

Lieb-Robinson bounds, Arveson spectrum and Haag-Ruelle scattering theory for gapped quantum spin systems

Sven Bachmann¹, Wojciech Dybalski², and Pieter Naaijkens³

¹*Mathematisches Institut der Universität München, Theresienstrasse 39, D-80333 München, Germany,
E-mail: sven.bachmann@math.lmu.de*

²*Zentrum Mathematik, Technische Universität München, D-85747 Garching, Germany,
E-mail: dybalski@ma.tum.de*

³*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstrasse 2, D-30167 Hannover,
Germany, E-mail: pieter.naijkens@itp.uni-hannover.de*

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Abstract

We consider translation invariant gapped quantum spin systems satisfying the Lieb-Robinson bound and containing single-particle states in a ground state representation. Following the Haag-Ruelle approach from relativistic quantum field theory, we construct states describing collisions of several particles, and define the corresponding S -matrix. We also obtain some general restrictions on the shape of the energy-momentum spectrum. For the purpose of our analysis we adapt the concepts of almost local observables and energy-momentum transfer (or Arveson spectrum) from relativistic QFT to the lattice setting. The Lieb-Robinson bound, which is the crucial substitute of strict locality from relativistic QFT, underlies all our constructions. Our results hold, in particular, in the Ising model in strong transverse magnetic fields.

1 Introduction

Since many important physical experiments involve collision processes, our understanding of physics relies to a large extent on scattering theory. The multitude of possible experimental situations is reflected by a host of theoretical descriptions of scattering processes, see for example [64]. One theoretical approach, due to Haag and Ruelle [34, 68], proved to be particularly convenient in the context of quantum systems with infinitely many degrees of freedom. Originally developed in axiomatic relativistic Quantum Field Theories (QFT), Haag-Ruelle (HR) scattering theory was adapted to various non-relativistic QFT and models of Quantum Statistical Mechanics. However, the most appealing aspect of HR theory, which is the conceptual clarity of its assumptions, typically gets lost in a non-relativistic setting: some authors use physically less transparent Euclidean or stochastic assumptions [9, 49], other resort to model-dependent computations [2]. In this paper we show that there is an exception to this rule: For a class of gapped quantum spin systems satisfying the Lieb-Robinson bound and admitting single-particle states in their ground-state representations, HR scattering theory can be developed in a natural, model-independent manner parallel to its original relativistic version.

To explain the content of this paper it is convenient to adopt a general framework which encompasses both local relativistic theories and spin systems: Let Γ be the abelian group of space translations with $\Gamma = \mathbb{R}^d$ in the relativistic case and $\Gamma = \mathbb{Z}^d$ for lattice theories. We denote by $\widehat{\Gamma}$

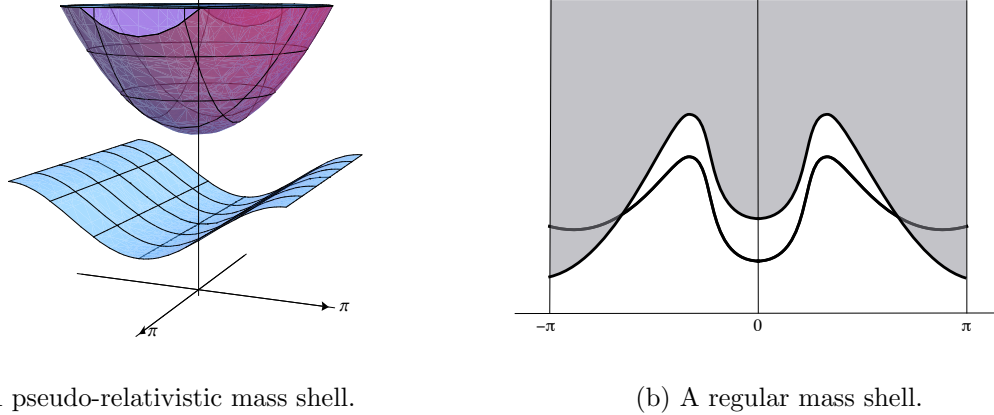


Figure 1: Two examples of the energy-momentum spectra within the scope of this paper: (1a) The lower blue surface is a pseudo-relativistic mass shell, since its variations are much smaller than the lower mass gap. This mass shell is not regular as it has a flat direction. (1b) The solid line, dipping into the shaded region, is a regular mass shell, since its second derivative vanishes at most at isolated points. This mass shell is not pseudo-relativistic, since it has large variations compared to the lower mass gap. On both figures $(E, p) = (0, 0)$ is an isolated eigenvalue. (Cf. Subsection 4.1 for precise definitions).

the Pontryagin dual of Γ , see [66], which in the latter case is the torus S_1^d and is sometimes called the Brillouin zone. We consider a C^* -dynamical system (τ, \mathfrak{A}) , where τ is a representation of the group of space-time translations $\mathbb{R} \times \Gamma$ in the automorphism group of a unital C^* -algebra \mathfrak{A} . We fix a τ -invariant ground state ω on \mathfrak{A} (or a vacuum state in the relativistic terminology) and proceed to its Gelfand-Naimark-Segal (GNS) representation $(\pi, \mathcal{H}, \Omega)$. This allows us to represent the observables as bounded operators on a Hilbert space \mathcal{H} , such that the ground state ω is given by the vector state Ω , see e.g. [35, Sect. III.2] for the construction. Let U be a unitary representation of $\mathbb{R} \times \Gamma$ implementing τ in π . The Stone-Naimark-Ambrose-Godement (SNAG) theorem [66] gives a spectral measure from which U can be recovered. We call the support of this measure its spectrum and denote it by $\text{Sp } U$. In more physical terms it is nothing but the energy-momentum spectrum and its elements will be denoted (E, p) . We always have $(E, p) = (0, 0) \in \text{Sp } U$, since this is the eigenvalue corresponding to the eigenvector Ω . We will always assume that this eigenvalue is isolated from the rest of the spectrum.

We say that π contains single-particle states, if there is an isolated mass shell $\mathfrak{h} \subset \text{Sp } U$. \mathfrak{h} should be the graph of a smooth function $p \mapsto \Sigma(p)$, defined on an open subset of $\widehat{\Gamma}$, whose Hessian matrix is almost everywhere non-zero. As Σ is the dispersion relation of the particle, the latter condition says that the group velocity $p \mapsto \nabla \Sigma(p)$ is constant at most on sets of measure zero. If also the determinant of the Hessian matrix is almost everywhere non-zero, that is $p \mapsto \nabla \Sigma(p)$ is invertible almost everywhere, then we say that the mass shell is *regular*. If the difference of any two distinct vectors on the mass shell is outside of the energy-momentum spectrum, then we say that the mass shell is *pseudo-relativistic*. We note that in relativistic theories, where the mass shells are given by the hyperboloids $\Sigma(p) = \sqrt{p^2 + m^2}$, both properties are trivially satisfied. In the context of spin systems, whose mass shells can a priori have arbitrary shape, we will assume that at least one of the two conditions holds. It is clear from Figure 1 that we can cover a wide range of different spectral situations. We refer to Subsection 4.1 for precise definitions and further discussion of regular and pseudo-relativistic mass shells, and to Section 6 for a proof that such mass shells actually appear in the concrete example of the Ising model in strong transverse magnetic fields in various space

dimensions.

Single-particle states in \mathcal{H} are elements of the spectral subspace of the mass shell \mathfrak{h} introduced above. To obtain states in \mathcal{H} describing several particles one needs to ‘multiply’ such single-particle states. This is performed following a prescription familiar from the Fock space: one identifies creation operators of individual particles and acts with their product on the vacuum. In our context ‘creation operators’ are elements of \mathfrak{A} , well localized in space and with definite energy-momentum transfer. Thus acting on the ground state Ω they create single-particle states with prescribed localization in phase-space. We recall that the energy-momentum transfer (or the Arveson spectrum) of $A \in \mathfrak{A}$, denoted $\text{Sp}_A \tau$, is defined as the smallest subset of $\mathbb{R} \times \widehat{\Gamma}$ s.t. the expression

$$\tau_f(A) := (2\pi)^{-\frac{d+1}{2}} \int dt d\mu(x) \tau_{(t,x)}(A) f(t, x), \quad f \in L^1(\mathbb{R} \times \Gamma), \quad (1.1)$$

vanishes for any f whose Fourier transform is supported outside of this set. (Here $d\mu$ is the Haar measure of Γ). We note that for $\Gamma = \mathbb{R}^d$ the Arveson spectrum of A is simply the support of the (inverse) Fourier transform of the distribution $(t, x) \mapsto \tau_{(t,x)}(A)$. The utility of this concept derives from the energy-momentum transfer relation

$$\pi(A_1)P(\Delta)\mathcal{H} \subset P(\overline{\Delta + \text{Sp}_{A_1}\tau})\mathcal{H}, \quad (1.2)$$

where P is the spectral measure of U defined on Borel subsets $\Delta \subset \text{Sp}U$. In particular, setting $\Delta = \{(0, 0)\} \in \mathbb{R} \times \widehat{\Gamma}$ and choosing A_1 s.t. $\text{Sp}_{A_1}\tau$ is a small neighbourhood of a point $(\Sigma(p_1), p_1)$ on the mass shell, we obtain a candidate for a creation operator of a particle whose energy and momentum are close to this point. Such operators with their energy-momentum transfer in a prescribed set are easy to obtain by setting $A_1 = \tau_f(A)$ and making use of the fact that $\text{Sp}_{\tau_f(A)}\tau \subset \text{supp } \widehat{f}$. However, to ensure localization of the created particle in space we need more structure: A minimal requirement seems to be the existence of a norm-dense *-subalgebra of *almost local* observables $\mathfrak{A}_{\text{a-loc}} \subset \mathfrak{A}$ s.t. for any $A, A_1, A_2 \in \mathfrak{A}_{\text{a-loc}}$

$$\tau_{(t,x)}(A) \in \mathfrak{A}_{\text{a-loc}}, \quad (t, x) \in \mathbb{R} \times \Gamma, \quad (1.3)$$

$$\tau_f(A) \in \mathfrak{A}_{\text{a-loc}}, \quad f \in S(\mathbb{R} \times \Gamma), \quad (1.4)$$

$$[\tau_{x_1}(A_1), \tau_{x_2}(A_2)] = O(\langle x_1 - x_2 \rangle^{-\infty}), \quad (1.5)$$

where $S(\mathbb{R} \times \Gamma)$ are Schwartz class functions and in (1.5) a rapid decay of the norm of the commutator is meant, cf. Appendix D. Postponing further discussion of $\mathfrak{A}_{\text{a-loc}}$ to a later part of this introduction, we note that with the above input the HR construction goes through in the usual way: In view of (1.4), we can set $B_1^* := \pi(A_1)$, where A_1 has its energy-momentum transfer near the point (E_1, p_1) of the mass shell and in addition is almost local. Next, we pick a positive-energy wave packet

$$g_{1,t}(x) = (2\pi)^{-\frac{d}{2}} \int_{\widehat{\Gamma}} dp e^{-i\Sigma(p)t + ipx} \widehat{g}_1(p), \quad \widehat{g}_1 \in C^\infty(\widehat{\Gamma}), \quad (1.6)$$

describing the free evolution of the particle in question. (Here dp is the Haar measure on $\widehat{\Gamma}$ and \widehat{g}_1 is supported near p_1). The HR creation operator, given by

$$B_{1,t}^*(g_{1,t}) := (2\pi)^{-\frac{d}{2}} \int d\mu(x) B_{1,t}^*(x) g_{1,t}(x), \quad B_{1,t}^*(x) = U(t, x) B_1^* U(t, x)^*, \quad (1.7)$$

creates from Ω a time-independent single-particle state Ψ_1 . Given a collection Ψ_1, \dots, Ψ_n of such single-particle states, corresponding to distinct points $(\Sigma(p_1), p_1), \dots, (\Sigma(p_n), p_n)$ on the mass shell,

s.t. also the velocities $\nabla\Sigma(p_1), \dots, \nabla\Sigma(p_n)$ are distinct¹, the n -particle scattering state is constructed as follows

$$\Psi_1^{\text{out}} \times \dots \times \Psi_n^{\text{out}} := \lim_{t \rightarrow \infty} B_{1,t}^*(g_{1,t}) \cdots B_{n,t}^*(g_{n,t})\Omega, \quad (1.8)$$

where the existence of the limit follows via Cook's method (see Theorem XI.4 of [64] or the proof of Theorem 4.8 below) and property (1.5). One obtains in addition that the scattering states span a subspace in \mathcal{H} naturally isomorphic to the symmetric Fock space, and thus they can be interpreted as configurations of independent bosons. This latter step requires that the mass shell is regular or pseudo-relativistic, and that multiples of Ω are the only translation invariant vectors in \mathcal{H} . In the language of the physical spins, these states correspond to collective, strongly interacting excitations above the ground state, which become asymptotically independent from each other for large times. As we shall see, mass shells of particles with fermionic or anyonic statistics do not appear in the energy-momentum spectrum of a translation invariant ground state (or a vacuum state). Nevertheless, variants of HR theory for fermions and anyons have been developed in the relativistic setting. These would also be valid for lattice field theories such as lattice fermions. In Section 7 we discuss prospects of adapting these constructions to spin systems on a lattice, where fermions or anyons arise only as collective quasi-particle excitations. The existence of anyonic excitations in such systems is believed to be generically associated to topologically ordered ground states (cf. [30, 44]).

Let us now come back to the requirement that \mathfrak{A} should contain a norm-dense $*$ -algebra $\mathfrak{A}_{\text{a-loc}}$ of almost local observables, whose elements satisfy properties (1.3)–(1.5). In relativistic theories this requirement is met as follows: Recall that in this context \mathfrak{A} is the C^* -inductive limit of a net of observables $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ labelled by open bounded regions \mathcal{O} of *space-time* \mathbb{R}^{d+1} . Due to the covariance property

$$\tau_{(t,x)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + (t, x)) \quad (1.9)$$

the $*$ -algebra of all local observables $\mathfrak{A}_{\text{loc}}$ is invariant under the action of τ and thus satisfies (1.3). Locality of the net, which says that observables localized in spacelike-separated regions commute, implies (1.5). However, property (1.4) cannot be expected, unless f is compactly supported. To ensure (1.4), one proceeds to a slightly larger $*$ -algebra consisting of $A \in \mathfrak{A}$ s.t.

$$A - A_r = O(r^{-\infty}), \text{ for some } A_r \in \mathfrak{A}(\mathcal{O}_r), \text{ where } \mathcal{O}_r := \{ (t, x) \mid |x| + |t| < r \}. \quad (1.10)$$

That is, A can be approximated in norm by observables localized in double-cones, centered at zero, of radius r , up to an error vanishing faster than any inverse power of r . It is easy to see that this algebra of almost local observables $\mathfrak{A}_{\text{a-loc}}$, introduced for the first time in [4], satisfies (1.3)–(1.5). In particular, invariance under τ follows from the invariance of $\mathfrak{A}_{\text{loc}}$ and isotony of the net.

In spin systems the existence of a $*$ -algebra $\mathfrak{A}_{\text{a-loc}}$ of almost local observables, satisfying (1.3)–(1.5), is less obvious. We recall that here the algebra \mathfrak{A} is the C^* -inductive limit of a net of local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda)$ labelled by bounded subsets of *space* (more precisely finite subsets of the lattice $\Gamma = \mathbb{Z}^d$). As these time-zero algebras are covariant only under space translations, the $*$ -algebra $\mathfrak{A}_{\text{loc}}$ of all local observables is usually not invariant under time evolution². Thus the algebra of almost local observables $\mathfrak{A}_{\text{a-loc}}$, defined by replacing \mathcal{O}_r with a ball of radius r in \mathbb{Z}^d in (1.10), has a priori no reason to be time invariant. However, and this is essential in our paper, using the Lieb-Robinson bound one can show that it actually *is* invariant. This bound can be schematically stated as follows

$$\|[\tau_t(A), B]\| \leq C_{A,B} e^{\lambda(v_{\text{LR}}t - d(A,B))}, \quad A, B \in \mathfrak{A}_{\text{loc}}, \quad (1.11)$$

¹In the relativistic case for the dimension of space $d \geq 3$ the velocities of particles do not have to be distinct [35].

²Except for special cases such as for Hamiltonians consisting of commuting interactions of bounded range.

where $d(A, B)$ is the distance between the localization regions of A and B , $v_{\text{LR}} > 0$ is called the Lieb-Robinson velocity and $\lambda > 0$ is a constant. With the help of this estimate we show in Theorem 3.10 that $\mathfrak{A}_{\text{a-loc}}$ satisfies the crucial properties (1.3), (1.4). Property (1.5) follows directly from the definition of $\mathfrak{A}_{\text{a-loc}}$. The Lieb-Robinson bound provides us via these results a crucial tool to sufficiently localise single-particle excitations. This opens the route to HR scattering theory for spin systems as described above, where particularly in the construction of multi-particle states, the localisation of single-particle states is essential.

Apart from its relevance to scattering theory, almost locality combined with Arveson spectrum is a powerful technique which appears in many contexts in relativistic QFT (see e.g. [18]). To demonstrate its flexibility we include several results, partially or completely independent of scattering theory, which concern the shape of $\text{Sp}U$: First, we show that velocity of a particle, defined as the gradient of its dispersion relation $p \mapsto \nabla\Sigma(p)$, is always bounded by the Lieb-Robinson velocity v_{LR} appearing in (1.11). Second, we verify additivity of the spectrum: If $\tilde{p}_1, \tilde{p}_2 \in \text{Sp}U$ then also $\tilde{p}_1 + \tilde{p}_2 \in \text{Sp}U$, where the addition is understood in $\mathbb{R} \times \hat{\Gamma}$. This result is a counterpart of a well known fact from the relativistic setting [35]. Third, we check that gaps in the spectrum of the finite-volume Hamiltonians cannot close in the thermodynamic limit. We hope that the concepts of almost locality and Arveson spectrum will find further interesting applications in the setting of spin systems.

