, S)$. From the latter, there exist $y, z \in S$, such that $y \prec_1 x$ and $z \prec_2 x$. By (SPU$_\oplus^+$), it then follows that either $y \prec_{1 \oplus 2} x$ or $z \prec_{1 \oplus 2} x$, contradicting $x \in \min(\preceq_{1 \oplus 2}, S)$. Thus, $\min(\preceq_{1 \oplus 2}, S) \subseteq \min(\preceq_1, S) \cup \min(\preceq_2, S)$, as required.

(c) **From** (LB$_\oplus$) **to** (WPU$_\oplus^+$)**:** We derive the contrapositive of (WPU$_\oplus^+$), namely:

   If $y \prec_{1 \oplus 2} x$ and $y \prec_{1 \oplus 2} z$, then $y \prec_1 x$ or $y \prec_2 z$

Assume then that $y \prec_{1 \oplus 2} x$ and $y \prec_{1 \oplus 2} z$. It follows from this that $\min(\preceq_{1 \oplus 2}, \{x, y, z\}) \subseteq \{y\}$. By (LB$_\oplus$), we then recover either (i) $\min(\preceq_1, \{x, y, z\}) \subseteq \{y\}$ or (ii) $\min(\preceq_2, \{x, y, z\}) \subseteq \{y\}$. Assume (i). It follows that $y \prec_1 x$. Assume (ii). It follows that $y \prec_2 z$. Hence, either $y \prec_1 x$ or $y \prec_2 z$, as required.

(d) From $(\text{WPU}^+_\oplus)$ **to** $(\text{LB}_\oplus)$**:** Assume for reductio that $(\text{LB}_\oplus)$ fails, so that there exist an x and a y such that $y \in \min(\preceq_1, S)$ and $z \in \min(\preceq_2, S)$, but $y, z \notin \min(\preceq_{1\oplus2}, S)$. From the latter, there exist an x_1 and x_2 such that $x_1, x_2 \in S$, $x_1 \prec_{1\oplus2} y$ and $x_2 \prec_{1\oplus2} z$. Since $\preceq_{1\oplus2}$ is a total preorder, we may assume that there exists an x such that $x \in S$, $x \prec_{1\oplus2} y$ and $x \prec_{1\oplus2} z$. By $(\text{WPU}^+_\oplus)$, we then have either $x \prec_1 y$ or $x \prec_2 z$, contradicting our assumption that $y \in \min(\preceq_1, S)$ and $z \in \min(\preceq_2, S)$. $\qquad\qquad\square$

Theorem 1. $\oplus$ *is a TeamQueue combinator iff it is a basic combinator that satisfies* $(\text{SPU}^+_\oplus)$, $(\text{WPU}^+_\oplus)$ *and the following 'no overtaking' property;*

$(\text{NO}_\oplus)$ *For $i \neq j$, if $x \prec_i y$ and $z \preceq_j y$, then either $x \prec_{1\oplus2} y$ or $z \preceq_{1\oplus2} y$*

Proof: We prove the following claim:

Corollary 2. $\oplus$ *satisfies* $(\text{SPU}^+_\oplus)$, $(\text{WPU}^+_\oplus)$ *and* $(\text{NO}_\oplus)$ *iff it satisfies*

$(\text{F}_\oplus)$ $\min(\preceq_{1\oplus2}, S)$ *is equal to either* $\min(\preceq_1, S)$, $\min(\preceq_2, S)$ *or* $\min(\preceq_1, S) \cup \min(\preceq_2, S)$

Given this, the desired result then follows from Proposition 6 below.

(a) From $(\text{SPU}^+_\oplus)$**,** $(\text{WPU}^+_\oplus)$ **and** $(\text{NO}_\oplus)$ **to** $(\text{F}_\oplus)$**:** We know that $\min(\preceq_{1\oplus2}, S) \subseteq \min(\preceq_1, S) \cup \min(\preceq_2, S)$ from $(\text{UB}_\oplus)$, which was shown to be equivalent to $(\text{SPU}^+_\oplus)$ in Proposition 5. Now if the converse holds, i.e. $\min(\preceq_1, S) \cup \min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$, then we have $\min(\preceq_{1\oplus2}, S) = \min(\preceq_1, S) \cup \min(\preceq_2, S)$ and we are done. So assume $\min(\preceq_1, S) \cup \min(\preceq_2, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$. Then either $\min(\preceq_1, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$ or $\min(\preceq_2, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$. Let's assume $\min(\preceq_1, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$. We will show that this implies $\min(\preceq_{1\oplus2}, S) = \min(\preceq_2, S)$, which will suffice. (If instead we assume $\min(\preceq_2, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$, then

the same reasoning will show $\min(\preceq_{1\oplus2}, S) = \min(\preceq_1, S)$, which also suffices.) Since $\min(\preceq_1, S) \not\subseteq \min(\preceq_{1\oplus2}, S)$, let $x \in \min(\preceq_1, S)$ but $x \notin \min(\preceq_{1\oplus2}, S)$.

