Subfactors and unitary R-matrices

Gandalf Lechner

November 11, 2019

Abstract

This is an extended abstract from a talk at the Oberwolfach workshop “Subfactors and Applications” in October 2019. It summarizes some results from [2] (joint work with Roberto Conti) and [5, 4].

The Yang-Baxter equation is a cubic equation for a linear map \( R \in V \otimes V \to V \otimes V \) on the tensor square of a vector space \( V \), namely

\[(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R), \quad (\text{YBE})\]

where 1 is the identity on \( V \). This equation and its variants come from quantum physics, but also play a central role in various branches of mathematics, for instance in knot theory, quantum groups/Hopf algebras, and braid groups. Further recent interest in the solutions of the YBE stems from topological quantum computing [6].

Despite this widespread interest in the YBE, no satisfactory understanding of its solutions has been reached. In this talk, a new approach to the YBE was presented, based on operator algebras and subfactors [2]. We restrict to the case of most interest in applications, namely the case where \( V \) is a finite-dimensional Hilbert space and \( R \) is unitary. Such “R-matrices” exist in any dimension \( d = \dim V \), simple examples being the identity 1 on \( V \otimes V \), the tensor flip \( F(v \otimes w) = w \otimes v \), diagonal R-matrices, and Gaussian R-matrices. The (unknown) set of all R-matrices of dimension \( d \) is denoted \( R(d) \).

The general strategy of our approach is to start from an arbitrary R-matrix \( R \in R(d) \) with base space \( V \) and derive operator-algebraic data (such as endomorphisms, subfactors, indices) from it that inform us about \( R \). The main structural elements of our approach can be summarized in the following diagram:

\[
\begin{align*}
\varphi(N) & \subset \mathcal{N} \\
\cup & \cup \\
\varphi(\mathcal{L}_R) & \subset \mathcal{L}_R \\
\cap & \cap \\
\lambda_R(N) & \subset \mathcal{N}
\end{align*}
\]

Starting at the top of the diagram, \( N \) is the hyperfinite II\(_1\) factor realised as an infinite tensor product \( N = \bigotimes_{n \geq 1} \text{End}V \), weakly closed w.r.t. the normalised trace \( \tau = \bigotimes_{n \geq 1} \frac{\text{Tr}_V}{d} \),

\*Cardiff University, School of Mathematics, Cardiff, CF24 4AG, UK. E-mail: LechnerG@Cardiff.ac.uk
and equipped with the shift \( \varphi : \mathcal{N} \to \mathcal{N}, \varphi(x) = 1 \otimes x \). We identify finite tensor powers \( \text{End} V^\otimes n \) with their natural embeddings into \( \mathcal{N} \), so that \( R \in \mathcal{N} \) and the YBE reads \( \varphi(R) R \varphi(R) = R \varphi(R) R \).

The second line of the diagram is about the braid group structure: As is well known, any \( R \in \mathcal{R}(d) \) defines a group homomorphism \( \rho_R \) from the infinite braid group \( B_{\infty} \) into the unitary group of \( \mathcal{N} \) by mapping the standard generators \( b_n, n \in \mathbb{N} \), of \( B_{\infty} \) to \( \varphi^{n-1}(R) \in \mathcal{N} \). The von Neumann algebra generated by this representation is denoted \( \mathcal{L}_R \).

The third line of the diagram introduces the Yang-Baxter endomorphism \( \lambda_R \in \text{End} \mathcal{N} \). It is defined in such a way that it restricts to the shift \( \varphi \) on \( \mathcal{N} \). Explicitly,

\[
\lambda_R : \mathcal{N} \to \mathcal{N}, \quad \lambda_R(x) := w\lim_{n \to \infty} R \cdots \varphi^n(R) x \varphi^n(R^*) \cdots R^*.
\]

This definition is natural also from the point of view of the Cuntz algebra\(^1\). As particular examples, we note that the identity \( R \)-matrix gives the identity endomorphism, \( \lambda_1 = \text{id}_\mathcal{N} \), and the flip \( F \) gives the canonical endomorphism, \( \lambda_F = \varphi \).

Let us list a few results from [2] (joint work with Roberto Conti):

1. \( \mathcal{L}_R \) is a factor (II\(_1\) for non-trivial \( R \)). This provides us with three subfactors (I) \( \lambda_R(\mathcal{N}) \subset \mathcal{N} \), (II) \( \varphi(\mathcal{L}_R) \subset \mathcal{L}_R \), and (III) \( \mathcal{L}_R \subset \mathcal{N} \) derived from \( R \).

2. Subfactors (I),(II) have always finite index \( \leq d^2 \), but (III) may have infinite index. Its relative commutant coincides with the fixed point algebra \( \mathcal{N}^{\lambda_R} \).

3. The subfactors (I), (II) can be iterated by taking powers of \( \lambda_R \) and \( \varphi \), respectively. One has \( R \in \varphi^2(\mathcal{L}_R)' \cap \mathcal{L}_R \subset \lambda_R^2(\mathcal{N})' \cap \mathcal{N} \). Hence, for any non-trivial \( R \)-matrix, \( \lambda_R^2 \) is reducible and \( \lambda_R \) is not an automorphism [1].

4. Both squares in (**) are commuting squares. Denoting the \( \tau \)-preserving conditional expectation \( \mathcal{N} \to \lambda_R(\mathcal{N}) \) by \( E_R \), and the associated left inverse of \( \lambda_R \) by \( \phi_R := \lambda_R^{-1} \circ E_R \), this implies \( \phi_R(x) = \phi_F(x) \), \( x \in \mathcal{L}_R \).