Having summarized the content of this paper, we proceed now to a more systematic comparison of our work with the literature: Definition and properties (1.3), (1.4), (1.5) of the algebra of almost local observables for spin systems are in fact not new. Nearing the completion of this work, Klaus Fredenhagen pointed out to us the Diplom thesis of Schmitz [70], which contains such results. Also the bound $\nabla\Sigma(p) \leq v_{\text{LR}}$ is proven in [70]. Yet, since our proofs of these facts are different than Schmitz', and also the reference [70] is not easily accessible, we decided to keep the complete discussion of these results in our paper. We also stress that the overlap between our work and Schmitz' does not go beyond the technical facts mentioned above. In particular, there is no discussion of scattering theory in [70].

Let us now turn to the works which do concern scattering theory for lattice systems: In the special case of the ferromagnetic Heisenberg model it is possible to develop scattering theory adapting arguments from many body quantum mechanics [33, 40]. For certain perturbations of non-interacting gapped lattice models collision theory was established by Yarotsky, exploiting the special form of the finite volume Hamiltonians [76]. Malyshev discusses particle excitations in the Ising model in external fields (and more generally, in so-called Markov random fields), implementing HR ideas [50]. Barata and Fredenhagen have developed HR scattering theory for Euclidean lattice field theories [9] on a $d+1$ -dimensional lattice. Clustering estimates play an important role in the last three references, and in fact they also appear in early proofs of the original HR theorem. In our analysis we avoid clustering estimates altogether (in the case of pseudo-relativistic mass shells) or derive them from the Lieb-Robinson bound via [57] (in the case of regular mass shells).

Let us now discuss briefly more recent literature, centered around the Lieb-Robinson bound: In [36] certain operators were constructed which create from the ground state single-particle states of a given momentum, up to controllable errors. Although technically quite different, this finite volume result inspired the present investigation as it leaves little doubt that the Lieb-Robinson bound is a sufficient input to develop HR theory for spin systems containing mass shells. Our counterpart of the main result from [36] is Lemma 4.15 below. We further note that [36] gives a theoretical justification that a matrix product state ansatz describes single-particle states efficiently, see [73] for more details.

Our paper is built up as follows. In Section 2 we introduce the standard concepts and tools of the theory of quantum spin systems. In Section 3, which still has a preliminary character, we

introduce two concepts which may be less well known to experts in quantum spin systems: almost locality and the Arveson spectrum. (A more general perspective on this latter concept is given in Appendix A). Section 4 is the heart of the paper: there we develop the Haag-Ruelle scattering theory for lattice systems. In Section 5 we include some results on shape of the spectrum obtained with the methods of this paper. Examples of concrete systems satisfying our assumptions are given in Section 6. Finally, in Section 7, we comment on possible future directions. Our conventions and some more technical proofs are relegated to the appendices.

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2 Framework and preliminaries

2.1 Quasilocal algebra \mathfrak{A} and space translations

Let $\Gamma = \mathbb{Z}^d$ be the set labelling the sites of the system, which we equip with a metric $|\cdot|$. We define by $\mathcal{P}(\Gamma)$ the set of all subsets and by $\mathcal{P}_{\text{fin}}(\Gamma)$ the set of all finite subsets of Γ . Also, for $\Lambda, \Lambda_1, \Lambda_2 \in \mathcal{P}_{\text{fin}}(\Gamma)$ we set

$$\text{diam}(\Lambda) := \sup\{|x_1 - x_2| \mid x_1, x_2 \in \Lambda\}, \quad (2.1)$$

$$\text{dist}(\Lambda_1, \Lambda_2) := \inf\{|x_1 - x_2| \mid x_1 \in \Lambda_1, x_2 \in \Lambda_2\}, \quad (2.2)$$

$$\Lambda^r := \{x \in \Gamma \mid \text{dist}(x, \Lambda) \leq r\}, \quad (2.3)$$

and $|\Lambda| := \sum_{x \in \Lambda} 1$ is the volume of Λ . We denote by $Z = \{0\}$ the set consisting of one point at the origin, so that Z^r is the ball of radius r centered at the origin.

We assume that at each site $x \in \Gamma$ there is an ℓ -dimensional quantum spin whose observables are elements of $M_\ell(\mathbb{C})$, the set of complex $\ell \times \ell$ matrices, where ℓ is independent of x . For $\Lambda \in \mathcal{P}_{\text{fin}}(\Gamma)$ we set

$$\mathfrak{A}(\Lambda) := \bigotimes_{x \in \Lambda} M_\ell(\mathbb{C}). \quad (2.4)$$

For $\Lambda_1, \Lambda_2 \in \mathcal{P}_{\text{fin}}(\Gamma)$ such that $\Lambda_1 \subset \Lambda_2$, we have the natural embedding $\mathfrak{A}(\Lambda_1) \subset \mathfrak{A}(\Lambda_2)$ given by identifying $A \in \mathfrak{A}(\Lambda_1)$ with $A \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathfrak{A}(\Lambda_2)$. Thus we obtain a net $\Lambda \mapsto \mathfrak{A}(\Lambda)$ which is local in the sense that for $\Lambda_1, \Lambda_2 \in \mathcal{P}_{\text{fin}}(\Gamma)$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ we have

$$[\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = 0. \quad (2.5)$$

We define

$$\mathfrak{A}_{\text{loc}} := \bigcup_{\Lambda \in \mathcal{P}_{\text{fin}}(\Gamma)} \mathfrak{A}(\Lambda), \quad (2.6)$$

and the quasilocal algebra \mathfrak{A} as a C^* -completion of $\mathfrak{A}_{\text{loc}}$.

We denote by $\Gamma \ni x \mapsto \tau_x$ the natural representation of the group of translations from Γ in automorphisms of \mathfrak{A} .

2.2 Interactions

We define an interaction as a map $\Phi : \mathcal{P}_{\text{fin}}(\Gamma) \rightarrow \mathfrak{A}$, s.t. $\Phi(\Lambda)$ is self-adjoint and $\Phi(\Lambda) \in \mathfrak{A}(\Lambda)$. The interaction is said to be translation invariant if

$$\Phi(\Lambda + x) = \tau_x(\Phi(\Lambda)). \quad (2.7)$$

We also introduce a family of local Hamiltonians: For any $\Lambda \in \mathcal{P}_{\text{fin}}(\Gamma)$ we set

$$H_\Lambda := \sum_{X \subset \Lambda} \Phi(X). \quad (2.8)$$

H_Λ induce the local dynamics $\{\tau_t^\Lambda\}_{t \in \mathbb{R}}$ given by

$$\tau_t^\Lambda(A) := e^{iH_\Lambda t} A e^{-iH_\Lambda t}, \quad A \in \mathfrak{A}. \quad (2.9)$$

As stated in Corollary 2.2 below, the global dynamics $\tau_t := \lim_{\Lambda \nearrow \Gamma} \tau_t^\Lambda$ can be defined for a large class of interactions which we now specify.

Let $F : [0, \infty) \rightarrow (0, \infty)$ be a non-increasing function such that

$$\|F\| := \sum_{x \in \Gamma} F(|x|) < \infty, \quad (2.10)$$

$$C := \sup_{x, y \in \Gamma} \sum_{z \in \Gamma} \frac{F(|z-x|)F(|y-z|)}{F(|y-x|)} < \infty. \quad (2.11)$$

For $\Gamma = \mathbb{Z}^d$ one can choose $F(x) = (1+x)^{-d-\varepsilon}$ for any $\varepsilon > 0$ and $C \leq 2^{d+\varepsilon+1} \sum_{x \in \Gamma} (1+|x|)^{d+\varepsilon}$, see [56]. Then, for any $\lambda > 0$, $F_\lambda(r) := \exp(-\lambda r)F(r)$ satisfies the same conditions with $\|F_\lambda\| \leq \|F\|$ and $C_\lambda \leq C$. The formula

$$\|\Phi\|_\lambda := \sup_{x \in \Gamma} \sum_{X \ni x, 0} \frac{\|\Phi(X)\|}{F_\lambda(|x|)} \quad (2.12)$$

defines a norm on the set of translation invariant interactions. We let \mathcal{B}_λ be the set of interactions such that $\|\Phi\|_\lambda < \infty$.

2.3 Lieb-Robinson bounds and the existence of dynamics

The fast decay of interactions from \mathcal{B}_λ implies that the associated dynamics satisfies the Lieb-Robinson bounds [47, 56] stated in Theorem 2.1 below. Beyond its physically natural interpretation as a finite maximal velocity of propagation of correlations through the system, the Lieb-Robinson bounds have many useful structural corollaries such as the existence of the dynamics in the infinite system. This article will further emphasize that they provide an analogue of the speed of light in the framework of quantum spin systems (cf. Corollary 5.2). Another instance of that analogy can already be found in the exponential clustering property of [38, 57] stated in Theorem 2.4 below.

For interactions $\Phi \in \mathcal{B}_\lambda$ we have the following Lieb-Robinson bounds [47, 56]:

Theorem 2.1. *Let $\Lambda_1, \Lambda_2, \Lambda$ be finite subsets of Γ such that $\Lambda_1, \Lambda_2 \subset \Lambda$, and let $A \in \mathfrak{A}(\Lambda_1)$, $B \in \mathfrak{A}(\Lambda_2)$. Moreover, assume that there is $\lambda > 0$ such that $\Phi \in \mathcal{B}_\lambda$. Then*

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_\lambda} e^{2\|\Phi\|_\lambda C_\lambda |t|} \sum_{w \in \Lambda_1} \sum_{z \in \Lambda_2} F_\lambda(|w-z|) \quad (2.13)$$

for all $t \in \mathbb{R}$ and uniformly in Λ . Thus defining the Lieb-Robinson velocity as $v_\lambda := \frac{2\|\Phi\|_\lambda C_\lambda}{\lambda}$ we have

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_\lambda} \min(|\Lambda_1|, |\Lambda_2|) \|F\| e^{-\lambda(\text{dist}(\Lambda_1, \Lambda_2) - v_\lambda |t|)}. \quad (2.14)$$

It is well-known [56, 67] that the Lieb-Robinson bound allows for the extension of the dynamics from the local to the quasi-local algebra by a direct proof of the Cauchy property of the sequence $\tau_t^{\Lambda_n}(A)$ for any increasing and absorbing sequence of finite subsets $\Lambda_n \subset \Gamma$.

Corollary 2.2. *Assume that there is $\lambda > 0$ such that $\Phi \in \mathcal{B}_\lambda$. Then there exists a strongly continuous one parameter group of automorphisms of \mathfrak{A} , $\{\tau_t\}_{t \in \mathbb{R}}$, such that for all $A \in \mathfrak{A}_{\text{loc}}$,*

$$\lim_{n \rightarrow \infty} \|\tau_t^{\Lambda_n}(A) - \tau_t(A)\| = 0,$$

*independently of the choice of the absorbing and increasing sequence Λ_n . Moreover, there exists a *-derivation δ such that $\tau_t = e^{t\delta}$.*

2.4 Ground states and clustering

We start with a definition of a ground state of a C^* -dynamical system (\mathfrak{A}, τ) :

Definition 2.3. *Let ω be a state on (\mathfrak{A}, τ) s.t. $\omega \circ \tau_t = \omega$ for all t . Let $(\pi, \mathcal{H}, \Omega)$ be the corresponding GNS representation and $t \mapsto U(t)$ the unitary representation of time translations implementing $t \mapsto \tau_t$ in π , which satisfies $U(t)\Omega = \Omega$ for all t . Let H be the generator of U (the Hamiltonian) i.e. $U(t) = e^{itH}$.*

1. *We say that ω is a ground state and π a ground state representation if H is positive.*
2. *We say that π has a lower mass gap if $\{0\}$ is an isolated eigenvalue of H .*
3. *We say that π has a unique ground state vector if $\{0\}$ is a simple eigenvalue of H (corresponding up to phase to the eigenvector Ω).*
4. *We say that ω is translation invariant if $\omega \circ \tau_x = \omega$ for all $x \in \Gamma$.*

We note here that our definition of a ground state is in fact equivalent to the more algebraic one given by the inequality

$$-i\omega(A^*\delta(A)) \geq 0, \quad A \in D(\delta). \quad (2.15)$$

See [12, Prop. 5.3.19].

Finally, we recall that the approximate locality provided by the Lieb-Robinson bound yields exponential clustering in the ground state for Hamiltonians with a lower mass gap. The statement below follows from Theorem 4.1 of [58]. We will use it in the proof of Theorem 4.9.

Theorem 2.4. *Let ω be a ground state whose GNS representation has a lower mass gap and a unique ground state vector. Then, under the assumptions of Theorem 2.1, for any local observables $A \in \mathfrak{A}(\Lambda_1)$, $B \in \mathfrak{A}(\Lambda_2)$, s.t. $\text{dist}(\Lambda_1, \Lambda_2) \geq 1$ we have*

$$|\omega(AB) - \omega(A)\omega(B)| \leq C\|A\|\|B\| \min(|\Lambda_1|, |\Lambda_2|)e^{-\mu \text{dist}(\Lambda_1, \Lambda_2)}, \quad (2.16)$$

where $C, \mu > 0$ are independent of $A, B, \Lambda_1, \Lambda_2$ within the above restrictions.

In the remainder of this paper we will always consider a lattice system with a translation invariant interaction $\Phi \in \mathcal{B}_\lambda$, $\lambda > 0$, so that Lieb-Robinson bounds apply, and a translation invariant ground state with a GNS representation $(\pi, \mathcal{H}, \Omega)$.

3 Arveson spectrum and almost local observables

In this section, which is still preparatory, we introduce the concepts of Arveson spectrum and almost local observables. In Section 4 we will show their utility in the context of scattering theory. The Arveson spectrum is a topic in the spectral analysis of groups of automorphisms. A more extensive account can be found in [60].

3.1 Space-time translations

We claim that for any $x \in \Gamma$, $t \in \mathbb{R}$ and $A \in \mathfrak{A}$,

$$\tau_t \circ \tau_x(A) = \tau_x \circ \tau_t(A). \quad (3.1)$$

Since τ is an action of Γ this is equivalent to $\tau_{-x} \circ \tau_t \circ \tau_x(A) = \tau_t(A)$ for all $A \in \mathfrak{A}$. By Corollary 2.2 and because τ_x is an automorphism we have

$$\tau_{-x} \circ \tau_t \circ \tau_x(A) = \lim_{n \rightarrow \infty} \tau_{-x} \circ \tau_t^{\Lambda_n} \circ \tau_x(A) = \lim_{n \rightarrow \infty} \tau_t^{\Lambda_n - x}(A), \quad (3.2)$$

for $A \in \mathfrak{A}_{\text{loc}}$, where in the last step the translation invariance of the interaction (2.7) is used, and the limits denote convergence in norm. Since the right hand side converges irrespective of the choice of exhausting sequence, and $\mathfrak{A}_{\text{loc}}$ is norm-dense in \mathfrak{A} , the claim follows. Hence, we can consistently define

$$\mathbb{R} \times \Gamma \ni (t, x) \mapsto \tau_{(t,x)} := \tau_t \circ \tau_x \quad (3.3)$$

as a strongly-continuous representation of the group of space-time translations in automorphisms of \mathfrak{A} .

Remark 3.1. *For the sake of clarity, we will sometimes write $\tau^{(1)}$, $\tau^{(d)}$, τ to distinguish the respective groups of automorphisms:*

$$\mathbb{R} \ni t \mapsto \tau_t, \quad \Gamma \ni x \mapsto \tau_x, \quad \mathbb{R} \times \Gamma \ni (t, x) \mapsto \tau_{(t,x)}. \quad (3.4)$$

In the following lemma we construct the unitary representation of space-time translations U in the GNS representation of a translation invariant ground state ω .

Lemma 3.2. *Suppose that the interaction is translation invariant, see (2.7), and ω is a translation invariant ground state. Let $\Gamma \ni x \mapsto U(x)$ be a unitary representation of translations implementing τ_x in π , which satisfies $U(x)\Omega = \Omega$, and let $U(t)$ be as in Definition 2.3. Then we can consistently define a unitary representation of space-time translations*

$$\mathbb{R} \times \Gamma \ni (t, x) \mapsto U(t, x) = U(t)U(x) = U(x)U(t). \quad (3.5)$$

This representation implements $(t, x) \mapsto \tau_{(t,x)}$ in π and satisfies $U(t, x)\Omega = \Omega$ for all $(t, x) \in \mathbb{R} \times \Gamma$.