We first derive $\min(\preceq_{1\oplus2}, S) \subseteq \min(\preceq_2, S)$. Let $y \in \min(\preceq_{1\oplus2}, S)$ and assume for reductio that $y \notin \min(\preceq_2, S)$. Then $\exists z \in S$ such that $z \prec_2 y$. From $y \in \min(\preceq_{1\oplus2}, S)$, we know that $y \preceq_{1\oplus2} z$. From $x \in \min(\preceq_1, S)$, we also know that $x \preceq_1 y$. From $z \prec_2 y$, $y \preceq_{1\oplus2} z$ and $x \preceq_1 y$, we can deduce by ($NO_\oplus$)that $x \preceq_{1\oplus2} y$, in contradiction with $x \notin \min(\preceq_{1\oplus2}, S)$. Hence, $y \in \min(\preceq_2, S)$, as required.

We now derive $\min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$. Let $y \in \min(\preceq_2, S)$ and assume, for reductio, that $y \notin \min(\preceq_{1\oplus2}, S)$. From $x, y \notin \min(\preceq_{1\oplus2}, S)$, $\exists z \in S$, such that $z \prec_{1\oplus2} x$ and $z \prec_{1\oplus2} y$. Then, from ($WPU_\oplus^+$), we have either $z \prec_1 x$ or $z \prec_2 y$. If $z \prec_1 x$, then we contradict $x \in \min(\preceq_1, S)$. If $z \prec_2 y$, then we contradict $y \in \min(\preceq_2, S)$. Either way, we obtaina contradiction, so $y \in \min(\preceq_{1\oplus2}, S)$, as required.

(b)(i) **From** ($F_\oplus$) **to** ($SPU_\oplus^+$)**:** From ($F_\oplus$), we know that ($UB_\oplus$) holds, i.e., $\forall S$, $\min(\preceq_{1\oplus2}, S) \subseteq \min(\preceq_1, S) \cup \min(\preceq_2, S)$. This is equivalent to ($SPU_\oplus^+$) by Proposition 5.

(b)(ii) **From** ($F_\oplus$) **to** ($WPU_\oplus^+$)**:** From ($F_\oplus$), we know that, $\forall S$, either $\min(\preceq_1, S) \subseteq \min(\preceq_{1\oplus2}, S)$ or $\min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$. This is the property ($LB_\oplus$) and we already proved in relation to Proposition 5 that it entails ($WPU_\oplus^+$).

(b)(iii) **From** ($F_\oplus$) **to** ($NO_\oplus$)**:** From ($F_\oplus$), we know that, $\forall S$, $i \neq j$, either $\min(\preceq_{1\oplus2}, S) \subseteq \min(\preceq_i, S)$ or $\min(\preceq_j, S) \subseteq \min(\preceq_{1\oplus2}, S)$. Now assume $x \prec_i y$, $y \preceq_{1\oplus2} x$, $z \preceq_j y$ and, for reductio, $y \prec_{1\oplus2} z$. From $y \preceq_{1\oplus2} x$ and $y \prec_{1\oplus2} z$, we obtain$y \in \min(\preceq_{1\oplus2}, \{x, y, z\})$ but from $x \prec_i y$, we obtain$y \notin \min(\preceq_i, \{x, y, z\})$. Hence $\min(\preceq_{1\oplus2}, \{x, y, z\}) \not\subseteq \min(\preceq_i, \{x, y, z\})$. From this and the property cited at the beginning of this paragraph, we obtain $\min(\preceq_j, \{x, y, z\}) \subseteq \min(\preceq_{1\oplus2}$

$, \{x, y, z\})$. We also know from $(F_\oplus)$ that $\min(\preceq_{1\oplus2}, \{x, y, z\}) \subseteq \min(\preceq_1, \{x, y, z\}) \cup \min(\preceq_2, \{x, y, z\})$. Hence, since $y \in \min(\preceq_{1\oplus2}, \{x, y, z\})$ and $y \notin \min(\preceq_i, \{x, y, z\})$, we obtain $y \in \min(\preceq_j, \{x, y, z\})$. Hence, since $z \preceq_j y$, $z \in \min(\preceq_j, \{x, y, z\})$ and so, from $\min(\preceq_j, \{x, y, z\}) \subseteq \min(\preceq_{1\oplus2}, \{x, y, z\})$, $z \in \min(\preceq_{1\oplus2}, \{x, y, z\})$, contradicting $y \prec_{1\oplus2} z$. Hence $z \preceq_{1\oplus2} y$, as required. $\square$

Proposition 6. $\oplus$ *is a TeamQueue combinator iff it is a basic combinator that satisfies the following 'factoring' property, for all $S \subseteq W$:*

$(F_\oplus)$ $\qquad \min(\preceq_{1\oplus2}, S)$ *is equal to either* $\min(\preceq_1, S)$, $\min(\preceq_2, S)$ *or* $\min(\preceq_1, S) \cup \min(\preceq_2, S)$

Proof:

(a) **Right-to-left direction:** Let $\oplus$ be any combinator that satisfies those properties. We must specify a sequence $\langle a_{\preceq_1, \preceq_2}(i) \rangle_{i \in \mathbb{N}}$ for each ordered pair $\langle \preceq_1, \preceq_2 \rangle$ such that (i) the sequence satisfies properties (a1) and (a2) and (ii) $\oplus_a = \oplus$.