An interesting object to consider is \( \phi_R(R) \). This is an element of \( \varphi(\mathcal{L}_R)' \cap \mathcal{L}_R \), which thanks to (4) coincides with the (normalised) left partial trace \( \phi_F(R) \) of \( R \). We therefore have explicit elements of the relative commutant, and a connection from operator-algebraic structures to concrete properties of \( R \). One finds [2]:

5. Let \( R \in \mathcal{R} \). Then the left and right partial traces of \( R \) coincide and are normal elements of \( \text{End} V \).

6. Define the character \( \tau_R \) of an \( R \)-matrix as the map \( \tau_R : B_{\infty} \to \mathbb{C}, \tau_R := \tau \circ \rho_R \). If two \( R \)-matrices \( R, S \in \mathcal{R}(d) \) have the same character, then \( \phi_R(R) \) and \( \phi_S(S) \) are unitarily equivalent.

7. Any \( R \)-matrix with spectrum contained in a disc of radius less than \( 1 - 2^{-1/4} \) is trivial\(^2\).

---

1 Viewing \( R \in \mathcal{R}(d) \) as a unitary in \( \mathcal{O}_d \) yields a canonically associated endomorphism \( \lambda_R \) of \( \mathcal{O}_d \). This endomorphism gives (**) by extension to a type III\(_1/d\) factor \( \mathcal{M} \supset \mathcal{N} \) and restriction.

2 This result has its origin in an estimate on the Jones index \([\mathcal{N} : \lambda_R(\mathcal{N})]\) in terms of \( \phi_R(R) \).
Item (6) suggests to consider R-matrices up to the natural equivalence relation $R \sim S$ given by coinciding characters and dimensions of R-matrices. Then $\phi_R(R)$ is an invariant for $\sim$, and in the involutive case ($R^2 = 1$), it is even a complete invariant: $R \sim S \iff \phi_R(R) \equiv \phi_S(S)$ [4]. In the general non-involutive case, the partial trace is not a complete invariant.

As the last section in this overview, let us consider the problem of classifying all R-matrices up to the equivalence $\sim$ and announce some results from the upcoming article [5]. We consider here the case that the spectrum of $R$ has cardinality 2, and normalise it to $\sigma(R) = \{-1, q\}$, $|q| = 1$, $q \neq -1$. In this situation, the representation $\rho_R$ factors through the Hecke algebra $H_\infty(q)$, and we moreover have [5]:

(8) If $R \in \mathcal{R}(d)$ has no two opposite eigenvalues $\mu, -\mu$ in its spectrum, then $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is irreducible and $\tau_R$ is a (positive) Markov trace.

Hence for $q \neq 1$, any R-matrix gives a positive Markov trace on $H_\infty(q)$. We may therefore use Wenzl’s classification of positive Markov traces on $H_\infty(q)$ [7]. Recall that his results state in particular that for a positive Markov trace to exist, one must have $q \in \{1, e^{2\pi i / \ell} : \ell \in \{4, 5, \ldots\}\}$, and at fixed $\ell$, there exist finitely many possible Markov traces. In our Yang-Baxter setting, these possibilities are severely restricted [5]:

(9) Let $R$ be an R-matrix with spectrum $\{-1, q\}$, $q \neq 1$, and eigen projection $P$ for the eigenvalue $-1$. Then $q \in \{\pm i, e^{i\pi/3}\}$. If $q = \pm i$, then $\tau(P) = \frac{1}{2}$, and if $q = e^{\pm i\pi/3}$, then $\tau(P) = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. Two such R-matrices, $R, S$ are equivalent (in the sense of $\sim$) if they have the same spectrum $(q)$, dimension $(d)$, and trace $(\tau(P))$.

The above result does not imply that all the possible combinations of eigenvalues $q$ and traces $\tau(P)$ are indeed realised. We have found explicit R-matrices realising the combinations $(q = \pm i, \tau(P) = \frac{1}{2})$, $(q = e^{i\pi/3}, \tau(P) = \frac{1}{3})$, $(q = e^{i\pi/3}, \tau(P) = \frac{2}{3})$ and conjecture that the last possibility, $(q = e^{i\pi/3}, \tau(P) = \frac{1}{2})$, is not realised by any R-matrix. This is in line with observations made by Galindo, Hong, and Rowell [3], but so far no proof of this conjecture exists.

It is instructive to compare these findings with the situation at $q = 1$, which is completely different. For $q \neq 1$, we always have irreducible $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, and the equivalence takes a simple form (it is given by the three parameters $d, q, \tau(P)$). For $q = 1$, on the other hand, $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is reducible except for the special cases $R \sim \pm 1, \pm F$, and the equivalence is more involved (it is given by the unitary equivalence class of $\phi_R(R)$). The case $q = 1$ corresponds to $R$ being involutive and $\rho_R$ factoring through the infinite symmetric group. In that case, a complete and explicit classification of R-matrices up to equivalence exists: R-matrices are parameterised by pairs of Young diagrams with $d$ boxes in total, corresponding to the positive and negative eigenvalues of $\phi_R(R)$ [4]. We also mention that in this case, the index $[\mathcal{L}_R : \varphi(\mathcal{L}_R)]$ is a rational typically non-integer number.
References