Proof. We note that for $A, B \in \mathfrak{A}_{\text{loc}}$, for all $x \in \Gamma$ and $t \in \mathbb{R}$,

$$\begin{aligned} \langle \pi(A)\Omega, U(x)U(t)\pi(B)\Omega \rangle &= \langle \pi(A)\Omega, \pi(\tau_x \circ \tau_t(B))\Omega \rangle = \langle \pi(A)\Omega, \pi(\tau_t \circ \tau_x(B))\Omega \rangle \\ &= \langle \pi(A)\Omega, U(t)U(x)\pi(B)\Omega \rangle, \end{aligned} \quad (3.6)$$

since we treat translation invariant interactions and hence (3.1) holds. Thus we have shown that $U(x)U(t) = U(t)U(x)$ on \mathcal{H} . \square

Recall that the dual group of $\Gamma = \mathbb{Z}^d$ is the d -dimensional torus $\widehat{\Gamma} = S_1^d$ (the Brillouin zone). By the SNAG theorem [66], there exists a spectral measure dP on $\mathbb{R} \times \widehat{\Gamma}$ with values in \mathcal{H} s.t.

$$U(t, x) = \int_{\mathbb{R} \times \widehat{\Gamma}} e^{iEt - ipx} dP(E, p). \quad (3.7)$$

The following is a special case of Definition A.1 in the case of unitaries, see Subsection A.2.

Definition 3.3. We define $\text{Sp}U$ as the support of dP .

By definition of the ground state we have that $\text{Sp}U \subset \mathbb{R}_+ \times \widehat{\Gamma}$.

3.2 Arveson spectrum

In this subsection we give a very brief overview of basic concepts of spectral analysis of automorphisms groups. For a more systematic discussion we refer to Appendix A.

Let $A \in \mathfrak{A}$ and $f \in L^1(\mathbb{R} \times \Gamma)$, $g \in L^1(\Gamma)$, $h \in L^1(\mathbb{R})$. Then, using the space-time translation automorphisms $(t, x) \mapsto \tau_{(t,x)}$ we set

$$\tau_f(A) := (2\pi)^{-\frac{d+1}{2}} \sum_{x \in \Gamma} \int dt \tau_{(t,x)}(A) f(t, x), \quad (3.8)$$

$$\tau_g^{(d)}(A) := (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} \tau_x(A) g(x), \quad (3.9)$$

$$\tau_h^{(1)}(A) := (2\pi)^{-\frac{1}{2}} \int dt \tau_t(A) h(t). \quad (3.10)$$

These are again elements of \mathfrak{A} by the strong continuity of τ , which is a consequence of the Lieb-Robinson bound. Now we define:

Definition 3.4. The Arveson spectrum of $A \in \mathfrak{A}$ w.r.t. τ , denoted $\text{Sp}_A \tau$, is the smallest closed subset of $\mathbb{R} \times \widehat{\Gamma}$ with the property that $\tau_f(A) = 0$ for any $f \in L^1(\mathbb{R} \times \Gamma)$ s.t. its Fourier transform \widehat{f} is supported outside of this set. The Arveson spectra of A w.r.t. $\tau^{(1)}$ and $\tau^{(d)}$, denoted $\text{Sp}_A \tau^{(1)}$ and $\text{Sp}_A \tau^{(d)}$, are defined analogously.

Remark 3.5. $\text{Sp}_A \tau$ can also be called the energy-momentum transfer of A , cf. relation (3.15) below.

Items (3.11)–(3.14) of the following proposition are easy and well known consequences of Definition 3.4. Equation (3.15) is a special case of Theorem 3.5 of [5]. For the reader's convenience, we give an elementary argument in Appendix B, which is based on the proof of Theorem 3.26 of [26].

Proposition 3.6. We have for any $A \in \mathfrak{A}$, $(t, x) \in \mathbb{R} \times \Gamma$, $f \in L^1(\mathbb{R} \times \Gamma)$, $g \in L^1(\Gamma)$:

$$\text{Sp}_{A^* \tau} = -\text{Sp}_A \tau, \quad (3.11)$$

$$\text{Sp}_{\tau_{(t,x)}(A) \tau} = \text{Sp}_A \tau, \quad (3.12)$$

$$\text{Sp}_{\tau_f(A) \tau} \subset \text{Sp}_A \tau \cap \text{supp } \widehat{f}, \quad (3.13)$$

$$\text{Sp}_{\tau_g^{(d)}(A) \tau} \subset \text{Sp}_A \tau \cap (\mathbb{R} \times \text{supp } \widehat{g}). \quad (3.14)$$

Moreover, with P as defined in (3.7), we have the energy-momentum transfer relation

$$\pi(A)P(\Delta) = P(\overline{\Delta + \text{Sp}_A \tau})\pi(A)P(\Delta) \quad (3.15)$$

for any Borel subset $\Delta \subset \mathbb{R} \times \widehat{\Gamma}$.

By (3.13), the smearing operation (3.8) allows one to construct observables A with energy-momentum transfers contained in arbitrarily small sets. Such observables will be needed in Section 4 to create single-particle states from the ground state. Since particles are localized excitations, these observables should have good localization properties. However, in view of Proposition 3.7 below, if $A \in \mathfrak{A}_{\text{loc}}$ is s.t. $\text{Sp}_A \tau$ is a small neighbourhood of some point $(E, p) \in \mathbb{R} \times \widehat{\Gamma}$ then $A \in \mathfrak{C}I$ and $(E, p) = (0, 0)$. Therefore, in the next subsection we introduce a slightly larger class of *almost local observables* which are still essentially localized in bounded regions of space-time but contains observables whose Arveson spectra are in arbitrarily small sets. The class of almost local observables is invariant under time translations and under the smearing operation (3.8).

Proposition 3.7. *Let $A \in \mathfrak{A}_{\text{loc}}$, $A \notin \mathfrak{C}I$. Then $\text{Sp}_A \tau^{(d)} = \widehat{\Gamma}$.*

Proof. The proof follows a remark in the Introduction of [17]. Since \mathfrak{A} is a simple algebra and A is not a multiple of the identity, there is a $B \in \mathfrak{A}_{\text{loc}}$ such that $[\tau_x(A), B] \neq 0$ for some $x \in \Gamma$. Define $T(x) := [\tau_x(A), B]$ which is supported, as an operator valued function of x , in a bounded region of Γ by locality. Its Fourier transform, $\widehat{T}(p) := (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} e^{-ip \cdot x} [\tau_x(A), B]$, which is a function on $\widehat{\Gamma} = S_1^d$, is such that $\text{supp } \widehat{T} = \widehat{\Gamma}$. This can be seen by interpreting the Fourier transform \widehat{T} as a periodic function on \mathbb{R}^d and verifying that it is real-analytic and non-zero. Now suppose that there is some open $U \subset \widehat{\Gamma}$ s.t. $U \cap \text{Sp}_A \tau^{(d)} = \emptyset$. Then, by definition, for any $g \in L^1(\Gamma)$ s.t. $\text{supp } \widehat{g} \subset U$ we have $\tau_g^{(d)}(A) = 0$. Since $[\tau_g^{(d)}(A), B] = (2\pi)^{-\frac{d}{2}} \int \widehat{T}(-p) \widehat{g}(p) dp$, this contradicts $\text{supp } \widehat{T} = \widehat{\Gamma}$. \square

3.3 Almost local observables

In this subsection we introduce a convenient class of *almost local observables*. We recall that in the context of relativistic QFT almost local observables were first defined in [4]. This class is larger than $\mathfrak{A}_{\text{loc}}$ but its elements have much better localization properties than arbitrary elements of \mathfrak{A} (see Lemma 3.9 below). The relevance of almost local observables to our investigation comes from Theorems 3.10 and 3.11 below.

As mentioned in the Introduction, the class of almost local observables has been studied before in the context of lattice systems in the Diplom thesis of Schmitz [70]. Counterparts of Lemma 3.9 and Theorem 3.10 below can be found in this work. Since our proofs are different, and also the reference [70] is not readily accesible, we present these results here in detail.

Definition 3.8. *We say that $A \in \mathfrak{A}$ is almost local if there exists a sequence $\mathbb{R}_+ \ni r \mapsto A_r \in \mathfrak{A}(Z^r)$ s.t.*

$$A - A_r = O(r^{-\infty}). \quad (3.16)$$

The $$ -algebra of almost local observables is denoted by $\mathfrak{A}_{\text{a-loc}}$.*

In contrast to local observables, the commutator of two elements of $\mathfrak{A}_{\text{a-loc}}$ need not identically vanish if one of them is translated sufficiently far from the other. Instead we have a rapid decay of commutators:

Lemma 3.9. *Let $A_i \in \mathfrak{A}_{\text{a-loc}}$, $i = 1, 2$. Then, for $y \in \Gamma$,*

$$[A_1, \tau_y(A_2)] = O(\langle y \rangle^{-\infty}), \quad (3.17)$$

where the notation $\langle y \rangle^{-\infty}$ is defined in Appendix D.

Proof. Making use of almost locality of A_i , we find $A_{i,r} \in \mathfrak{A}(Z^r)$ s.t. $A_i - A_{i,r} = O(r^{-\infty})$. Thus we get

$$[A_1, \tau_y(A_2)] = [A_{1,r}, \tau_y(A_{2,r})] + O(r^{-\infty}). \quad (3.18)$$

Setting $r = \varepsilon \langle y \rangle$, we obtain that for sufficiently small $\varepsilon > 0$ and $\langle y \rangle$ sufficiently large

$$Z^r \cap (Z^r + y) = \emptyset, \quad (3.19)$$

which concludes the proof. \square

The following theorem gives important invariance properties of $\mathfrak{A}_{\text{a-loc}}$. We note that they are in general not true for $\mathfrak{A}_{\text{loc}}$. For precise definitions of the Schwartz classes of functions on a lattice, $S(\Gamma)$ and $S(\mathbb{R} \times \Gamma)$, we refer to Appendix D.

Theorem 3.10. *Let $A \in \mathfrak{A}_{\text{a-loc}}$. Then*

- (a) $\tau_{(t,x)}(A) \in \mathfrak{A}_{\text{a-loc}}$ for all $(t, x) \in \mathbb{R} \times \Gamma$, and
- (b) $\tau_f(A), \tau_g^{(d)}(A), \tau_h^{(1)}(A) \in \mathfrak{A}_{\text{a-loc}}$ for all $f \in S(\mathbb{R} \times \Gamma)$, $g \in S(\Gamma)$ and $h \in S(\mathbb{R})$.

In contrast to relativistic QFT, for lattice systems this result is not automatic. We give a proof in Appendix C.

To conclude this section we give a lattice variant of an important result of Buchholz [17], which will be helpful in Section 4 (cf. Lemma 4.5 (c)). This result illustrates how the Arveson spectrum and almost locality can be used in combination.

Theorem 3.11. *Let Δ be a compact subset of $\text{Sp}U$, $A \in \mathfrak{A}$ and $\text{Sp}_A \tau$ be a compact subset of $(-\infty, 0) \times \hat{\Gamma}$. Then, for any $\Lambda \in \mathbf{P}_{\text{fin}}(\Gamma)$,*

$$\|P(\Delta) \sum_{x \in \Lambda} \pi(\tau_x(A^* A)) P(\Delta)\| \leq C \sum_{x \in (\Lambda - \Lambda)} \|[A^*, \tau_x(A)]\|, \quad (3.20)$$

where C is independent of Λ and $\Lambda - \Lambda = \{x_1 - x_2 \mid x_1, x_2 \in \Lambda\} \subset \Gamma$. If in addition A is almost local, we can conclude that

$$\|P(\Delta) \sum_{x \in \Gamma} \pi(\tau_x(A^* A)) P(\Delta)\| < \infty. \quad (3.21)$$

Proof. First, we note that by positivity of the spectrum of H , our assumption on $\text{Sp}_A \tau$ and the compactness of Δ , relation (3.15) gives that $A^n P(\Delta) = 0$ for $n \in \mathbb{N}$ sufficiently large. Now to obtain (3.20) we apply Lemma 2.2 of [17], noting that its proof remains valid if integrals over compact subsets of \mathbb{R}^d are replaced with sums over finite subsets of Γ .

If A is almost local, by Lemma 3.9 the sum on the r.h.s. of (3.20) can be extended from $\Lambda - \Lambda$ to Γ . As a consequence also the sum on the l.h.s. can be extended from Λ to Γ and it still defines a bounded operator (as a strong limit of an increasing net of bounded operators which is uniformly bounded). \square

We remark that relation (3.20) actually holds in any representation in which space-time translations are unitarily implemented and the Hamiltonian is positive (not necessarily a ground state representation).

4 Scattering theory

A procedure to construct wave operators and the S -matrix, which we adapt to lattice systems in this section, is known in local relativistic QFT as the Haag-Ruelle theory [34, 68]. Our presentation here is close to [27, 29] which in turn profited from [3, 18, 39].

Let us start with stating our **standing assumptions** for this section:

1. We consider a lattice system given by a quasi-local algebra \mathfrak{A} and a translation invariant interaction $\Phi \in \mathcal{B}_\lambda$, $\lambda > 0$. Thus the space-time translations act on \mathfrak{A} by the group of automorphisms $\mathbb{R} \times \Gamma \ni (t, x) \mapsto \tau_{(t,x)}$ of equation (3.3).
2. We consider a translation invariant ground state ω of \mathfrak{A} and its GNS representation $(\pi, \mathcal{H}, \Omega)$. The space-time translation automorphisms τ are implemented in π by the group of unitaries U constructed in Lemma 3.2.
3. $\text{Sp}U$ contains an isolated simple eigenvalue $\{0\}$ (with eigenvector Ω) and an isolated mass shell \mathfrak{h} (see Definition 4.1 below).
4. The mass shell is either pseudo-relativistic or regular (see Definition 4.1 below).

There is no general criterion that ensures the validity of the assumptions in a given quantum lattice system. In fact, proving the very existence of an isolated mass shell is in general a difficult problem. However, there are models where our standing assumptions can be checked in full rigour, see Section 6. In other models isolated mass shells have been shown to exist numerically [36].

4.1 Single-particle subspace

A class of localized excitations of the ground state (particles) is carried by *mass shells* in $\text{Sp}U$. We define a mass shell as follows:

Definition 4.1. *Let ω be a translation invariant ground state of a system (\mathfrak{A}, τ) with translation invariant interaction. We say that $\mathfrak{h} \subset \mathbb{R} \times \widehat{\Gamma}$ is a mass shell if*

- (a) $\mathfrak{h} \subset \text{Sp}U$, $P(\mathfrak{h}) \neq 0$, where P is the spectral projection of (3.7).
- (b) There is an open subset $\Delta_{\mathfrak{h}} \subset \widehat{\Gamma}$ and a real-valued function $\Sigma \in C^\infty(\Delta_{\mathfrak{h}})$ such that

$$\mathfrak{h} = \{ (\Sigma(p), p) \mid p \in \Delta_{\mathfrak{h}} \}.$$

We shall call Σ the dispersion relation and denote by $D^2\Sigma(p) := [\partial_i \partial_j \Sigma(p)]_{i,j=1,\dots,d}$ its Hessian.

- (c) The set $\mathcal{T} := \{ p \in \Delta_{\mathfrak{h}} \mid D^2\Sigma(p) = 0 \}$ has Lebesgue measure zero.

Moreover, we say that:

1. A mass shell is *isolated*, if for any $p \in \Delta_{\mathfrak{h}}$ there is $\varepsilon > 0$ such that³

$$([\Sigma(p) - \varepsilon, \Sigma(p) + \varepsilon] \times \{p\}) \cap \text{Sp}U = \{(\Sigma(p), p)\}. \quad (4.1)$$

2. A mass shell is *regular*, if the set $\mathcal{X} := \{ p \in \Delta_{\mathfrak{h}} \mid \det D^2\Sigma(p) = 0 \}$ has Lebesgue measure zero.

³To cover mass shells which dip into the continuous spectrum, as in Figure 1b, we allow for a different $\varepsilon > 0$ for each $p \in \Delta_{\mathfrak{h}}$.

3. A mass shell is pseudo-relativistic, if $(\mathfrak{h} - \mathfrak{h}) \cap \text{Sp}U = \{0\}$.

Finally, we define $\Delta'_\mathfrak{h} := \Delta_\mathfrak{h} \setminus \mathcal{X}$.

Given a mass shell, we define the corresponding single-particle subspace in a natural way:

Definition 4.2. Let \mathfrak{h} be a mass shell in the sense of Definition 4.1. The corresponding single-particle subspace is given by $\mathcal{H}_\mathfrak{h} := P(\mathfrak{h})\mathcal{H}$.

Let us add some comments on Definition 4.1: Requirement (c) prevents the *group velocity* $\nabla\Sigma$ from being constant on a set of momenta of non-zero Lebesgue measure. This condition ensures that configurations of particles with distinct velocities form a dense subspace. We will make use of this fact in Subsection 4.4, see Lemma 4.19. We note that for $\Delta_\mathfrak{h} = \widehat{\Gamma}$, and Σ a real-analytic function, condition (c) holds if and only if Σ is non-constant. In fact, if (c) is violated, then, by analyticity, the second derivative of Σ vanishes identically on $\widehat{\Gamma}$. Thus interpreting Σ as a function on \mathbb{R}^d , it is periodic in all variables and linear. This is only possible if Σ is constant. We will use this observation in the proof of Proposition 6.5.

For the construction of scattering states in Theorem 4.8 we only need to assume that the mass shell is isolated. However, to show the Fock space structure of scattering states in Theorem 4.9 we need to assume in addition that it is either regular or pseudo-relativistic. Regularity means that the momentum-velocity relation $p \mapsto \nabla\Sigma(p)$ is invertible almost everywhere, in particular Σ has no flat directions. In $d = 1$ obviously every mass shell is regular, but for $d > 1$ it is not easy to verify this condition in models. Therefore we introduced the alternative class of pseudo-relativistic mass shells described in 3. of Definition 4.1. Since 0 is an isolated eigenvalue, Property 3. can always be ensured at the cost of shrinking the set $\Delta_\mathfrak{h}$ and is compatible with flat directions of Σ . We note that mass shells in relativistic theories with lower mass gaps always satisfy this property, which justifies the name ‘pseudo-relativistic’.