Assume that $\langle S_1, S_2, \ldots, S_n \rangle$ represents $\preceq_{1\oplus2}$. Then we specify $a_{\preceq_1, \preceq_2}$ by setting, for all i,

$$j \in a_{\preceq_1, \preceq_2}(i) \text{ iff } \min(\preceq_j, \bigcap_{k<i} S_k^c) \subseteq S_i(= \min(\preceq_{1\oplus2}, \bigcap_{k<i} S_k^c))$$

Regarding (i), $a_{\preceq_1, \preceq_2}$ satisfies (a1) since $\oplus$ satisfies $(F_\oplus)$ and (a2) since $\oplus$ satisfies $(HI_\oplus)$.

Regarding (ii), let $\langle T_1, T_2, \ldots, T_m \rangle$ represent $\preceq_{1\oplus_a2}$. We prove by induction that $T_i = S_i$. Regarding $i = 1$: The result follows from $(HI_\oplus)$. Regarding the inductive step: Assume $T_j = S_j$, $\forall j < i$. We want to show $T_i = S_i$. By construction, $T_i = \bigcup_{j \in a_{\preceq_1, \preceq_2}(i)} \min(\preceq_j, \bigcap_{k<i} S_k^c)$. So we need to show $\min(\preceq_{1\oplus2}, \bigcap_{k<i} S_k^c) = \bigcup_{j \in a_{\preceq_1, \preceq_2}(i)} \min(\preceq_j, \bigcap_{k<i} S_k^c)$. This follows from $(F_\oplus)$.

(b) **Left-to-right direction:** Given Corollary 2, which we established in the course of the proof of Theorem 1 and which states that $(\mathrm{F}_\oplus)$ is equivalent to the conjunction of $(\mathrm{SPU}_\oplus^+)$, $(\mathrm{WPU}_\oplus^+)$ and $(\mathrm{NO}_\oplus)$, it suffices to show that $\oplus_a$ satisfies each of these three properties.

  (i) **Regarding** $(\mathrm{SPU}_\oplus^+)$**:** We prove the contrapositive. Suppose $y \preceq_{1\oplus 2} x$ and $y \preceq_{1\oplus 2} z$ and suppose $\langle S_1, S_2, \ldots, S_n \rangle$ represents $\preceq_{1\oplus 2}$. Assume $y \in S_i = \bigcup_{j \in a_{\preceq_1, \preceq_2}(i)} \min(\preceq_j, \bigcap_{k<i} S_k^c) \subseteq \min(\preceq_1, \bigcap_{k<i} S_k^c) \cup \min(\preceq_2, \bigcap_{k<i} S_k^c)$. Assume $y \in \min(\preceq_1, \bigcap_{k<i} S_k^c)$. Since $y \preceq_{1\oplus 2} x$, we know that $x \in \bigcap_{k<i} S_k^c$, hence $y \preceq_1 x$, as required. Similarly, if $y \in \min(\preceq_2, \bigcap_{k<i} S_k^c)$, then $y \preceq_2 z$.

  (ii) **Regarding** $(\mathrm{WPU}_\oplus^+)$**:** We prove the contrapositive. Suppose $y \prec_{1\oplus 2} x$ and $y \prec_{1\oplus 2} z$. Assume $y \in S_i$. Since $y \prec_{1\oplus 2} x$ and $y \prec_{1\oplus 2} z$, we know that $x, z \in \bigcap_{k<i} S_k^c \cap S_i^c$. Now, we know that $S_i \subseteq$ equals one of $\min(\preceq_1, \bigcap_{k<i} S_k^c)$, $\min(\preceq_2, \bigcap_{k<i} S_k^c)$ or $\min(\preceq_1, \bigcap_{k<i} S_k^c) \cup \min(\preceq_2, \bigcap_{k<i} S_k^c)$. We consider each case in turn:

    (1) **Case in which** $S_i = \min(\preceq_1, \bigcap_{k<i} S_k^c)$**:** From $y \in S_i$ and $x \in \bigcap_{k<i} S_k^c \cap S_i^c$, we have $y \prec_1 x$, as required.

    (2) **Case in which** $S_i = \min(\preceq_2, \bigcap_{k<i} S_k^c)$**:** From $y \in S_i$ and $z \in \bigcap_{k<i} S_k^c \cap S_i^c$, we have $y \prec_2 z$, as required.

    (3) **Case in which** $S_i = \min(\preceq_1, \bigcap_{k<i} S_k^c) \cup \min(\preceq_2, \bigcap_{k<i} S_k^c)$**:** Either we have $y \in \min(\preceq_1, \bigcap_{k<i} S_k^c)$, in which case $y \prec_1 x$, or $y \in \min(\preceq_2, \bigcap_{k<i} S_k^c)$, in which case $y \prec_2 z$.

  (iii) **Regarding** $(\mathrm{NO}_\oplus)$**:** We show: If $x \prec_i y$, $y \preceq_{1\oplus 2} x$ and $y \prec_{1\oplus 2} z$, then $y \prec_j z$, $i \neq j$, $i, j \in \{1, 2\}$. Assume $y \in S_t$. Then, from $y \preceq_{1\oplus 2} x$ and $y \prec_{1\oplus 2} z$, we have $x, z \in \bigcap_{k<t} S_t^c$ and furthermore $z \in S_t^c$. We know that S_t equals one of $\min(\preceq_1, \bigcap_{k<t} S_k^c)$, $\min(\preceq_2, \bigcap_{k<t} S_k^c)$ or $\min(\preceq_1, \bigcap_{k<t} S_k^c) \cup \min(\preceq_2, \bigcap_{k<t} S_k^c)$. From $x \prec_i y$, we know that $y \notin \min(\preceq_i, \bigcap_{k<t} S_k^c)$, hence we must have $y \in \min(\preceq_j, \bigcap_{k<t} S_k^c)$. Furthermore, we are left with either $S_t =$