To conclude this subsection, we discuss briefly positive-energy wave packets describing the free evolution of a single particle with a dispersion relation $\Delta_\mathfrak{h} \ni p \mapsto \Sigma(p)$ introduced in Definition 4.1. Such a wave packet is given by

$$g_t(x) := (2\pi)^{-\frac{d}{2}} \int_{\Delta_\mathfrak{h}} dp e^{-i\Sigma(p)t + ipx} \widehat{g}(p), \quad (4.2)$$

where $\widehat{g} \in C^\infty(\widehat{\Gamma})$ with $\text{supp } \widehat{g} \subset \Delta_\mathfrak{h}$. Its velocity support is defined as

$$V(g) := \{ \nabla\Sigma(p) \mid p \in \text{supp } \widehat{g} \}. \quad (4.3)$$

Some parts of the following proposition have appeared already in [76]. It is a generalization of the (non)-stationary phase method to wave packets on a lattice.

Proposition 4.3. Let $\chi_+ \in C_0^\infty(\mathbb{R}^d)$ be equal to one on $V(g)$ and vanish outside of a slightly larger set and let $\chi_- = 1 - \chi_+$. We write $\chi_{\pm,t}(x) := \chi_\pm(x/t)$. Then we have

$$\|\chi_{+,t}g_t\|_1 = O(t^d), \quad \|\chi_{-,t}g_t\|_1 = O(t^{-\infty}), \quad \|g_t\|_1 = O(t^d). \quad (4.4)$$

Assuming in addition that $\text{supp } \widehat{g} \subset \Delta'_\mathfrak{h}$ (i.e. $\det D^2\Sigma \neq 0$ on the support of \widehat{g}) we have

$$\sup_{x \in \Gamma} |g_t(x)| = O(t^{-d/2}), \quad \|g_t\|_1 = O(t^{d/2}). \quad (4.5)$$

Proof. To prove (4.4) we use decomposition (A.20) to express g_t as a finite sum of functions to which the non-stationary phase method (the Corollary of Theorem XI.14 of [64]) applies. Thus

$$|(\chi_{-,t}g_t)(x)| = O(\langle |x| + |t| \rangle^{-\infty}), \quad (4.6)$$

either because of the corollary or because $\chi_{-,t}$ vanishes in the cases the corollary does not apply. It follows that $\|\chi_{-,t}g_t\|_1 = O(t^{-\infty})$. Now since χ_+ is compactly supported, we obtain $\|\chi_{+,t}g_t\|_1 = O(t^d)$.

To prove (4.5) we use again decomposition (A.20) in order to apply the stationary phase method (Corollary of Theorem XI.15 of [64]). This gives $\sup_{x \in \Gamma} |g_t(x)| = O(t^{-d/2})$. By decomposing $g_t(x) = \chi_{+,t}(x)g_t(x) + \chi_{-,t}(x)g_t(x)$, using $\|\chi_{-,t}g_t\|_1 = O(t^{-\infty})$, $\sup_{x \in \Gamma} |g_t(x)| = O(t^{-d/2})$ and the fact that χ_+ is compactly supported we get $\|g_t\|_1 = O(t^{d/2})$. \square

4.2 Haag-Ruelle creation operators

In this subsection we will define and study properties of certain ‘creation operators’ from $\pi(\mathfrak{A}_{\text{a-loc}})$ which create elements of the single-particle subspace $\mathcal{H}_{\mathfrak{h}} = P(\mathfrak{h})\mathcal{H}$ from the ground state vector Ω .

Definition 4.4. Let $A^* \in \mathfrak{A}_{\text{a-loc}}$ be s.t. $\text{Sp}_{A^*}\tau \subset (0, \infty) \times \widehat{\Gamma}$ is compact and $\text{Sp}_{A^*}\tau \cap \text{Sp}U \subset \mathfrak{h}$. Let g_t be a wave packet given by (4.2). We say that:

1. $B^* := \pi(A^*)$ is a creation operator. $\text{Sp}_{A^*}\tau = \text{Sp}_{\pi^{-1}(B^*)}\tau$ will be called the energy-momentum transfer of B^* .
2. $B_t^*(g_t) := \pi(\tau_t \circ \tau_{g_t}^{(d)}(A^*))$ is the Haag-Ruelle (HR) creation operator.

Recall that \mathfrak{A} is simple so that π is faithful, and thus 1. is well-defined. Furthermore, setting $B_t^*(x) := U(t, x)B^*U(t, x)^*$, we can equivalently write

$$B_t^*(g_t) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} B_t^*(x)g_t(x), \quad (4.7)$$

consistently with the notation $\pi(A^*)(g) := \pi(\tau_g^{(d)}(A^*))$ from Appendix D.

We also note that the map $t \mapsto B_t^*$ is smooth in the norm topology for any creation operator and the respective derivatives are again creation operators. Indeed, since $\text{Sp}_{A^*}\tau$ is compact, we have that $A^* = \tau_f(A^*)$ for any $f \in S(\mathbb{R} \times \Gamma)$ such that \widehat{f} is equal to 1 on $\text{Sp}_{A^*}\tau$. It follows that $\tau_t(A^*) = \tau_{f_t}(A^*)$, where $f_t(s, x) = f(s - t, x)$. Hence,

$$\partial_t^n \tau_t(A^*) = (-1)^n \tau_{\partial_s^n f_t}(A^*), \quad (4.8)$$

and $t \mapsto \tau_t(A^*)$ is C^∞ . The claim follows, since π is continuous.

By Proposition 3.6, $\text{Sp}_{\tau_t \circ \tau_{g_t}^{(d)}(A^*)}\tau \subset \text{Sp}_{A^*}\tau$ thus by the energy-momentum transfer relation (3.15)

$$B_t^*(g_t)\Omega \in \mathcal{H}_{\mathfrak{h}}, \quad (B_t^*(g_t))^*\Omega = 0. \quad (4.9)$$

On the other hand it follows from Theorem 3.10 that $\tau_t \circ \tau_{g_t}^{(d)}(A^*) \in \mathfrak{A}_{\text{a-loc}}$, thus $B_t^*(g_t)$ creates a well-localized excitation (particle) from the vacuum. The expression $\tau_t \circ \tau_{g_t}^{(d)}$ amounts to comparing the interacting forward evolution τ_t with the free backward evolution $\tau_{g_t}^{(d)}$. The goal of scattering theory is to show that they match at asymptotic times. In part (a) of the next lemma we show that at the single-particle level these two evolutions match already at finite times.

Lemma 4.5. *Let $B_t^*(g_t)$ be a HR creation operator and χ_\pm be as in Proposition 4.3. We have*

- (a) $B_t^*(g_t)\Omega = B^*(g)\Omega = P(\mathfrak{h})B^*(g)\Omega,$
- (b) $\partial_t(B_t^*(g_t)) = (\partial_t B_t^*)(g_t) + B_t^*(\partial_t g_t),$
- (c) $\|B_t^*(g_t)P(\Delta)\| = O(1)$ for any compact $\Delta \subset \text{Sp}U,$
- (d) $\|B_t^*(g_t)\|, \|B_t^*(\chi_{+,t}g_t)\| = O(t^d), \|B_t^*(\chi_{-,t}g_t)\| = O(t^{-\infty}),$

where in (b) $\partial_t B_t^*$ (resp. $\partial_t g_t$) is again a creation operator (resp. a wave packet).

Proof. Let us show (a): Making use of (3.7) and (4.9) we get

$$\begin{aligned}
B_t^*(g_t)\Omega &= (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} g_t(x)U(t,x)B^*\Omega \\
&= (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} g_t(x) \int_{\mathfrak{h}} e^{iEt-ipx} dP(E,p)P(\mathfrak{h})B^*\Omega \\
&= \int_{\mathfrak{h}} e^{i(E-\Sigma(p))t} \widehat{g}(p) dP(E,p)P(\mathfrak{h})B^*\Omega = B^*(g)\Omega.
\end{aligned} \tag{4.10}$$

Here in the third step we applied the Fourier inversion formula (cf. Appendix D). In the last step we used that $E = \Sigma(p)$ in the region of integration and thus the expression is t -independent. Then we reversed the steps to conclude the proof of (a).

Part (b) is an obvious computation, which is legitimate since $t \mapsto B_t^*$ is smooth in norm as we argued above.

Part (c) is a consequence of Theorem 3.11: Let $\Psi, \Phi \in \mathcal{H}$ be unit vectors, with $\Phi \in \text{Ran}P(\Delta)$. Then by the Cauchy-Schwarz inequality

$$|\langle \Psi, B_t^*(g_t)\Phi \rangle| \leq (2\pi)^{-\frac{d}{2}} \left(\sum_{x \in \Gamma} |\langle \Psi, B_t^*(x)\Phi \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{x \in \Gamma} |g_t(x)|^2 \right)^{\frac{1}{2}}. \tag{4.11}$$

By Parseval's theorem, the second factor on the r.h.s. above is a time-independent constant. As for the first factor, we have

$$\sum_{x \in \Gamma} |\langle \Psi, B_t^*(x)\Phi \rangle|^2 \leq \sum_{x \in \Gamma} \langle \Psi, P(\Delta')B_t^*(x)B_t(x)P(\Delta')\Psi \rangle \leq \|P(\Delta') \sum_{x \in \Gamma} B^*(x)B(x)P(\Delta')\|, \tag{4.12}$$

where in the first step we made use of the fact that $B^* = \pi(A^*)$, where $\text{Sp}_{A^*}\tau$ is compact, and of the energy-momentum transfer relation (3.15), to introduce the projection on a compact set $\Delta' \supset (\Delta + \text{Sp}_{A^*}\tau) \cap \text{Sp}U$. Since A^* is almost local, and

$$\text{Sp}_{A^*}\tau = -\text{Sp}_{A^*}\tau \subset (-\infty, 0) \times \widehat{\Gamma}, \tag{4.13}$$

the expression on the r.h.s. of (4.12) is finite by Theorem 3.11.

Part (d) of the lemma follows from Proposition 4.3 and the obvious estimate $\|B^*(g')\| \leq \|B^*\| \|g'\|_1, g' \in L^1(\Gamma)$. \square

The next lemma concerns the decay of commutators of HR creation operators associated with wave packets with disjoint velocity supports.

Lemma 4.6. *Let $B_{1,t}^*(g_{1,t}), B_{2,t}^*(g_{2,t}), B_{3,t}^*(g_{3,t})$ be HR creation operators s.t. $V(g_1) \cap V(g_2) = \emptyset$ and $V(g_3)$ arbitrary. Then*

- (a) $[B_{1,t}^*(g_{1,t}), B_{2,t}^*(g_{2,t})] = O(t^{-\infty})$,
- (b) $[B_{1,t}^*(g_{1,t}), [B_{2,t}^*(g_{2,t}), B_{3,t}^*(g_{3,t})]] = O(t^{-\infty})$.

The statements remains valid if some of the HR creation operators are replaced with their adjoints.

Proof. To prove (a) we decompose $g_{i,t} = \chi_{i,+}g_{i,t} + \chi_{i,-}g_{i,t}$, $i = 1, 2$, where $\chi_{i,\pm}$ appeared in Proposition 4.3. Then, by Lemma 4.5, we get

$$[B_{1,t}^*(g_{1,t}), B_{2,t}^*(g_{2,t})] = [B_{1,t}^*(\chi_{1,+}g_{1,t}), B_{2,t}^*(\chi_{2,+}g_{2,t})] + O(t^{-\infty}). \quad (4.14)$$

Now for each n there is a constant $c_n > 0$ such that the following estimate holds,

$$\begin{aligned} \|[B_{1,t}^*(\chi_{1,+}g_{1,t}), B_{2,t}^*(\chi_{2,+}g_{2,t})]\| &= \left\| \sum_{x_1, x_2 \in \Gamma} \chi_{1,+}(x_1) \chi_{2,+}(x_2) [B_{1,t}^*(x_1), B_{2,t}^*(x_2)] \right\| \\ &\leq \sum_{x_1, x_2 \in \Gamma} |\chi_{1,+}(x_1/t)| |\chi_{2,+}(x_2/t)| \frac{c_n}{\langle x_1 - x_2 \rangle^n}, \end{aligned} \quad (4.15)$$

where in the second line the existence of c_n follows from almost locality, Lemma 3.9, and the fact that $|g_i(t, x)| = O(1)$ uniformly in x . Since $\chi_{1,+}, \chi_{2,+}$ are approximate characteristic functions of $V(g_1), V(g_2)$, they may be chosen with compact, disjoint supports. Therefore, by changing variables to $y_1 := x_1/t, y_2 := x_2/t$ we obtain that (4.15) is $O(t^{2d-n})$ and hence $O(t^{-\infty})$ since $n \in \mathbb{N}$ is arbitrary.

To prove (b) we decompose $\widehat{g}_3(p) = \widehat{g}_{3,1}(p) + \widehat{g}_{3,2}(p)$, using a smooth partition of unity, s.t. $V(g_{3,1}) \cap V(g_1) = \emptyset$ and $V(g_{3,2}) \cap V(g_2) = \emptyset$. Now the statement follows from part (a) and the Jacobi identity. \square

The next lemma is a counterpart of the relation $a(f)a^*(f)\Omega = \Omega\langle\Omega, a(f)a^*(f)\Omega\rangle$ which holds for the usual creation and annihilation operators on the Fock space. We will need it to establish the Fock space structure of scattering states. This is the only result of this subsection which requires that the mass shell is pseudo-relativistic or regular.

Lemma 4.7. *Let $B_{1,t}^*(g_{1,t}), B_{2,t}^*(g_{2,t})$ be HR creation operators.*

- (a) *If \mathfrak{h} is pseudo-relativistic, and the energy-momentum transfers of $B_{1,t}^*, B_{2,t}^*$ are contained in a sufficiently small neighbourhood of \mathfrak{h} then*

$$(B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t})\Omega = \Omega\langle\Omega, (B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t})\Omega\rangle. \quad (4.16)$$

- (b) *If \mathfrak{h} is regular and $\text{supp } \widehat{g}_1, \text{supp } \widehat{g}_2 \subset \Delta'_{\mathfrak{h}}$, then*

$$(B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t})\Omega = \Omega\langle\Omega, (B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t})\Omega\rangle + O(t^{-d/2}). \quad (4.17)$$

Proof. Part (a) follows immediately from $(\mathfrak{h} - \mathfrak{h}) \cap \text{Sp } U = \{0\}$ and the energy-momentum transfer relation (3.15).

To prove (b) we will use the clustering property stated in Theorem 2.4 and a similar strategy as in [16, p.169-171]: Let $A_i \in \mathfrak{A}_{\text{a-loc}}$, $i = 1, 2, 3, 4$, $A_i(x_i) := \tau_{x_i}(A_i)$ and $A_{i,r} \in \mathfrak{A}(Z^r)$ be s.t. $\|A_i - A_{i,r}\| = O(r^{-\infty})$. Now we write $P(\{0\})^\perp = 1 - |\Omega\rangle\langle\Omega|$ and consider the function

$$F(x_1, x_2, x_3, x_4) := \langle\Omega, \pi([A_1(x_1), A_2(x_2)])P(\{0\})^\perp \pi([A_3(x_3), A_4(x_4)])\Omega\rangle. \quad (4.18)$$

We will show that this function is rapidly decreasing in relative variables, i.e.

$$F(x_1, x_2, x_3, x_4) = O(\langle x_1 - x_2 \rangle^{-\infty}) O(\langle x_3 - x_4 \rangle^{-\infty}) O(\langle x_2 - x_3 \rangle^{-\infty}). \quad (4.19)$$

By almost locality of A_i and Lemma 3.9, we obtain rapid decay in $x_1 - x_2$ and $x_3 - x_4$. Thus it suffices to show

$$F(x_1, x_2, x_3, x_4) = O(\langle x_2 - x_3 \rangle^{-\infty}). \quad (4.20)$$

To this end, we first write

$$F(x_1, x_2, x_3, x_4) := \langle \Omega, \pi([A_{1,r}(x_1), A_{2,r}(x_2)]) P(\{0\})^\perp \pi([A_{3,r}(x_3), A_{4,r}(x_4)]) \Omega \rangle + O(r^{-\infty}). \quad (4.21)$$

Denoting the first term on the r.h.s. of (4.21) by $F_r(x_1, x_2, x_3, x_4)$, we obtain from Theorem 2.4:

$$|F_r(x_1, x_2, x_3, x_4)| \leq C \chi(X_1^r \cap X_2^r \neq \emptyset) \chi(X_3^r \cap X_4^r \neq \emptyset) \max_{i=1,\dots,4} |X_i^r| e^{-\mu \text{dist}(X_1^r \cup X_2^r, X_3^r \cup X_4^r)}, \quad (4.22)$$

where we set $X_i^r := Z^r + x_i = \{x_i\}^r$ and $\chi(K)$ is equal to one if the condition K is satisfied and zero otherwise. $X_1^r \cap X_2^r \neq \emptyset$ and $X_3^r \cap X_4^r \neq \emptyset$ imply that

$$|x_1 - x_2| \leq 2r, \quad |x_3 - x_4| \leq 2r. \quad (4.23)$$

Moreover,

$$\begin{aligned} \text{dist}(X_1^r \cup X_2^r, X_3^r \cup X_4^r) &= \inf\{|w_a - w_b| \mid w_a \in X_1^r \cup X_2^r, w_b \in X_3^r \cup X_4^r\} \\ &= \inf_{\substack{m \in 1,2 \\ n \in 3,4}} \inf\{|w_a - w_b| \mid w_a \in X_m^r, w_b \in X_n^r\} \\ &\geq \inf_{\substack{m \in 1,2 \\ n \in 3,4}} |x_m - x_n| - 2r \\ &\geq |x_2 - x_3| - 6r, \end{aligned} \quad (4.24)$$

where in the last step we made use of (4.23). Thus noting that $|X_i^r| = |Z^r| \leq (2r)^d$ we can write

$$|F_r(x_1, x_2, x_3, x_4)| \leq C(2r)^d e^{6\mu r} e^{-\mu|x_2-x_3|}. \quad (4.25)$$

Setting $r = \varepsilon \langle x_2 - x_3 \rangle$, $\varepsilon > 0$ sufficiently small, and making use (4.21) we obtain (4.20) and therefore (4.19).

Now we are ready to prove (4.17). Since $(B_{1,t}^*(g_{1,t}))^* \Omega = 0$, we have

$$\|P(\{0\})^\perp (B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t}) \Omega\|^2 \leq \sum_{x_1, \dots, x_4 \in \Gamma} |F(x_1, x_2, x_3, x_4)| |g_{1,t}(x_1) g_{2,t}(x_2) g_{1,t}(x_3) g_{2,t}(x_4)|. \quad (4.26)$$

By introducing relative variables $x_{i,j} = x_i - x_j$, and using (4.19), we get

$$\|P(\{0\})^\perp (B_{1,t}^*(g_{1,t}))^* B_{2,t}^*(g_{2,t}) \Omega\|^2 \leq t^{-d} \sum_{x_{1,2}, x_{2,3}, x_{3,4} \in \Gamma} O(\langle x_{1,2} \rangle^{-\infty}) O(\langle x_{2,3} \rangle^{-\infty}) O(\langle x_{3,4} \rangle^{-\infty}). \quad (4.27)$$

Here we exploited regularity of the mass shell which gives $\|g_{i,t}\|_1 = O(t^{d/2})$ and $|g_{i,t}(x)| = O(t^{-d/2})$ uniformly in $x \in \Gamma$. (We applied the former estimate to $g_{1,t}(x_1)$, and the latter to the remaining three wave packets). This concludes the proof. \square

4.3 Scattering states and their Fock space structure

In this subsection we prove the main results of this paper which are the existence of Haag-Ruelle scattering states for lattice systems (Theorem 4.8) and their Fock space structure (Theorem 4.9). Using these two results we will define the wave operators and the scattering matrix in the next subsection. The method of proof of the next theorem follows [3].

Theorem 4.8. *Let B_1^*, \dots, B_n^* be creation operators and g_1, \dots, g_n be positive energy wave packets with disjoint velocity supports. Then there exists the n -particle scattering state given by*

$$\Psi^{\text{out}} = \lim_{t \rightarrow \infty} B_{1,t}^*(g_{1,t}) \dots B_{n,t}^*(g_{n,t}) \Omega. \quad (4.28)$$

Moreover, let S_n be the set of permutations of an n -element set. Then, for any $\sigma \in S_n$,

$$\Psi^{\text{out}} = \lim_{t \rightarrow \infty} B_{\sigma(1),t}^*(g_{\sigma(1),t}) \dots B_{\sigma(n),t}^*(g_{\sigma(n),t}) \Omega. \quad (4.29)$$

Finally, let $\tilde{B}_{i,t}^*(\tilde{g}_i)$, $i = 1, \dots, n$ be HR creation operators satisfying the same assumptions and let $\tilde{\Psi}^{\text{out}}$ be the corresponding scattering state. If $\tilde{B}_{i,t}^*(\tilde{g}_i) \Omega = B_{i,t}^*(g_{i,t}) \Omega$, for all $i = 1, \dots, n$, and $V(\tilde{g}_i) \cap V(g_j) = \emptyset$ for all $i \neq j$, then $\tilde{\Psi}^{\text{out}} = \Psi^{\text{out}}$.

Proof. We prove the existence of scattering states by Cook's method: Let $\{\Psi_t\}_{t \in \mathbb{R}}$ be the approximating sequence on the r.h.s. of (4.28). Due to the Cauchy criterion for convergence and the bound $\|\Psi_{t_2} - \Psi_{t_1}\| \leq \int_{t_1}^{t_2} dt \|\partial_t \Psi_t\|$, it suffices to show that the derivative

$$\partial_t \Psi_t = \sum_{i=1}^n B_{1,t}^*(g_{1,t}) \dots \partial_t (B_{i,t}^*(g_{i,t})) \dots B_{n,t}^*(g_{n,t}) \Omega \quad (4.30)$$

is integrable in norm. By Lemma 4.5 (a), $\partial_t (B_{i,t}^*(g_{i,t}))$ annihilates Ω . Thus we can commute this expression to the right until it acts on Ω and it is enough to show that the resulting terms with the commutators are $O(t^{-\infty})$. This follows from Lemma 4.6 (a) and Lemma 4.5 (b), (d).

Since any permutation can be decomposed into a product of adjacent transpositions, the estimate

$$B_{1,t}^*(g_{1,t}) \dots B_{i,t}^*(g_{i,t}) B_{i+1,t}^*(g_{i+1,t}) \dots \Omega = B_{1,t}^*(g_{1,t}) \dots B_{i+1,t}^*(g_{i+1,t}) B_{i,t}^*(g_{i,t}) \dots \Omega + O(t^{-\infty}), \quad (4.31)$$

which holds by Lemma 4.6 (a) and Lemma 4.5 (d), is sufficient to prove the symmetry of Ψ^{out} .

Finally, by iterating the relation

$$\begin{aligned} B_{1,t}^*(g_{1,t}) \dots B_{n,t}^*(g_{n,t}) \Omega &= B_{1,t}^*(g_{1,t}) \dots B_{n-1,t}^*(g_{n-1,t}) \tilde{B}_{n,t}^*(\tilde{g}_{n,t}) \Omega \\ &= \tilde{B}_{n,t}^*(\tilde{g}_{n,t}) B_{1,t}^*(g_{1,t}) \dots B_{n-1,t}^*(g_{n-1,t}) \Omega + O(t^{-\infty}), \end{aligned} \quad (4.32)$$

which follows again from Lemma 4.6 (a) and Lemma 4.5 (d), and taking the limit $t \rightarrow \infty$, we obtain that Ψ^{out} coincides with the scattering state $\tilde{\Psi}^{\text{out}}$. \square

By the last part of Theorem 4.8, the scattering state Ψ^{out} depends only on the time independent single-particle vectors $\Psi_i = B_{i,t}^*(g_{i,t}) \Omega$, and possibly the velocity supports $V(g_i)$. The latter dependence can easily be excluded making use of formula (4.33) below. It easily follows from formula (4.33)

below. Anticipating this fact we will write $\Psi^{\text{out}} =: \Psi_1^{\text{out}} \times \dots \times \Psi_n^{\text{out}}$.

Theorem 4.9. *Suppose that the mass shell is pseudo-relativistic. Let Ψ^{out} and $\tilde{\Psi}^{\text{out}}$ be two scattering states defined as in Theorem 4.8 with n and \tilde{n} particles, respectively. Then*

$$\langle \tilde{\Psi}^{\text{out}}, \Psi^{\text{out}} \rangle = \delta_{n, \tilde{n}} \sum_{\sigma \in S_n} \langle \tilde{\Psi}_1, \Psi_{\sigma_1} \rangle \cdots \langle \tilde{\Psi}_n, \Psi_{\sigma_n} \rangle, \quad (4.33)$$

$$U(t, x)(\Psi_1^{\text{out}} \times \cdots \times \Psi_n^{\text{out}}) = (U(t, x)\Psi_1)^{\text{out}} \times \cdots \times (U(t, x)\Psi_n)^{\text{out}}, \quad (t, x) \in \mathbb{R} \times \Gamma, \quad (4.34)$$

where S_n is the set of permutations of an n -element set.

If the mass shell is regular, the same conclusion holds if $\text{supp } \hat{g}_i, \text{supp } (\hat{g}_i)$, $i = 1, \dots, n$ are supported in $\Delta'_\mathfrak{h}$.

Remark 4.10. *We stress that in (4.33) we only need $V(g_i) \cap V(g_j) = \emptyset$ and $V(\tilde{g}_i) \cap V(\tilde{g}_j) = \emptyset$ for $i \neq j$, where g_i, \tilde{g}_i enter the definition of $\Psi^{\text{out}}, \tilde{\Psi}^{\text{out}}$ as specified in (4.28). $V(g_i)$ and $V(\tilde{g}_j)$ may overlap.*

Proof. We first consider the case $n = \tilde{n}$. To prove (4.33), we set for simplicity of notation $B_i(t) := (B_{i,t}^*(g_{i,t}))^*$, $\tilde{B}_j(t) := (\tilde{B}_{j,t}^*(\tilde{g}_{j,t}))^*$ and denote by $\Psi_t, \tilde{\Psi}_t$ the approximating sequences of $\Psi^{\text{out}}, \tilde{\Psi}^{\text{out}}$. We assume that (4.33) holds for $n-1$ and compute

$$\begin{aligned} \langle \tilde{\Psi}_t, \Psi_t \rangle &= \langle \Omega, \tilde{B}_n(t) \cdots \tilde{B}_1(t) B_1(t)^* \cdots B_n(t)^* \Omega \rangle \\ &= \sum_{k=1}^n \langle \Omega, \tilde{B}_n(t) \cdots \tilde{B}_2(t) B_1(t)^* \cdots [\tilde{B}_1(t), B_k(t)^*] \cdots B_n(t)^* \Omega \rangle \\ &= \sum_{k=1}^n \sum_{l=k+1}^n \langle \Omega, \tilde{B}_n(t) \cdots \tilde{B}_2(t) B_1(t)^* \cdots \check{k} \cdots [[\tilde{B}_1(t), B_k(t)^*], B_l(t)^*] \cdots B_n(t)^* \Omega \rangle \\ &\quad + \sum_{k=1}^n \langle \Omega, \tilde{B}_n(t) \cdots \tilde{B}_2(t) B_1(t)^* \cdots \check{k} \cdots B_n(t)^* \tilde{B}_1(t) B_k(t)^* \Omega \rangle, \end{aligned} \quad (4.35)$$

where \check{k} indicates that $B_k(t)^*$ is omitted from the product. The terms involving double commutators vanish in the limit $t \rightarrow \infty$ by Lemma 4.6 (b) and Lemma 4.5 (d). To treat the last term on the r.h.s. of (4.35) we have to consider two cases.

For the first case, suppose that the mass shell is pseudo-relativistic. Recall that $B^* = \pi(A^*)$ and set $B_f^* := \pi(\tau_f(A^*))$, $f \in S(\mathbb{R} \times \Gamma)$. Choosing \hat{f} supported in a small neighbourhood $O \subset \hat{\Gamma}$ of a subset of \mathfrak{h} , we obtain that the energy-momentum transfer of B_f^* is contained in O (cf. Proposition 3.6). Demanding in addition that \hat{f} is equal to one on $\mathfrak{h} \cap \text{Sp}_{A^*} \tau$ we can ensure that $B_{f,t}^*(g_t) \Omega = B_t^*(g_t) \Omega$ (cf. the proof of Lemma 4.5 (a)). Thus, in view of the last part of Theorem 4.8, we can assume without loss that all the HR creation operators involved in (4.35) satisfy the assumptions of Lemma 4.7 (a). Then the last term on the r.h.s. of (4.35) factorizes as follows

$$\sum_{k=1}^n \langle \Omega, \tilde{B}_n(t) \cdots \tilde{B}_2(t) B_1(t)^* \cdots \check{k} \cdots B_n(t)^* \Omega \rangle \langle \Omega, \tilde{B}_1(t) B_k(t)^* \Omega \rangle \quad (4.36)$$

by Lemma 4.7 (a). Now by the induction hypothesis the expression above factorizes in the limit $t \rightarrow \infty$ and gives (4.33).

For the other case, let us assume that the mass shell is regular. By the energy-momentum transfer relation (3.15) and Lemma 4.5 (c) we obtain that

$$|\Omega\rangle \langle \Omega| \tilde{B}_n(t) \cdots \tilde{B}_2(t) B_1(t)^* \cdots \check{k} \cdots B_n(t)^* = O(1). \quad (4.37)$$

Consequently, by Lemma 4.7 (b), the last term on the r.h.s. of (4.35) equals

$$\sum_{k=1}^n \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_2(t) B_1(t)^* \dots \check{k} \dots B_n(t)^* \Omega \rangle \langle \Omega, \tilde{B}_1(t) B_k(t)^* \Omega \rangle + O(t^{-d/2}). \quad (4.38)$$

Now the inductive argument is concluded as in the previous case.

If $n \neq \tilde{n}$, a similar argument implies that the scalar product is zero, since the reduction will yield eventually an expectation value of a product of either only creation or only annihilation operators, which vanishes. We omit the details.

This gives (4.33). Relation (4.34) follows from (4.33) and Lemma 4.5 (a). \square

Remark 4.11. *We conclude this subsection with two technical comments:*

1. *In estimate (4.37) above we made crucial use of the bound from Lemma 4.5 (c), which relies on Theorem 3.11. Weaker bounds from Lemma 4.5 (d) would clearly not suffice to conclude the proof along the above lines. Without this sharper bound available, in the regular case we would have to proceed via cumbersome ‘truncated vacuum expectation values’ of arbitrary order, which burden all the textbook presentations of the Haag-Ruelle theory known to us (see e.g. [3, 64]).*
2. *It is clear that the uniqueness of the vacuum vector Ω is not needed in the proof of existence of scattering states (Theorem 4.8). It is less clear how to generalize the proof of the Fock space structure (Theorem 4.9) to the case of degenerate vacua and we leave this question open here. We think, however, that this can be accomplished by following the remarks from the last paragraph of Section 5 of [16].*

4.4 Wave operators and the S -matrix

Up to now we left aside the question if any of the scattering states Ψ^{out} is different from zero. In this subsection we will demonstrate that scattering states in fact span a subspace of \mathcal{H} which is naturally isomorphic to the symmetric Fock space over $\mathcal{H}_{\mathfrak{h}}$. This observation allows to define the wave operators and the S -matrix.

We denote by $\Gamma(\mathcal{H}_{\mathfrak{h}})$ the symmetric Fock space over the single-particle subspace $\mathcal{H}_{\mathfrak{h}}$ and by $a^*(f), a(f)$, $f \in \mathcal{H}_{\mathfrak{h}}$, the corresponding creation and annihilation operators. We also define the unitary representation of translations on the single-particle subspace

$$U_{\mathfrak{h}}(t, x) = U(t, x)|_{\mathcal{H}_{\mathfrak{h}}}. \quad (4.39)$$

Its second quantization $(t, x) \mapsto \Gamma(U_{\mathfrak{h}}(t, x))$ is a unitary representation of space-time translations on $\Gamma(\mathcal{H}_{\mathfrak{h}})$.

Definition 4.12. *We say that an isometry $W^{\text{out}} : \Gamma(\mathcal{H}_{\mathfrak{h}}) \rightarrow \mathcal{H}$ is an outgoing wave operator if*

$$W^{\text{out}}\Omega = \Omega, \quad (4.40)$$

$$W^{\text{out}}(a^*(\Psi_1) \dots a^*(\Psi_n)\Omega) = \Psi_1^{\text{out}} \times \dots \times \Psi_n^{\text{out}}, \quad (4.41)$$

$$U(t, x) \circ W^{\text{out}} = W^{\text{out}} \circ \Gamma(U_{\mathfrak{h}}(t, x)), \quad (4.42)$$

for any collection of $\Psi_1, \dots, \Psi_n \in \mathcal{H}_{\mathfrak{h}}$ as in Theorem 4.8 and any $(t, x) \in \mathbb{R} \times \Gamma$.

An incoming wave operator W^{in} is defined analogously, by taking the limit $t \rightarrow -\infty$ in Theorem 4.8. The S -matrix is an isometry on $\Gamma(\mathcal{H}_{\mathfrak{h}})$ given by

$$S = (W^{\text{out}})^* W^{\text{in}}. \quad (4.43)$$

The main result of this section is the following theorem:

Theorem 4.13. *If the isolated mass shell \mathfrak{h} is either pseudo-relativistic or regular then the wave operators and the S -matrix exist and are unique.*

Proof. We proceed similarly as in the proof of Proposition 6.6 of [29]. Let $\Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}})$ be the n -particle subspace of $\Gamma(\mathcal{H}_{\mathfrak{h}})$ and let $\mathcal{F} \subset \Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}})$ be the subspace spanned by vectors $a^*(\Psi_1) \dots a^*(\Psi_n)\Omega$, Ψ_i as in Theorem 4.8. Due to (4.29,4.33), there exists a unique isometry $W_n^{\text{out}} : \mathcal{F} \rightarrow \mathcal{H}$ s.t.

$$W_n^{\text{out}}(a^*(\Psi_1) \dots a^*(\Psi_n)\Omega) = \Psi_1^{\text{out}} \times \dots \times \Psi_n^{\text{out}}. \quad (4.44)$$

By (4.34) it satisfies

$$U(t, x) \circ W_n^{\text{out}} = W_n^{\text{out}} \circ \Gamma^{(n)}(U_{\mathfrak{h}}(t, x)), \quad (4.45)$$

where $\Gamma^{(n)}(U_{\mathfrak{h}}(t, x))$ is the restriction of $\Gamma(U_{\mathfrak{h}}(t, x))$ to $\Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}})$. Thus it suffices to prove that \mathcal{F} is dense in $\Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}})$. Indeed, in that case W_n^{out} extends uniquely to an isometry $W_n^{\text{out}} : \Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}}) \rightarrow \mathcal{H}$, and $W^{\text{out}} := \bigoplus_{n \in \mathbb{N}_0} W_n^{\text{out}}$ is the unique outgoing wave operator in the sense of Definition 4.12. The incoming wave operator W^{in} is defined analogously, by taking limits $t \rightarrow -\infty$ in Theorems 4.8 and 4.9.