$\min(\preceq_j, \bigcap_{k<t} S_k^c)$ or $S_t = \min(\preceq_1, \bigcap_{k<t} S_k^c) \cup \min(\preceq_2, \bigcap_{k<t} S_k^c)$.
In either case, since $z \in S_t^c$, we must have $y \prec_j z$, as required. □

Proposition 7. *Given* $(\mathrm{DOM}_\oplus)$, $\oplus$ *is a TeamQueue combinator iff it is a basic combinator that satisfies* $(\mathrm{SPU}_\oplus^+)$ *and* $(\mathrm{WPU}_\oplus^+)$.

Proof: We show that, given $(\mathrm{DOM}_\oplus)$, if $\oplus$ satisfies $(\mathrm{SPU}_\oplus^+)$ and $(\mathrm{WPU}_\oplus^+)$, then it satisfies $(\mathrm{NO}_\oplus)$ and hence, by Theorem 1 and Proposition 8, it is a TeamQueue combinator.

Suppose $x \prec_i y$, $y \preceq_{1\oplus2} x$ and $z \preceq_j y$, with $i \neq j$. We must show $z \preceq_{1\oplus2} y$. If we can show $z \preceq_i y$, then we can conclude $z \preceq_{1\oplus2} y$ from $(\mathrm{WPU}_\oplus)$. So suppose for reductio that $y \prec_i z$. From $(\mathrm{DOM}_\oplus)$, $\exists S$, such that, $\forall u, v \in S$, $u \preceq_1 v$ iff $u \preceq_2 v$ and $\forall u, v \in S^c$, $u \preceq_1 v$ iff $u \preceq_2 v$. From $z \preceq_j y$ and $y \prec_i z$, it must be the case that $y \in S$ and $z \in S^c$. If $x \in S$, then from $x \prec_i y$, we obtain $x \prec_j y$ and so $x \prec_{1\oplus2} y$ from $(\mathrm{SPU}_\oplus)$, contradicting $y \preceq_{1\oplus2} x$. If $x \in S^c$, then, since $x \prec_i y \prec_i z$ and $z \in S^c$, $x \prec_j z$. So from this and $z \preceq_j y$, we obtain $x \prec_j y$ and so again $x \prec_{1\oplus2} y$ from $(\mathrm{SPU}_\oplus)$, contradicting $y \preceq_{1\oplus2} x$. Hence, it must be that $z \preceq_i y$, as required. □

Proposition 8. *Given* $(\mathrm{DOM}_\oplus)$, $\oplus$ *satisfies* $(\mathrm{SPU}_\oplus^+)$ *and* $(\mathrm{WPU}_\oplus^+)$ *iff it satisfies* $(\mathrm{SPU}_\oplus)$ *and* $(\mathrm{WPU}_\oplus)$, *respectively.*

Proof: We prove this by demonstrating the equivalence, given $(\mathrm{DOM}_\oplus)$, of $(\mathrm{SPU}_\oplus)$ and $(\mathrm{WPU}_\oplus)$ with $(\mathrm{UB}_\oplus)$ and $(\mathrm{LB}_\oplus)$, respectively, which we have shown (see Proposition 5) to be equivalent to $(\mathrm{SPU}_\oplus^+)$ and $(\mathrm{WPU}_\oplus^+)$, respectively. Regarding $(\mathrm{SPU}_\oplus)$ and $(\mathrm{UB}_\oplus)$, our proof is direct. Regarding $(\mathrm{WPU}_\oplus)$ and $(\mathrm{LB}_\oplus)$, we first show that $(\mathrm{WPU}_\oplus)$ is equivalent to the following weakening of $(\mathrm{LB}_\oplus)$:

$(\mathrm{LB}_\oplus^-)$ $\quad \min(\preceq_1, S) \cap \min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$

before showing that $(\text{LB}_\oplus^-)$ is equivalent to $(\text{LB}_\oplus)$ under the domain restriction $(\text{DOM}_\oplus)$.

(a)(i) **From $(\text{UB}_\oplus)$ to $(\text{SPU}_\oplus)$:** The result follows from the fact that $x \preceq y$ iff $x \in \min(\preceq, \{x, y\})$.

(a)(ii) **From $(\text{SPU}_\oplus)$ to $(\text{UB}_\oplus)$:** We must show that $\min(\preceq_{1\oplus2}, S) \subseteq \min(\preceq_1, S) \cup \min(\preceq_2, S)$. Assume $(\text{DOM}_\oplus)$, $(\text{SPU}_\oplus)$ and that there exists an x, such that $x \in \min(\preceq_{1\oplus2}, S)$ but, for contradiction, that $x \notin \min(\preceq_1, S) \cup \min(\preceq_2, S)$. From the latter, there exist $y_1, y_2 \in S$, such that (i) $y_1 \prec_1 x$ and (ii) $y_2 \prec_2 x$. From the minimality of x, (iii) $x \preceq_{1\oplus2} y_1$ and (iv) $x \preceq_{1\oplus2} y_2$. From (i) and (iii) on the one hand and (ii) and (iv) on the other, by $(\text{SPU}_\oplus)$, we recover (v) $x \preceq_2 y_1$ and (vi) $x \preceq_1 y_2$, respectively. The conjunctions of (i) and (vi), i.e. $y_1 \prec_1 x \preceq_1 y_2$, and of (ii) and (v), i.e. $y_2 \prec_2 x \preceq_2 y_1$, however, jointly contradict $(\text{DOM}_\oplus)$, since the latter entails that there exist no x, y_1, y_2 such that $y_1 \prec_1 x \preceq_1 y_2$ but $y_2 \prec_2 x \preceq_2 y_1$. Hence $x \in \min(\preceq_1, S) \cup \min(\preceq_2, S)$, as required.

(b)(i) **From $(\text{WPU}_\oplus)$ to $(\text{LB}_\oplus^-)$:** Let $x \in \min(\preceq_1, S) \cap \min(\preceq_2, S)$ and assume for reductio that $x \notin \min(\preceq_{1\oplus2}, S)$. Then there exists $y \in S$ such that $y \prec_{1\oplus2} x$. By $(\text{WPU}_\oplus)$, either $y \prec_1 x$ or $y \prec_2 x$. Assume $y \prec_1 x$ (the other case is analogous). Then $x \notin \min(\preceq_1, S)$ and hence $x \notin \min(\preceq_1, S) \cap \min(\preceq_2, S)$. Contradiction. Hence, $x \in \min(\preceq_{1\oplus2}, S)$, as required.

(b)(ii) **From $(\text{LB}_\oplus^-)$ to $(\text{WPU}_\oplus)$:** Suppose $x \preceq_1 y$ and $x \preceq_2 y$. Then $x \in \min(\preceq_1, \{x, y\}) \cap \min(\preceq_2, \{x, y\})$. Assume for reductio that $y \prec_{1\oplus2} x$. Then $x \notin \min(\preceq_{1\oplus2}, \{x, y\})$, so, from $(\text{LB}_\oplus^-)$, $x \notin \min(\preceq_1, \{x, y\}) \cap \min(\preceq_2, \{x, y\})$. Contradiction. Hence $x \preceq_{1\oplus2} y$, as required.

(c)(i) **From $(\text{LB}_\oplus)$ to $(\text{LB}_\oplus^-)$:** Obvious.

(c)(ii) **From $(\text{LB}_\oplus^-)$ to $(\text{LB}_\oplus)$:** Assume that $(\text{LB}_\oplus)$ doesn't hold. Then there exists an S such that $\min(\preceq_1, S) \not\subseteq \min(\preceq_{1\oplus2}, S)$ and $\min(\preceq_2$

$, S) \nsubseteq \min(\preceq_{1\oplus2}, S)$. So there exist $x, y \in S$ such that $x \in \min(\preceq_1, S)$, $y \in \min(\preceq_2, S)$ and $x, y \notin \min(\preceq_{1\oplus2}, S)$. Hence there exists $z \in S$ such that $z \prec_{1\oplus2} x$ and $z \prec_{1\oplus2} y$. By (LB$_\oplus^-$), equivalently by (WPU$_\oplus$), we know from $z \prec_{1\oplus2} x$ that either $z \prec_1 x$ or $z \prec_2 x$. From this and the fact that $x \in \min(\preceq_1, S)$, we recover the result that $z \prec_2 x$. Similarly, we also recover $z \prec_1 y$. So we obtain the following pattern: $x \preceq_1 z \prec_1 y$ and $y \preceq_2 z \prec_2 x$. But this is not possible given (DOM$_\oplus$). Hence (LB$_\oplus$) holds, as required. $\qquad\square$

Theorem 2. $\oplus_{\mathrm{STQ}}$ *is the only basic combinator that satisfies both* (SPU$_\oplus^+$) *and the following 'Parity' constraint:*

(PAR$_\oplus$)   *If* $x \prec_{1\oplus2} y$ *then for each* $i \in \{1,2\}$ *there exists* z *such that*
$$x \sim_{1\oplus2} z \text{ and } z \prec_i y$$

Proof: We need to show that if $\oplus$ satisfies (SPU$_\oplus^+$) and (PAR$_\oplus$), for any $\preceq_1, \preceq_2$, we have $\preceq_{1\oplus2} = \preceq_{1\oplus_{\mathrm{STQ}}2}$. Assume that $\preceq_{1\oplus2}$ and $\preceq_{1\oplus_{\mathrm{STQ}}2}$ are represented by $\langle S_1, S_2, \ldots, S_m \rangle$ and $\langle T_1, T_2, \ldots, T_n \rangle$ respectively. We will prove, by induction on i, that $S_i = T_i$, $\forall i$. Assume $S_j = T_j$, $\forall j < i$. We must show $S_i = T_i$.

(a) **Regarding $S_i \subseteq T_i$:** Let $x \in S_i$, so that $x \preceq_{1\oplus2} y$, $\forall y \in \bigcap_{j<i} S_j^{\mathsf{c}}$. Assume for reductio that $x \notin T_i$. Since $x \in S_i$, we know that $x \in \bigcap_{j<i} S_j^{\mathsf{c}} = \bigcap_{j<i} T_j^{\mathsf{c}}$. Hence, since $x \notin T_i$ and, by construction of $\preceq_{1\oplus_{\mathrm{STQ}}2}$, there exists $y_1 \in \bigcap_{j<i} T_j^{\mathsf{c}}$ such that $y_1 \prec_1 x$ and there exists $y_2 \in \bigcap_{j<i} T_j^{\mathsf{c}}$ such that $y_2 \prec_2 x$. Then, by (SPU$_\oplus^+$), either $y_1 \prec_{1\oplus2} x$ or $y_2 \prec_{1\oplus2} x$, in both cases contradicting $x \preceq_{1\oplus2} y$, $\forall y \in \bigcap_{j<i} S_j^{\mathsf{c}}$. Hence $x \in T_i$, as required.