Let us now show density of \mathcal{F} . We denote by $P_{\mathfrak{h}}$ the spectral measure of $U_{\mathfrak{h}}$ and define the product spectral measure on $(\mathbb{R} \times \widehat{\Gamma})^{\times n}$ with values in $\mathcal{H}_{\mathfrak{h}}^{\otimes n}$ (the non-symmetrized n -particle subspace):

$$d\tilde{P}_{\mathfrak{h}}((E_1, p_1), \dots, (E_n, p_n)) := dP_{\mathfrak{h}}(E_1, p_1) \otimes \dots \otimes dP_{\mathfrak{h}}(E_n, p_n). \quad (4.46)$$

Clearly, $\tilde{P}_{\mathfrak{h}}$ is supported in $\mathfrak{h}^{\times n}$ and $\tilde{P}_{\mathfrak{h}}(\mathfrak{h}^{\times n})\mathcal{H}_{\mathfrak{h}}^{\otimes n} = \mathcal{H}_{\mathfrak{h}}^{\otimes n}$. Using Lemma 4.17 below, it is easy to see that

$$\overline{\mathcal{F}} = \Theta_s \circ \tilde{P}_{\mathfrak{h}}(\mathfrak{h}^{\times n} \setminus D)\mathcal{H}_{\mathfrak{h}}^{\otimes n}, \quad (4.47)$$

where $\Theta_s : \mathcal{H}_{\mathfrak{h}}^{\otimes n} \mapsto \Gamma^{(n)}(\mathcal{H}_{\mathfrak{h}})$ is the orthogonal projection on the subspace of symmetrized n -particle vectors and

$$D := \{((\Sigma(p_1), p_1), \dots, (\Sigma(p_n), p_n)) \in \mathfrak{h}^{\times n} \mid \nabla \Sigma(p_i) = \nabla \Sigma(p_j) \text{ for some } i \neq j\}. \quad (4.48)$$

We have to subtract D in (4.47) to account for the disjointness of velocity supports of vectors in \mathcal{F} . By Lemma 4.19 below, the auxiliary set

$$D_0 := \{p_1, \dots, p_n \in \Delta_{\mathfrak{h}} \mid \nabla \Sigma(p_i) = \nabla \Sigma(p_j) \text{ for some } i \neq j\} \quad (4.49)$$

has Lebesgue measure zero, which implies finally that $\tilde{P}_{\mathfrak{h}}(D) = 0$ by Lemma 4.16 below. With this and equation (4.47), the proof is complete. \square

Remark 4.14. *Although the construction of asymptotic states requires disjoint velocity supports, Theorem 4.13 shows that, up to isometry, their span is dense in the bosonic Fock space. In particular, n -particle product states, or condensates, can be obtained as limits of n -particle asymptotic states with disjoint velocity supports.*

We now state and prove the lemmas used in the proof of the above theorem.

Lemma 4.15. *Let $\Delta \subset \mathfrak{h}$ be an open bounded set and let \mathcal{K} be a subspace of \mathcal{H} spanned by vectors of the form $B^*\Omega$, where B^* is a creation operator in the sense of Definition 4.4 s.t. $\text{Sp}_{\pi^{-1}(B^*)} \tau \cap \mathfrak{h} \subset \Delta$. Then \mathcal{K} is dense in $P(\Delta)\mathcal{H}$.*

Proof. We pick an arbitrary $A^* \in \mathfrak{A}_{\text{loc}}$ and $f \in S(\mathbb{R} \times \Gamma)$ s.t. $\text{supp } \widehat{f} \cap \text{Sp } U \subset \Delta$. Then $B^* = \pi(\tau_f(A^*))$ is a creation operator as specified in the lemma and

$$B^*\Omega = \int_{\Delta} \widehat{f}(E, p) dP(E, p) \pi(A^*)\Omega \in P(\Delta)\mathcal{H}_{\mathfrak{h}}. \quad (4.50)$$

Approximating with \widehat{f} the characteristic function of Δ and exploiting cyclicity of Ω we conclude the proof. \square

Lemma 4.16. *Let $O \subset \Delta_{\mathfrak{h}}$ and $\mathfrak{h}_O := \{(\Sigma(p), p) \in \mathfrak{h} \mid p \in O\}$. If O has Lebesgue measure zero then $P(\mathfrak{h}_O) = 0$.*

Proof. We proceed similarly as in the first part of the proof of Proposition 2.2 from [18]. From Lemma 4.15 we know that vectors of the form $B^*\Omega$, where B^* is a creation operator in the sense of Definition 4.4, span a dense subspace of $\mathcal{H}_{\mathfrak{h}}$. By formula (3.7), we can write

$$\begin{aligned} \langle B^*\Omega, P(\mathfrak{h}_O)B^*\Omega \rangle &= \lim_{n \rightarrow \infty} \int_{\mathfrak{h}} \widehat{\chi}_{n, \mathfrak{h}_O}(E, p) \langle B^*\Omega, dP(E, p)B^*\Omega \rangle \\ &= \lim_{n \rightarrow \infty} \int \widehat{\chi}_{n, \mathfrak{h}_O}(\Sigma(p), p) \langle B^*\Omega, dP(E, p)B^*\Omega \rangle, \end{aligned} \quad (4.51)$$

where $\chi_{n, \mathfrak{h}_O} \in S(\mathbb{R} \times \Gamma)$ and $\widehat{\chi}_{n, \mathfrak{h}_O}$ approximate the characteristic function of \mathfrak{h}_O pointwise. We set $\widehat{\chi}_{n, O}(p) := \widehat{\chi}_{n, \mathfrak{h}_O}(\Sigma(p), p)$, which approximates pointwise the characteristic function of O . We get from (4.51)

$$\begin{aligned} \int \widehat{\chi}_{n, \mathfrak{h}_O}(\Sigma(p), p) \langle B^*\Omega, dP(E, p)B^*\Omega \rangle &= (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} \chi_{n, O}(x) \langle B^*\Omega, U(x)B^*\Omega \rangle \\ &= (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} \chi_{n, O}(x) \langle \Omega, [B, B^*(x)]\Omega \rangle \\ &= (2\pi)^{-\frac{d}{2}} \int_{\widehat{\Gamma}} \widehat{\chi}_{n, O}(p) \widehat{f}(p) dp, \end{aligned} \quad (4.52)$$

where we used the translation invariance of Ω and $B\Omega = 0$ in the second equality. In the last line, $\widehat{f}(x) := \langle \Omega, [B, B^*(x)]\Omega \rangle$ is a rapidly decreasing function by Lemma 3.9 and in the last step we made use of Parseval's theorem. Since $\widehat{\chi}_{n, O}$ converges pointwise to a characteristic function of a set of Lebesgue measure zero, we have shown $\|P(\mathfrak{h}_O)B^*\Omega\| = 0$ and the proof is complete. \square

Lemma 4.17. *Let $\Delta \subset \mathfrak{h}$ be an open bounded set and \mathcal{L} be a subspace of $\mathcal{H}_{\mathfrak{h}}$ spanned by vectors of the form $B_t^*(g_t)\Omega$. Here $B_t^*(g_t)$ is a HR creation operator such that $\text{supp } \widehat{g} \subset \Delta_{\mathfrak{h}} \setminus \mathcal{X}_0$, where \mathcal{X}_0 is a set of Lebesgue measure zero and $\{(\Sigma(p), p) \mid p \in \text{supp } \widehat{g}\} \subset \Delta$. Then \mathcal{L} is dense in $P(\Delta)\mathcal{H}_{\mathfrak{h}}$.*

Remark 4.18. *If the mass shell is pseudo-relativistic (resp. regular) we use this lemma with $\mathcal{X}_0 = \emptyset$ (resp. $\mathcal{X}_0 = \mathcal{X}$).*

Proof. First, we note that by Lemma 4.16 the set $\mathfrak{h}_{\mathcal{X}_0} := \{(\Sigma(p), p) \mid p \in \mathcal{X}_0\}$ satisfies $P(\mathfrak{h}_{\mathcal{X}_0}) = 0$, thus we can assume that Δ does not intersect with $\mathfrak{h}_{\mathcal{X}_0}$. Now we will argue by contradiction. Suppose that there is a non-zero vector $\Psi \in P(\Delta)\mathcal{H}_{\mathfrak{h}}$ which is orthogonal to all elements of \mathcal{L} . Now we find g , as in the statement of the lemma, s.t.

$$\int_{\Delta} \overline{\widehat{g}(p)} dP(E, p) \Psi \neq 0. \quad (4.53)$$

To this end we approximate with \widehat{g} the characteristic function of $\Delta_0 := \{p \in \Delta_{\mathfrak{h}} \mid (\Sigma(p), p) \in \Delta\}$ from inside. This is possible since we assumed that $\mathfrak{h}_{\mathcal{X}}$ does not intersect with Δ . Next, recall that \mathcal{K} , as defined in Lemma 4.15, consists of vectors of the form $B^*\Omega$, where B^* is a creation operator s.t. $\text{Sp}_{\pi^{-1}(B^*)}\tau \cap \mathfrak{h} \subset \Delta$. Since \mathcal{K} is dense in $P(\Delta)\mathcal{H}_{\mathfrak{h}}$, we can find B^* , within the above restrictions, s.t.

$$0 \neq \langle B^*\Omega, \int_{\Delta} \overline{\widehat{g}(p)} dP(E, p)\Psi \rangle = \langle B_t^*(g_t)\Omega, \Psi \rangle. \quad (4.54)$$

But the last expression is zero by definition of Ψ . \square

Note that the claim of the lemma holds a fortiori if $\text{supp } \widehat{g} \subset \Delta_{\mathfrak{h}}$.

Lemma 4.19. *Let Σ be a dispersion relation of a mass shell (not necessarily pseudo-relativistic or regular, see Definition 4.1). Then the set*

$$D_0 := \{p_1, \dots, p_n \in \Delta_{\mathfrak{h}} \mid \nabla\Sigma(p_i) = \nabla\Sigma(p_j) \text{ for some } i \neq j\} \quad (4.55)$$

has Lebesgue measure zero in $\widehat{\Gamma}^{\times n}$.

Proof. Note that D_0 is a union of sets

$$D_{0,i,j} := \{p_1, \dots, p_n \in \Delta_{\mathfrak{h}} \mid \nabla\Sigma(p_i) = \nabla\Sigma(p_j)\}, \quad 1 \leq i < j \leq n. \quad (4.56)$$

Suppose, by contradiction, that $D_{0,1,2}$ has non-zero Lebesgue measure. Then, for some $v \in \mathbb{R}^d$, the set

$$D_{0,v} := \{p_1, \dots, p_n \in \Delta_{\mathfrak{h}} \mid \nabla\Sigma(p_1) = v\} \quad (4.57)$$

also has non-zero Lebesgue measure. Otherwise computing the measure of $D_{0,1,2}$ as an iterated integral of its characteristic function we would get zero. Clearly, we have

$$D_{0,v} \subset \{p_1 \in \Delta_{\mathfrak{h}} \mid D^2\Sigma(p_1) = 0\} \times \{p_2, \dots, p_n \in \Delta_{\mathfrak{h}}\}, \quad (4.58)$$

hence $\mathcal{T} = \{p \in \Delta_{\mathfrak{h}} \mid D^2\Sigma(p) = 0\}$ has non-zero Lebesgue measure, which contradicts property (c) from Definition 4.1. \square

5 Shape of the spectrum

In this section we prove three results concerning the shape of $\text{Sp } U$. In the following, v_{λ} is the Lieb-Robinson velocity defined in Theorem 2.1.

Proposition 5.2, whose proof uses the scattering theory developed in the previous section, says that the velocity of a particle computed as $\nabla\Sigma(p)$ cannot be larger than the Lieb-Robinson velocity v_{λ} . This shows that v_{λ} plays a similar role in lattice systems as the velocity of light in relativistic theories. A different proof of this result, adopting methods from the proof of Proposition 2.1 of [18], appeared in [70]. In [63], v_{λ} was also shown to be a propagation bound for ‘signals’.

Proposition 5.3 says that $\text{Sp } U$ is a semi-group in $\mathbb{R} \times \widehat{\Gamma}$ i.e. if $q_1, q_2 \in \text{Sp } U$ then also $q_1 + q_2 \in \text{Sp } U$. A special case of this result actually follows from Theorem 4.9 and Lemma 4.17: if $q_1, \dots, q_n \in \mathfrak{h}$ then $q_1 + \dots + q_n \in \text{Sp } U$ (at least if $\mathcal{X} = \mathcal{T} = \emptyset$). However, the general proof below does not use scattering theory but follows directly from the clustering property (Theorem 2.4) and the energy-momentum transfer relation (3.15). This argument follows closely the proof from relativistic QFT, see Theorem 5.4.1 of [35].

Proposition 5.4 relates the spectrum of the finite volume Hamiltonians H_Λ to the spectrum of the Hamiltonian H in the thermodynamic limit. More precisely, H is the GNS Hamiltonian of the ground state which is the limit of the ground states of H_Λ . Our result says that gaps cannot close in the thermodynamic limit. This is of important practical interest, since quantum spin systems are usually defined in finite volume where explicit spectral estimates can be obtained. And indeed, it will be useful in the next section where we discuss concrete examples, to which the scattering theory of the previous section can be applied.

We start with the following lemma from [67], which we need for the proof of Proposition 5.2. It says that quantum spin systems satisfying the Lieb-Robinson bounds are asymptotically abelian w.r.t. space-time translations in the cone $\{|x| > v_\lambda t\}$ (cf. [70, Theorem II.7]).

Lemma 5.1. *Under the assumptions of Theorem 2.1 and for any $0 < \epsilon < v_\lambda^{-1}$, we have*

$$\|[\tau_t \circ \tau_x(A), B]\| \leq C(A, B, \lambda) e^{-\lambda(1-\epsilon v_\lambda)|x|} \quad (5.1)$$

in the cone $\mathcal{C}_\epsilon := \{(t, x) \mid |t| \leq \epsilon|x|\}$, where

$$C(A, B, \lambda) = 2\|A\|\|B\|C_\lambda^{-1}\|F\| \min\{|\Lambda_1|, |\Lambda_2|\} e^{\lambda \text{diam}(\Lambda_1 \cup \Lambda_2)}. \quad (5.2)$$

Proof. By Theorem 2.1 and the τ_x invariance of the norm,

$$\begin{aligned} \|[\tau_t \circ \tau_x(A), B]\| &\leq \frac{2\|A\|\|B\|}{C_\lambda} e^{2\|\Phi\|_\lambda C_\lambda |t|} e^{-\lambda|x|} \sum_{w \in \Lambda_1} \sum_{z \in \Lambda_2} e^{\lambda|z-w|} F(|z - (w-x)|) \\ &\leq \frac{2\|A\|\|B\|}{C_\lambda} \|F\| \min\{|\Lambda_1|, |\Lambda_2|\} e^{\lambda \text{diam}(\Lambda_1 \cup \Lambda_2)} \cdot e^{-\lambda(|x| - v_\lambda |t|)} \end{aligned} \quad (5.3)$$

and the lemma follows from the condition $|t| \leq \epsilon|x|$. \square

Proposition 5.2. *Under the standing assumptions of Section 4, we have the following: $|\nabla \Sigma(p)| \leq v_\lambda$ for any $p \in \Delta_\mathfrak{h}$.*

Proof. We prove this by contradiction. Suppose there is $p_0 \in \Delta_\mathfrak{h}$ s.t. $|\nabla \Sigma(p_0)| > v_\lambda$. Then, by smoothness of Σ , there is a neighbourhood O_{p_0} of p_0 , that is compactly contained in $\Delta_\mathfrak{h}$, and $0 < \epsilon < v_\lambda^{-1}$ s.t. $|\nabla \Sigma(p)| \geq \epsilon^{-1}$ for all $p \in O_{p_0}$. We consider the subset of the mass shell

$$\mathfrak{h}_{O_{p_0}} := \{(\Sigma(p), p) \mid p \in O_{p_0}\}, \quad (5.4)$$

which is open in \mathfrak{h} because the dependence $\Delta_\mathfrak{h} \ni p \mapsto (\Sigma(p), p) \in \mathfrak{h}$ is diffeomorphic and bounded since O_{p_0} is away from the boundary of $\Delta_\mathfrak{h}$ ⁴. To arrive at a contradiction we will show that the projection $P(\mathfrak{h}_{O_{p_0}})$ is zero.

By Lemma 4.17, vectors of the form

$$B^*(g)\Omega = B_t^*(g_t)\Omega, \quad \text{supp } \widehat{g} \subset O_{p_0}, \quad (5.5)$$

span a dense subspace in $P(\mathfrak{h}_{O_{p_0}})\mathcal{H}$. Thus it is enough to show that all these vectors are zero.