(ii) **Regarding $T_i \subseteq S_i$:** Let $x \in T_i$. Then, by construction of $\preceq_{1\oplus_{\mathrm{STQ}}2}$, we have $x \in \min(\preceq_1, \bigcap_{j<i} T_j^{\mathsf{c}}) \cup \min(\preceq_2, \bigcap_{j<i} T_j^{\mathsf{c}})$. Assume for reductio that $x \notin S_i$. We know that $x \in \bigcap_{j<i} T_j^{\mathsf{c}}$, so by the inductive hypothesis, $x \in \bigcap_{j<i} S_j^{\mathsf{c}}$. From this and $x \notin S_i$ we know that there exists a $y \in S_i$, such that $y \prec_{1\oplus2} x$. Then from (PAR$_\oplus$), there exist a $z_1 \in S_i$ such

that $z_1 \prec_1 x$ and a $z_2 \in S_i$ such that $z_2 \prec_2 x$. But this contradicts $x \in \min(\preceq_1, \bigcap_{j<i} T_j^c) \cup \min(\preceq_2, \bigcap_{j<i} T_j^c)$. Hence $x \in S_i$, as required.
□

Proposition 9. (PAR$_\oplus$) *is equivalent to:*

(SB$_\oplus$)   *If $x \prec_{1\oplus2} y$ for every $x \in S^c$, $y \in S$, then $\min(\preceq_1, S) \cup$*
         $\min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$

Proof:

(a) **From** (PAR$_\oplus$) **to** (SB$_\oplus$)**:** Assume that $x \prec_{1\oplus2} y$ for every $x \in S^c$, $y \in S$. We must show that $\min(\preceq_1, S) \cup \min(\preceq_2, S) \subseteq \min(\preceq_{1\oplus2}, S)$. So assume $x \in \min(\preceq_1, S) \cup \min(\preceq_2, S)$ but, for contradiction, $x \notin \min(\preceq_{1\oplus2}, S)$. Then $y \prec_{1\oplus2} x$ for some $y \in S$. From the latter, by (PAR$_\oplus$), we know that $z_1 \prec_1 x$ for some z_1 such that $y \preceq_{1\oplus2} z_1$ and $z_2 \prec_2 x$ for some z_2 such that $y \preceq_{1\oplus2} z_2$. Given our initial assumption, we can deduce from $y \preceq_{1\oplus2} z_1$, $y \preceq_{1\oplus2} z_2$ and $y \in S$ that $z_1, z_2 \in S$. But this, together with $z_1 \prec_1 x$ and $z_2 \prec_2 x$ contradicts $x \in \min(\preceq_1, S) \cup \min(\preceq_2, S)$. Hence $x \in \min(\preceq_{1\oplus2}, S)$, as required.

(ii) **From** (SB$_\oplus$) **to** (PAR$_\oplus$)**:** Suppose (PAR$_\oplus$) does not hold, i.e. $\exists x, y$, such that $x \prec_{1\oplus2} y$ and for no z do we have $x \sim_{1\oplus2} z$ and $z \prec_1 y$ (similar reasoning will apply if we replace $\prec_1$ by $\prec_2$ here). We will show that (SB$_\oplus$) fails, i.e. that $\exists S \subseteq W$, such that $x \prec_{1\oplus2} y$ for every $x \in S^c$, $y \in S$ and $\min(\preceq_1, S) \cup \min(\preceq_2, S) \not\subseteq \min(\preceq_{1\oplus2}, S)$.

Let $S = \{w \mid x \preceq_{1\oplus2} w\}$ (so that $S^c = \{w \mid w \prec_{1\oplus2} x\}$). Clearly $x \in S$ and, from $x \prec_{1\oplus2} y$, we know that $y \in S$ but $y \notin \min(\preceq_{1\oplus2}, S)$. Hence, to show $\min(\preceq_1, S) \cup \min(\preceq_2, S) \not\subseteq \min(\preceq_{1\oplus2}, S)$ and therefore that (SB$_\oplus$) fails, it suffices to show $y \in \min(\preceq_1, S)$. But if $y \notin \min(\preceq_1, S)$, then $z \prec_1 y$ for some $z \in S$, i.e. some z, such that $x \preceq_{1\oplus2} z$. Since $\preceq_{1\oplus2}$ is a TPO we may assume $x \sim_{1\oplus2} z$. This contradicts our initial

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assumption that for no z do we have $x \sim_{1 \oplus 2} z$ and $z \prec_1 y$. Hence $y \in \min(\preceq_1, S)$, as required. $\qquad\square$

Theorem 3. $\preceq_{1 \oplus_{\mathrm{STQ}} 2} \ \sqsupseteq \ \preceq_{1 \oplus 2}$, *for any TPOs $\preceq_1$ and $\preceq_2$, and any combinator $\oplus$ satisfying* $(\mathrm{SPU}_{\oplus}^{+})$.

Proof: Let $\langle T_1, \ldots, T_m \rangle$ be the ordered partition corresponding to $\preceq_{1 \oplus_{\mathrm{STQ}} 2}$. Let $\langle S_1, \ldots, S_n \rangle$ be the ordered partition corresponding to $\preceq_{1 \oplus 2}$. We must show that $\langle T_1, \ldots, T_m \rangle \sqsupseteq \langle S_1, \ldots, S_n \rangle$.

If $T_i = S_i$ for all i, then we are done. So let i be minimal such that $T_i \neq S_i$. We must show $S_i \subset T_i$. So let $y \in S_i$ and assume, for contradiction, that $y \notin T_i$. We know that $T_i \neq \varnothing$, since, otherwise, $\bigcup_{j<i} T_j = W$, hence $\bigcup_{j<i} S_j = W$ and so $S_i = \varnothing$, contradicting $S_i \neq T_i$. So let $x \in T_i$. Then $x \prec_{1 \oplus_{\mathrm{STQ}} 2} y$. So, by $(\mathrm{PAR}_{\oplus})$, (i) $\exists z_1$ such that $z_1 \sim_{1 \oplus_{\mathrm{STQ}} 2} x$ (i.e. $z_1 \in T_i$) and $z_1 \prec_1 y$ and (ii) $\exists z_2$ such that $z_2 \sim_{1 \oplus_{\mathrm{STQ}} 2} x$ (i.e. $z_2 \in T_i$) and $z_2 \prec_2 y$. Since $\oplus$ satisfies $(\mathrm{SPU}_{\oplus}^{+})$, it follows that either $z_1 \prec_{1 \oplus 2} y$ or $z_2 \prec_{1 \oplus 2} y$. Either way, $\exists z \in T_i$ such that $z \prec_{1 \oplus 2} y$. But if $z \prec_{1 \oplus 2} y$, then, since $y \in S_i$, $z \in \bigcup_{j<i} S_j = \bigcup_{j<i} T_j$, contradicting $z \in T_i$. Hence $y \in T_i$ and so $S_i \subset T_i$, as required. $\qquad\square$

Proposition 10. *Let $\oplus$ be an arbitrary TeamQueue combinator, let $*$ be an AGM revision operator and let $\div$ be such that $\preceq_{\Psi \div A}$ is defined from $*$ via* (Combi) *using $\oplus$. Then, for each $i = 1, 2, 3, 4$, if $*$ satisfies* $(\mathrm{C}i_{\preceq}^{*})$ *then $\div$ satisfies* $(\mathrm{C}i_{\preceq}^{\div})$.

Proof:

(a) **From** $(\mathrm{C}1_{\preceq}^{*})$ **to** $(\mathrm{C}1_{\preceq}^{\div})$**:** Let $x, y \in [\![\neg A]\!]$. We must show that $x \preceq_{\Psi \div A} y$ iff $x \preceq_{\Psi} y$. Note that from $(\mathrm{C}1_{\preceq}^{*})$, we have (1) $x \preceq_{\Psi * \neg A} y$ iff $x \preceq_{\Psi} y$. Regarding the left-to-right direction of the equivalence: Assume (2) $y \prec_{\Psi} x$. From (1) and (2), we recover (3) $y \prec_{\Psi * \neg A} x$. From (2) and (3), by $(\mathrm{SPU}_{\oplus})$, it follows that $y \prec_{\Psi \div A} x$, as required. Regarding

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the right-to-left-direction: Assume (4) $x \preceq_{\Psi} y$. From (1) and (4), we recover (5) $x \preceq_{\Psi * \neg A} y$. From (4) and (5), by (WPU$_{\oplus}$), it follows that $x \preceq_{\Psi \div A} y$, as required.

(b) From (C2$_{\preceq}^{*}$) **to** (C2$_{\preceq}^{\div}$)**:** Similar proof to the one given in (a).

(c) From (C3$_{\preceq}^{*}$) **to** (C3$_{\preceq}^{\div}$)**:** Let $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and (1) $x \prec_{\Psi} y$. We must show that $x \prec_{\Psi \div A} y$. From (C3$_{\preceq}^{*}$), we recover (2) $x \prec_{\Psi * \neg A} y$. From (1) and (2), by (SPU$_{\oplus}$), we then obtain $x \prec_{\Psi \div A} y$, as required.

(d) From (C4$_{\preceq}^{*}$) **to** (C4$_{\preceq}^{\div}$)**:** Let $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and (1) $x \preceq_{\Psi} y$. We must show that $x \preceq_{\Psi \div A} y$. From (C4$_{\preceq}^{*}$), we recover (2) $x \preceq_{\Psi * \neg A} y$. From (1) and (2), by (WPU$_{\oplus}$), we then obtain $x \preceq_{\Psi \div A} y$, as required. $\square$

Proposition 11. *Let $\oplus$ be a TeamQueue combinator, let $*$ be an AGM revision operator and let $\div$ be such that $\preceq_{\Psi \div A}$ is defined from $*$ via* (Combi) *using $\oplus$. If $*$ satisfies* (C1$_{\preceq}^{*}$) *and* (C2$_{\preceq}^{*}$) *then $\div$ satisfies* (PFI).