Since $(B^*(g))^*\Omega = 0$, we can write for any given $A' \in \mathfrak{A}(\Lambda')$, with Λ' arbitrary but finite,

$$\langle \pi(A')\Omega, B^*(g)\Omega \rangle = \langle \Omega, [\pi(A')^*, B_t^*(g_t)]\Omega \rangle = \langle \Omega, [\pi(A')^*, B_t^*(\chi_{+,t}g_t)]\Omega \rangle + O(t^{-\infty}), \quad (5.6)$$

⁴A priori $\Sigma(p)$ may tend to infinity when p approaches the boundary of the open set $\Delta_\mathfrak{h}$. Once proven, Proposition 5.2 excludes such behaviour, however.

where we applied Lemma 4.5 (d). Recall that $\chi_{+,t}(x) = \chi_+(x/t)$ where $\chi_+ \in C_0^\infty(\mathbb{R}^d)$ is supported in a small neighbourhood $V(g)^\delta$ of the velocity support $V(g)$. Clearly, we can assume that

$$V(g)^\delta \subset \{\nabla\Sigma(p) \mid p \in O_{p_0}\}, \quad (5.7)$$

since \widehat{g} is supported inside of O_{p_0} .

Recall that $B^* = \pi(A^*)$ for some $A^* \in \mathfrak{A}_{\text{a-loc}}$. Thus we can find $A_r \in \mathfrak{A}(Z^r)$ s.t. $\|A - A_r\| = O(r^{-\infty})$. Hence

$$\langle \Omega, [\pi(A')^*, B_t^*(\chi_{+,t}g_t)]\Omega \rangle = \langle \Omega, [\pi(A')^*, \pi(\tau_t(A_r^*))(\chi_{+,t}g_t)]\Omega \rangle + O(r^{-\infty}t^d), \quad (5.8)$$

where we used that $\|g_t\|_1 = O(t^d)$. Now we note that the sum

$$\tau_t(A_r^*)(\chi_{+,t}g_t) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} \tau_{(t,x)}(A_r^*)\chi_+(x/t)g_t(x) \quad (5.9)$$

extends over x in the set $X_t := \{x \in \Gamma \mid |x/t| \geq \epsilon^{-1}\}$. Clearly $\{(t,x) \mid x \in X_t\}$ is contained in the cone $\mathcal{C}_\epsilon := \{(t,x) \mid |t| \leq \epsilon|x|\}$. Thus, by Lemma 5.1, we have

$$\begin{aligned} |\langle \Omega, [\pi(A')^*, \pi(\tau_t(A_r^*))(\chi_{+,t}g_t)]\Omega \rangle| &\leq \|\widehat{g}\|_1 C(A', A_r^*, \lambda) \sum_{x \in \Gamma} e^{-\lambda(1-\epsilon v_\lambda)|x|} |\chi_+(x/t)| \\ &\leq \|\widehat{g}\|_1 C(A', A_r^*, \lambda)_{\mathcal{C}_\epsilon, v_\lambda} e^{-\frac{1}{2}\lambda(1-\epsilon v_\lambda)\epsilon^{-1}t}, \end{aligned} \quad (5.10)$$

where

$$C(A', A_r^*, \lambda) = 2\|A'\| \|A_r\| C_\lambda^{-1} \|F\| \min\{|A'|, |Z^r|\} e^{\lambda \text{diam}(\Lambda' \cup Z^r)}, \quad (5.11)$$

and we note that $\|A_r\| \leq \|A\| + O(r^{-\infty})$. Thus, setting $r(t) = t^\epsilon$, $\epsilon > 0$ sufficiently small, we obtain that the r.h.s. of (5.10) and error terms in (5.8), (5.6) tend to zero as $t \rightarrow \infty$. Since the l.h.s. of (5.6) is independent of t , we conclude that

$$\langle \pi(A')\Omega, B^*(g)\Omega \rangle = 0. \quad (5.12)$$

As this is true for any $A' \in \mathfrak{A}_{\text{loc}}$ we obtain that $B^*(g)\Omega = 0$ and the proof is complete. \square

Although conceptually very satisfactory, the bound provided by the proposition has the following practical flaw inherited from the proof of Theorem 2.1: the Lieb-Robinson velocity exhibited there, even after taking the infimum over the possible λ 's, only gives a crude upper bound, see Remark 2 in the original article [47] for a brief discussion.

Proposition 5.3. *Under the assumptions of Theorem 2.4, the following additivity property holds: Suppose that $q_1, q_2 \in \text{Sp } U$. Then $q_1 + q_2 \in \text{Sp } U$ where $+$ denotes the group operation in $\mathbb{R} \times \widehat{\Gamma}$.*

Proof. Let Δ be a (bounded) neighbourhood of $q_1 + q_2$. Let Δ_1, Δ_2 be (bounded) neighbourhoods of q_1, q_2 s.t. $\overline{\Delta_1} + \overline{\Delta_2} \subset \Delta$. We pick $A_1, A_2 \in \mathfrak{A}_{\text{a-loc}}$ s.t. $\text{Sp}_{A_1}\tau \subset \Delta_1$ and $\text{Sp}_{A_2}\tau \subset \Delta_2$. Then, by the energy-momentum transfer relation (3.15), we have for any $x \in \Gamma$

$$\Psi_x := \pi(\tau_x(A_1))\pi(A_2)\Omega \in P(\Delta)\mathcal{H}, \quad (5.13)$$

thus it suffices to show that some of these vectors are non-zero.

By almost locality, we can find $A_{i,r} \in \mathfrak{A}(Z^r)$ s.t. $(A_i - A_{i,r}) = O(r^{-\infty})$ in norm for $i = 1, 2$. Thus we have

$$\|\Psi_x\| = \|\Psi_{r,x}\| + O(r^{-\infty}), \text{ where } \Psi_{r,x} := \pi(\tau_x(A_{1,r}))\pi(A_{2,r})\Omega, \quad (5.14)$$

uniformly in x . By Theorem 2.4,

$$\lim_{|x| \rightarrow \infty} \|\Psi_{r,x}\| = \|\pi(A_{1,r})\Omega\| \|\pi(A_{2,r})\Omega\| = \|\pi(A_1)\Omega\| \|\pi(A_2)\Omega\| + O(r^{-\infty}). \quad (5.15)$$

By choosing r sufficiently large, we can conclude from (5.14), (5.15), that (5.13) is non-zero for sufficiently large $|x|$, provided that $\pi(A_1)\Omega, \pi(A_2)\Omega$ are non-zero.

To justify this last point, we pick $A_{0,i} \in \mathfrak{A}_{\text{loc}}$, $f_i \in S(\mathbb{R} \times \widehat{\Gamma})$, s.t. $\text{supp } \widehat{f}_i \subset \Delta_i$ and set $A_i = \tau_{f_i}(A_{0,i})$. Then we have

$$\pi(A_i)\Omega = \int \widehat{f}(E, p) dP(E, p) \pi(A_{0,i})\Omega. \quad (5.16)$$

Since with the spectral integrals above we can approximate $P(\Delta_i)$ (and $P(\Delta_i) \neq 0$ since $q_i \in \text{Sp } U$), by cyclicity of the ground state vector we can find $A_{0,i}$ s.t. $\pi(A_i)\Omega$ are non-zero. \square

As a preparation to the proof of Proposition 5.4, we discuss briefly finite-volume Hamiltonians and their ground states.

We first show how infinite volume ground states can be obtained from ground states of finite dimensional systems. We essentially follow the argument of [12, Prop. 5.3.25]. Let ψ_Λ be a ground state vector of H_Λ for the eigenvalue E_Λ , and define $\omega_\Lambda(A) := \langle \psi_\Lambda, A\psi_\Lambda \rangle$ for any $A \in \mathfrak{A}(\Lambda)$, which can be extended to the whole algebra. Banach-Alaoglu's theorem then ensures that there are weak-* limit points of the net ω_Λ as $\Lambda \nearrow \Gamma$, that we shall generically call ω . First,

$$\langle \psi_\Lambda, A^*[H_\Lambda, A]\psi_\Lambda \rangle = \langle \psi_\Lambda, A^*H_\Lambda A\psi_\Lambda \rangle - \langle \psi_\Lambda, A^*AH_\Lambda\psi_\Lambda \rangle \geq 0, \quad (5.17)$$

where we used in the first term that $H_\Lambda \geq E_\Lambda$ and in the second that $H_\Lambda\psi_\Lambda = E_\Lambda\psi_\Lambda$. Hence, ω_Λ is a ground state in the sense of (2.15) since $-i\delta_\Lambda = [H_\Lambda, \cdot]$ is the generator of the finite-volume dynamics. Furthermore, for any $A \in D(\delta)$ and any convergent (sub)sequence⁵ ω_{Λ_n} , there is a sequence $A_n \in \mathfrak{A}(\Lambda_n)$ such that $A_n \rightarrow A$ and $\delta_{\Lambda_n}(A_n) \rightarrow \delta(A)$. Hence,

$$-i\omega(A^*\delta(A)) = \lim_{n \rightarrow \infty} -i\omega_{\Lambda_n}(A_n^*\delta_{\Lambda_n}(A_n)) \geq 0, \quad (5.18)$$

so that the limiting state ω is a ground state in the general sense.

In this paper we are interested in models with isolated mass shells (cf. Definition 4.1). As we will see in the next section in an example, in lattice systems such mass shells may correspond to intervals in the spectrum of the GNS Hamiltonian, isolated from the rest of the spectrum by gaps⁶. Here again, such gaps can be deduced from the finite volume properties. The following proposition shows that a uniform spectral gap in finite volume cannot abruptly close in the GNS representation.

Proposition 5.4. *Let $E \in \mathbb{R}$ be such that for some $\epsilon > 0$, $\Lambda_0 \in \mathcal{P}_{\text{fin}}(\Gamma)$,*

$$(E_\Lambda + (E - \epsilon, E + \epsilon)) \cap \text{Sp}(H_\Lambda) = \emptyset$$

for all $\Lambda \supset \Lambda_0$. Then $E \notin \text{Sp}(H)$, where H is the GNS Hamiltonian of a weak- limit ω as above.*

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triplet of the limiting state ω . Let $f \in S(\mathbb{R})$ be such that \widehat{f} is supported in $(E - \epsilon, E + \epsilon)$. To prove the claim, it suffices to show that for any $A, B \in \mathfrak{A}_{\text{loc}}$,

$$\left\langle \pi(A)\Omega, \widehat{f}(H)\pi(B)\Omega \right\rangle = 0. \quad (5.19)$$

⁵A subsequence (rather than a subnet) can be found by exploiting the fact that the net converges in norm on any local algebra.

⁶Note that this is not possible in relativistic theories.

First, we write

$$\tau_f(B) = \frac{1}{\sqrt{2\pi}} \int dt f(t) \tau_t(B), \quad (5.20)$$

analogously to (3.10). (We skip the superscript (1) here as there is no danger of confusion). Next, we use the invariance of Ω to get

$$\widehat{f}(H)\pi(B)\Omega = \frac{1}{\sqrt{2\pi}} \int dt f(t) e^{iHt} \pi(B) e^{-iHt} \Omega = \frac{1}{\sqrt{2\pi}} \int dt f(t) \pi(\tau_t(B)) \Omega = \pi(\tau_f(B)) \Omega. \quad (5.21)$$

We further note that

$$\left\langle \pi(A)\Omega, \widehat{f}(H)\pi(B)\Omega \right\rangle = \omega(A^* \tau_f(B)). \quad (5.22)$$

Defining $\tau_f^\Lambda(B)$ analogously to (5.20), and recalling that $\|\tau_t^\Lambda(B) - \tau_t(B)\| \rightarrow 0$ as $\Lambda \nearrow \Gamma$, we obtain

$$\lim_{\Lambda \rightarrow \Gamma} \|\tau_f^\Lambda(B) - \tau_f(B)\| = 0. \quad (5.23)$$

This, and the weak* convergence $\omega_\Lambda \rightarrow \omega$ imply that for any $\delta > 0$, there is Λ_0 such that for all $\Lambda \supset \Lambda_0$,

$$\begin{aligned} \|A^* \tau_f(B) - A^* \tau_f^\Lambda(B)\| &< \delta/2, \\ |\omega(A^* \tau_f^\Lambda(B)) - \omega_\Lambda(A^* \tau_f^\Lambda(B))| &< \delta/2. \end{aligned} \quad (5.24)$$

Hence, by (5.22),

$$\left| \left\langle \pi(A)\Omega, \widehat{f}(H)\pi(B)\Omega \right\rangle - \omega_\Lambda(A^* \tau_f^\Lambda(B)) \right| < \delta. \quad (5.25)$$

But using the fact that $H_\Lambda \psi_\Lambda = E_\Lambda \psi_\Lambda$,

$$\omega_\Lambda(A^* \tau_f^\Lambda(B)) = \frac{1}{\sqrt{2\pi}} \int dt f(t) \omega_\Lambda \left(A^* e^{it(H_\Lambda - E_\Lambda)} B \right) = \omega_\Lambda \left(A^* \widehat{f}(H_\Lambda - E_\Lambda) B \right) = 0, \quad (5.26)$$

since \widehat{f} is supported in $(E - \epsilon, E + \epsilon)$ while, by assumption, this interval is outside the spectrum of $H_\Lambda - E_\Lambda$. This proves (5.19) and therefore the proposition. \square

6 Examples

In this section we show that the Ising model in a strong magnetic field ϵ^{-1} (given by (6.1) below) satisfies the standing assumptions of Section 4 and thus admits scattering theory developed in this paper. This model belongs to a larger family of systems which are perturbations of ‘classical’ models studied e.g. in [1, 22, 42], or in the same spirit but with results that are somewhat further away from what is required here, [11, 21, 43]. Here, we shall follow mostly [77] and the closely related [75], and apply concrete results of [76] and [62]. We conjecture that many other models of this family fit into the framework of Section 4, but complete proofs are missing. We will come back to this point in the later part of this section.

The Ising model (with open boundary conditions) is given by the following family of finite volume Hamiltonians

$$H_{\Lambda, \epsilon}^I := -\frac{1}{2} \sum_{x \in \Lambda} \left(\sigma_x^{(3)} - 1 \right) - \epsilon \sum_{(x, y) \in \mathcal{E}(\Lambda)} \sigma_x^{(1)} \sigma_y^{(1)}, \quad (6.1)$$

where $\mathcal{E}(\Lambda) := \{(x, y) \in \Lambda \times \Lambda \mid |y - x| = 1\}$ is the set of undirected edges of the lattice. When $\epsilon = 0$, the Hamiltonian is diagonal in the tensor product basis of eigenvectors of $\sigma^{(3)}$, and $\text{Sp}(H_{\Lambda, 0}^I) = \{n \in$

$\mathbb{N} \mid n \leq |\Lambda|$. The unique ground state is given by the vector $\otimes_{x \in \Lambda} |+\rangle$, with ground state energy $E_\Lambda^I = 0$ for all $\Lambda \in \mathcal{P}_{\text{fin}}(\Gamma)$. It is easy to see that this model satisfies the assumptions of Theorem 2.1 and thus the Lieb-Robinson bound holds.

In the course of our analysis in this section we will also need a version of the Ising model with periodic boundary conditions: for any $L \in \mathbb{N}$ we write $[-L, L] := \{-L, -L+1, \dots, L\}$ for an interval and $[-L, L]^{\times d}$ for a hypercube of side $2L$ centred at zero. The finite volume Hamiltonian of the hypercube $\Lambda = [-L, L]^{\times d}$ is given by

$$\tilde{H}_{\Lambda, \epsilon}^I := H_{\Lambda, \epsilon}^I - \epsilon \sum_{(x, y) \in \mathcal{F}(\partial\Lambda)} \sigma_x^{(1)} \sigma_y^{(1)}, \quad (6.2)$$

where the boundary $\partial\Lambda$ consists of all $x \in \Lambda$ which have nearest neighbours outside of Λ , $\mathcal{F}(\partial\Lambda) := \{(x, y) \in \partial\Lambda \times \partial\Lambda \mid y - x \in \{\pm 2Le_1, \dots, \pm 2Le_d\}\}$ and $\{e_i\}_{i=1, \dots, d}$ are primitive vectors of the lattice.

We note that due to the non-local character of the boundary terms, $\tilde{H}_{\Lambda, \epsilon}^I$ does not satisfy the assumptions of Theorem 2.1, and we cannot conclude the existence of its global dynamics $\tilde{\tau}_{\epsilon, t}$ directly from Corollary 2.2. Nevertheless, we have:

Lemma 6.1. *The global dynamics $\tilde{\tau}_{\epsilon, t}$ of the Ising model (6.2) with periodic boundary conditions exists and coincides with the global dynamics $\tau_{\epsilon, t}$ of the Ising model (6.1) with open boundary conditions. Hence, also the corresponding families of infinite volume ground states coincide.*

Proof. For any bounded X and $A \in \mathfrak{A}(X)$, let Λ_n be the smallest hypercube such that $\{x \in \mathbb{Z}^d \mid \text{dist}(x, X) \leq n\} \subset \Lambda_n$. Note that by locality it follows that $\delta_{\Lambda_1}(A) = \tilde{\delta}_{\Lambda_1}(A)$, where $\delta_{\Lambda_1}(A) := i[H_{\Lambda_1, \epsilon}^I, A]$ and $\tilde{\delta}$ is defined similarly. Note that by construction $\delta_{\Lambda_1}(\mathfrak{A}(X)) \subset \mathfrak{A}(\Lambda_1)$ and the same is true for $\tilde{\delta}_{\Lambda_n}$.