Proof: Assume that $*$ satisfies (C1$_{\preceq}^{*}$) and (C2$_{\preceq}^{*}$) and let $\div$ be the contraction operator defined from $*$ using a TeamQueue combinator. We saw above, in Proposition 10 that $\div$ will also satisfy (C1$_{\preceq}^{\div}$) and (C2$_{\preceq}^{\div}$). The desired result then immediately follows from the theorem established by Ramachandran *et al* (2011, Theorem 1), according to which every contraction function $\div$ obtained from a revision function $*$, such that $\div$ and $*$ satisfy HI, satisfies (PFI) if it also satisfies (C1$_{\preceq}^{\div}$) and (C2$_{\preceq}^{\div}$). $\square$

Proposition 12. *Let $*$ be any revision operator—such as the natural or restrained revision operator—satisfying* (C1$_{\preceq}^{*}$), (C2$_{\preceq}^{*}$), (C4$_{\preceq}^{*}$) *and the following property:*

If $x, y \notin \min(\preceq_{\Psi}, [\![A]\!])$ and $x \prec_{\Psi} y$, then $x \prec_{\Psi * A} y$

Let $\div$ be the contraction operator defined from $*$ via (Combi) *using* $\oplus_{\text{STQ}}$. *Then $\div$ is the natural contraction operator.*

Proof: Recall the definition of natural contraction:

$(\text{Def}\div_N)$ $x \preceq_{\Psi \div_N A} y$ iff

     (a) $x \in \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$, or

     (b) $x, y \notin \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$ and $x \preceq_\Psi y$

We must show that for any $x, y \in W$ and $A \in L$, $x \preceq_{\Psi \div A} y$ iff $x \preceq_{\Psi \div_N A} y$. We split into two cases.

(a) **Case in which $x \in \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$:** Then, by the definitions of $\div_N$ and $\div$, we have both $x \preceq_{\Psi \div A} y$ and $x \preceq_{\Psi \div_N A} y$, so the desired result holds.

(b) **Case in which $x \notin \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$:** Then by definition of $\div_N$, $x \preceq_{\Psi \div_N A} y$ iff both $y \notin \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$ and $x \preceq_\Psi y$. We now consider each direction of the equivalence to be demonstrated separately.

     (i) **From $x \preceq_{\Psi \div_N A} y$ to $x \preceq_{\Psi \div A} y$:** Suppose $x \preceq_{\Psi \div_N A} y$, and hence that both $y \notin \min(\preceq_\Psi, [\![\neg A]\!]) \cup \min(\preceq_\Psi, W)$ and $x \preceq_\Psi y$. Assume for reductio that $y \prec_{\Psi \div A} x$. By $(\text{PAR}_\oplus)$: if $y \prec_{\Psi \div A} x$, then there exists z such that $z \sim_{\Psi \div A} y$ and $z \prec_\Psi x$. Since $x \preceq_\Psi y$, we therefore also have $z \prec_\Psi y$. If $z \notin \min(\preceq_\Psi, [\![\neg A]\!])$, then from the postulate mentioned in the proposition, we obtain $z \prec_{\Psi * \neg A} y$ and then $z \prec_{\Psi \div A} y$ by $(\text{SPU}_\oplus)$. Contradiction. Hence we can assume $z \in \min(\preceq_\Psi, [\![\neg A]\!])$. From $x \preceq_\Psi y$, $y \prec_{\Psi \div A} x$ and $(\text{WPU}_\oplus)$, we know that $y \prec_{\Psi * \neg A} x$. From this, $(\text{C2}^*_{\preceq})$, $(\text{C4}^*_{\preceq})$ and $x \preceq_\Psi y$, we obtain $y \in [\![\neg A]\!]$. Hence, from $z \prec_\Psi y$ and $(\text{C1}^*_{\preceq})$, we recover $z \prec_{\Psi * \neg A} y$ and then $z \prec_{\Psi \div A} y$ by $(\text{SPU}_\oplus)$. Contradiction again. Hence $x \preceq_{\Psi \div A} y$, as required.

(ii) From $x \preceq_{\Psi \div A} y$ **to** $x \preceq_{\Psi \div_N A} y$**:** Assume that $x \preceq_{\Psi \div A} y$ and, for reductio, that $x \notin \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$ and either $y \prec_\Psi x$ or $y \in \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$. If the latter holds, then we know that $y \in \min(\preceq_{\Psi \div A}, W)$, by definition of $\div$. Hence, from this and $x \preceq_{\Psi \div A} y$, we also deduce that $x \in \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$, contradicting the assumption that $x \notin \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$. So assume that $y \notin \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$ and $y \prec_\Psi x$. From the latter and our assumption that $x \preceq_{\Psi \div A} y$, it follows by $(\mathrm{SPU}_\oplus)$ that $x \preceq_{\Psi * \neg A} y$. But it also follows from $y \notin \min(\preceq_\Psi, \llbracket \neg A \rrbracket) \cup \min(\preceq_\Psi, W)$ and $y \prec_\Psi x$ that $x, y \notin \min(\preceq_\Psi, \llbracket \neg A \rrbracket)$. We then recover, from the property mentioned in the proposition, the result that $x \preceq_\Psi y$, contradicting our assumption that $y \prec_\Psi x$. Hence, $x \preceq_{\Psi \div_N A} y$, as required. $\qquad\square$

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