Repeating the argument above it follows that $\delta_{\Lambda_n}^m(A) = \tilde{\delta}_{\Lambda_n}^m(A)$ for all $m \leq n$ and $A \in \mathfrak{A}(X)$. Hence,

$$\tau_{\epsilon, t}^{\Lambda_n}(A) - \tilde{\tau}_{\epsilon, t}^{\Lambda_n}(A) = \frac{t^n}{n!} \sum_{m > 0} \frac{t^m n!}{(m+n)!} \left(\delta_{\Lambda_n}^{m+n} - \tilde{\delta}_{\Lambda_n}^{m+n} \right) (A). \quad (6.3)$$

Now,

$$\delta_{\Lambda_n}^N(A) = \sum_{x_N, \dots, x_0} [\Phi(x_N, x_{N-1}), [\Phi(x_{N-1}, x_{N-2}), \dots, [\Phi(x_1, x_0), A] \dots]], \quad (6.4)$$

where the sum is over sequences such that $x_0 \in X$ and $|x_i - x_{i-1}| \leq 1$ for $1 \leq i \leq N$, and $\Phi(x_i, x_j)$ is either of the interaction terms of (6.1) and $\Phi(x_i, x_i) = \Phi(x_i)$ is understood. Since $\|\Phi(x_i, x_j)\| \leq \kappa$ for $\kappa = \max(1, \epsilon)$, it follows that

$$\|\delta_{\Lambda_n}^{m+n}(A)\| \leq (4d\kappa)^{m+n} \|A\| |X|, \quad (6.5)$$

and similarly for $\tilde{\delta}_{\Lambda_n}^{m+n}(A)$, with the exception that the paths can “wrap around” the hypercube. All in all,

$$\|\tau_{\epsilon, t}^{\Lambda_n}(A) - \tilde{\tau}_{\epsilon, t}^{\Lambda_n}(A)\| \leq 2 \|A\| |X| \frac{(4d\kappa)^n |t|^n}{n!} e^{4d\kappa|t|} \quad (6.6)$$

where we used that $n!/(n+m)! \leq 1/m!$.

By Corollary 2.2 we know that $\tau_{\epsilon, t}^{\Lambda_n}(A)$ converges to $\tau_{\epsilon, t}(A)$. With the above estimate it then follows that $\tilde{\tau}_{\epsilon, t}^{\Lambda_n}(A)$ also converges to $\tau_{\epsilon, t}(A)$ for all t in a compact interval. By the group property we conclude that the dynamics of the Ising model with periodic and open boundary conditions coincide in the thermodynamic limit.

It follows from the algebraic characterization of a ground state that also the sets of infinite volume ground states are equal. \square

In order to make this section more self-contained, we recall some known facts in the following three theorems before stating the main result of this section in Proposition 6.5.

Theorem 6.2 ([51, 76]). *Let $H_{\Lambda, \epsilon}^I$ be the Ising Hamiltonian. There is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$,*

(i) $H_{\Lambda, \epsilon}^I$ has a unique ground state $\psi_{\Lambda, \epsilon}^I$ with ground state energy $E_{\Lambda, \epsilon}^I$,

(ii) there is a constant $\gamma_\epsilon > 0$ such that $H_{\Lambda, \epsilon}^I \geq (E_{\Lambda, \epsilon}^I + \gamma_\epsilon) \left(1 - |\psi_{\Lambda, \epsilon}^I\rangle\langle\psi_{\Lambda, \epsilon}^I|\right)$, uniformly in Λ ,

(iii) there is a positive constant c such that

$$\text{Sp}(H_{\Lambda, \epsilon}^I - E_{\Lambda, \epsilon}^I) \subset \bigcup_{n \in \text{Sp}(H_{\Lambda, 0}^I)} \{n_\epsilon \mid |n_\epsilon - n| \leq cn_\epsilon\}, \quad (6.7)$$

(iv) there is a unique translation-invariant ground state ω_ϵ and $\omega_\epsilon(A) = \lim_{\Lambda \nearrow \Gamma} \langle \psi_{\Lambda, \epsilon}^I, A \psi_{\Lambda, \epsilon}^I \rangle$ for all $A \in \mathfrak{A}_{\text{loc}}$. Moreover, ω_ϵ is pure.

(i-iii) are Theorem 1 of [76]. Theorem 2 there establishes the convergence of finite volume ground states, $\bar{\omega}_\epsilon = \lim_{\Lambda \nearrow \Gamma} \langle \psi_{\Lambda, \epsilon}^I, \cdot \psi_{\Lambda, \epsilon}^I \rangle$. We further claim that $\bar{\omega}_\epsilon$ is translation invariant. Indeed, the limiting state is unique and independent of the choice of increasing and absorbing sequence $\Lambda_n \nearrow \Gamma$. Now, for any such sequence and any $x \in \mathbb{Z}^d$, the sequence $\Lambda_n - x$ also converges to Γ so that, making use of translation invariance of the interaction,

$$\bar{\omega}_\epsilon(\tau_x(A)) = \lim_{n \rightarrow \infty} \langle \psi_{\Lambda_n, \epsilon}^I, \tau_x(A) \psi_{\Lambda_n, \epsilon}^I \rangle = \lim_{n \rightarrow \infty} \langle \psi_{\Lambda_n - x, \epsilon}^I, A \psi_{\Lambda_n - x, \epsilon}^I \rangle = \bar{\omega}_\epsilon(A), \quad (6.8)$$

for any local A , which proves the claim. This, and the general uniqueness and purity of the translation-invariant ground state (Theorem 3 and Remark 4.8 of [51]) implies that $\bar{\omega}_\epsilon = \omega_\epsilon$, hence we obtain (iv). The GNS triple of ω_ϵ will be denoted by $(\pi_\epsilon, \mathcal{H}_\epsilon, \Omega_\epsilon)$ and the unitary action of space-time translations by U_ϵ . The GNS Hamiltonian of the Ising model will be called H_ϵ .

The notion of a ‘one-particle subspace’ of [76] is defined as a subspace $\mathcal{H}_1 \subset \mathcal{H}_\epsilon$, that is invariant under space-time translations (3.5) and such that there is a unitary operator $V : \mathcal{H}_1 \rightarrow L^2(\widehat{\Gamma})$ satisfying:

$$(VU_\epsilon(x)V^*\widehat{f})(p) = e^{ipx}\widehat{f}(p), \quad (6.9)$$

$$(VU_\epsilon(t)V^*\widehat{f})(p) = e^{im_\epsilon(p)t}\widehat{f}(p), \quad (6.10)$$

where we identified $L^2(\widehat{\Gamma})$ with the image of the Fourier transform (D.2). Note that this says that the group of unitaries $(t, x) \mapsto U_\epsilon(t, x)$ affords a spectral decomposition on $L^2(\widehat{\Gamma})$ in which the space-time translation operators act as multiplication by $e^{i(m_\epsilon(p)t + px)}$. Up to the sign of the momentum this corresponds to the characterization of a mass shell as a subset of the spectrum $\text{Sp}U_\epsilon$ in Definition 4.1 (a) as we will see in the proof of Proposition 6.5 below.

From (6.7), Proposition 5.4 and the explicit form of $\text{Sp}(H_{\Lambda, 0}^I)$ stated above, we conclude that

$$\text{Sp}H_\epsilon \subset \bigcup_{n \in \mathbb{N}_0} \{n_\epsilon \mid |n_\epsilon - n| \leq cn_\epsilon\}. \quad (6.11)$$

(Similar inclusion is obtained in Theorem 3 of [76] by different methods). Theorem 4 of [76] ensures that the $n = 1$ component of union on the r.h.s. of (6.11) contains a non-empty isolated subset of $\text{Sp}H_\epsilon$ which gives rise to a one-particle subspace:

Theorem 6.3 ([76]). *Let H_ϵ be the GNS Hamiltonian of the Ising model, and let $0 < \epsilon < \epsilon_0$ as above. Let*

$$P_\epsilon^1 = -\frac{1}{2\pi i} \int_{\mathcal{C}} (H_\epsilon - z)^{-1} dz \quad (6.12)$$

where \mathcal{C} is a circle centered at 1 and of radius $c\epsilon < r < 1 - 2c\epsilon$, and let \mathcal{G}_ϵ^1 be its range. Then,

(i) \mathcal{G}_ϵ^1 is a one-particle subspace,

(ii) the function $p \mapsto m_\epsilon(p)$ of (6.10) is real-analytic.

Finally, let us define

$$f_{\Lambda,\epsilon}^\beta(t) := \tilde{\omega}_{\Lambda,\epsilon}^\beta \left(\sigma_0^{(1)} e^{-t\tilde{H}_{\Lambda,\epsilon}^I} \sigma_0^{(1)} e^{t\tilde{H}_{\Lambda,\epsilon}^I} \right), \quad (6.13)$$

where $\tilde{\omega}_{\Lambda,\epsilon}^\beta$ is the Gibbs state over $\mathfrak{A}(\Lambda)$ associated to the Hamiltonian $\tilde{H}_{\Lambda,\epsilon}^I$ with periodic boundary conditions. Theorem 1 of [62] reads:

Theorem 6.4 ([62]). *There exists a Baire measure μ_ϵ depending analytically on ϵ , s.t. for ϵ small enough*

$$\lim_{\Lambda \nearrow \Gamma} \lim_{\beta \rightarrow \infty} f_{\Lambda,\epsilon}^\beta(t) = \int_0^\infty e^{-t\lambda} d\mu_\epsilon(\lambda). \quad (6.14)$$

Moreover, there are two reals $0 < a_{0,\epsilon} \leq a_{1,\epsilon}$ (and the latter inequality is strict for $\epsilon > 0$) such that $a_{1,\epsilon} - a_{0,\epsilon} = O(\epsilon)$ and μ_ϵ has no support in $(0, a_{0,\epsilon})$, while it is absolutely continuous in $[a_{0,\epsilon}, a_{1,\epsilon})$. The dependence $\epsilon \mapsto a_{0,\epsilon}$ is real analytic.

The following proposition is now an application of Theorems 6.2–6.4 to the present case.

Proposition 6.5. *There is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the infinite volume Ising model (6.1) satisfies the standing assumptions of Section 4. In particular, it has a unique ground state and an isolated mass shell in the sense of Definition 4.1 with $\Delta_{\mathfrak{h}} = S_1^d$. For any $d \geq 1$ this mass shell is pseudo-relativistic. For $d = 1$ it is also regular.*

Remark 6.6. *In order to ensure that the mass shell is pseudo-relativistic, ϵ_0 may depend on d . The question of regularity of the mass shell in the case $d > 1$ is not settled.*

Proof. First of all, we claim that the uniqueness and purity of the translation invariant ground state of Theorem 6.2 (iv) implies that π_ϵ has a unique (up to phase) normalized ground state vector. Indeed, suppose that $\Omega'_\epsilon \in \mathcal{H}_\epsilon$ is a unit vector such that $H_\epsilon \Omega'_\epsilon = 0$, while $\Omega'_\epsilon \neq \lambda \Omega_\epsilon$ for $\lambda \in S_1$. Then, the vector state ω'_ϵ associated to Ω'_ϵ satisfies

$$-i\omega'_\epsilon(A^* \delta_\epsilon(A)) = \langle \pi_\epsilon(A) \Omega'_\epsilon, [H_\epsilon, \pi_\epsilon(A)] \Omega'_\epsilon \rangle = \langle \pi_\epsilon(A) \Omega'_\epsilon, H_\epsilon \pi_\epsilon(A) \Omega'_\epsilon \rangle \geq 0, \quad (6.15)$$

since H_ϵ is a positive operator on the GNS Hilbert space. Hence ω'_ϵ is a ground state, and, by purity of ω_ϵ , we have $\omega'_\epsilon \neq \omega_\epsilon$ which is a contradiction. Furthermore, (ii) of Theorem 6.2 and Proposition 5.4 ensure that 0 is an isolated eigenvalue of the GNS Hamiltonian separated from the rest of the spectrum by a gap of at least γ_ϵ .

Let V be the unitary operator as in equations (6.9–6.10), corresponding to the one-particle subspace of Theorem 6.3. With Lemma 3.2, we obtain

$$(VU_\epsilon(t, x)V^* \hat{f})(p) = (VU_\epsilon(t, 0)V^* VU_\epsilon(0, x)V^* \hat{f})(p) = e^{im_\epsilon(p)t + ipx} \hat{f}(p). \quad (6.16)$$

Since V is unitary, this shows that $\mathfrak{h}_\epsilon := \{(m_\epsilon(-p), p) \mid p \in \hat{\Gamma}\} \subset \text{Sp}(U_\epsilon)$. By construction, \mathfrak{h}_ϵ has non-zero spectral measure. In other words, \mathfrak{h}_ϵ fulfills conditions (a), (b) of Definition 4.1. \mathfrak{h}_ϵ is isolated because it arose from an isolated part of $\text{Sp} H_\epsilon$, see the discussion below (6.11).

In order to check condition (c), we identify the dispersion relation Σ_ϵ with the function $p \mapsto m_\epsilon(-p)$, and we will ensure that this function is non-constant. Given that, (c) follows as explained below Definition 4.2. For $d = 1$ (c) is equivalent to regularity of the mass shell. For any $d \geq 1$, (6.11) implies that $\{0\}$ is isolated from the rest of the spectrum by a gap of the size at least $1 - c\epsilon$, while the energy component of any vector in $\mathfrak{h}_\epsilon - \mathfrak{h}_\epsilon$ is at most $2c\epsilon$, so that the mass shell is pseudo-relativistic for $\epsilon < (3c)^{-1}$.

The fact that the mass shell is non-constant follows from Theorem 6.4 as follows. As always, a tilde $\tilde{\cdot}$ will refer to objects obtained using periodic boundary conditions. Since $\tilde{\omega}_{\Lambda,\epsilon}^\beta$ is a $(\tilde{\tau}_\epsilon^\Lambda, \beta)$ -KMS state it satisfies the energy-entropy balance inequality

$$-i\tilde{\omega}_{\Lambda,\epsilon}^\beta(A^*\tilde{\delta}_\epsilon(A)) \geq \frac{1}{\beta}\tilde{\omega}_{\Lambda,\epsilon}^\beta(A^*A) \ln \frac{\tilde{\omega}_{\Lambda,\epsilon}^\beta(A^*A)}{\tilde{\omega}_{\Lambda,\epsilon}^\beta(AA^*)}, \quad (6.17)$$

and hence, with (2.15), the limit $\beta \rightarrow \infty$ yields a finite volume ground state $\langle \tilde{\psi}_{\Lambda,\epsilon}^I, \cdot \tilde{\psi}_{\Lambda,\epsilon}^I \rangle$ of the periodic Hamiltonian, see [12, Proposition 5.3.23] and its proof for details. With this and the spectral decomposition $\tilde{H}_{\Lambda,\epsilon}^I = \int \lambda d\tilde{P}_{\Lambda,\epsilon}^I(\lambda)$, the $\beta \rightarrow \infty$ limit of (6.13) reads

$$\lim_{\beta \rightarrow \infty} f_{\Lambda,\epsilon}^\beta(t) = \int_{\mathbb{R}} e^{-t(\lambda - \tilde{E}_{\Lambda,\epsilon}^I)} \langle \tilde{\psi}_{\Lambda,\epsilon}^I, \sigma_0^{(1)} d\tilde{P}_{\Lambda,\epsilon}^I(\lambda) \sigma_0^{(1)} \tilde{\psi}_{\Lambda,\epsilon}^I \rangle, \quad (6.18)$$

and by a change of variables,

$$\lim_{\beta \rightarrow \infty} f_{\Lambda,\epsilon}^\beta(t) = \int_0^\infty e^{-t\lambda} d\tilde{\mu}_{\Lambda,\epsilon}(\lambda), \quad (6.19)$$

where $d\tilde{\mu}_{\Lambda,\epsilon}(\lambda) = \langle \tilde{\psi}_{\Lambda,\epsilon}^I, \sigma_0^{(1)} d\tilde{Q}_{\Lambda,\epsilon}^I(\lambda) \sigma_0^{(1)} \tilde{\psi}_{\Lambda,\epsilon}^I \rangle$ and $d\tilde{Q}_{\Lambda,\epsilon}^I(\lambda)$ is the spectral measure of $\tilde{H}_{\Lambda,\epsilon}^I - \tilde{E}_{\Lambda,\epsilon}^I$. Let us denote $\hat{g}_t(\lambda) := \Theta(\lambda) \exp(-\lambda t) \in L^1(\mathbb{R})$, where $\Theta \in C^\infty(\mathbb{R})$ is such that $\Theta(\lambda) = 1$ for all $\lambda \geq 0$ and $\Theta(\lambda) = 0$ if $\lambda \leq -1$. Since g_t (the inverse Fourier transform of \hat{g}_t) decays faster than any polynomial, it is integrable. The right hand side above now reads

$$\begin{aligned} \int_0^\infty e^{-t\lambda} d\tilde{\mu}_{\Lambda,\epsilon}(\lambda) &= (2\pi)^{-\frac{1}{2}} \int_0^\infty \int_{\mathbb{R}} ds e^{is\lambda} g_t(s) \langle \tilde{\psi}_{\Lambda,\epsilon}^I, \sigma_0^{(1)} d\tilde{Q}_{\Lambda,\epsilon}^I(\lambda) \sigma_0^{(1)} \tilde{\psi}_{\Lambda,\epsilon}^I \rangle \\ &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} ds g_t(s) \langle \tilde{\psi}_{\Lambda,\epsilon}^I, \sigma_0^{(1)} \tilde{\tau}_{\epsilon,s}^\Lambda(\sigma_0^{(1)}) \tilde{\psi}_{\Lambda,\epsilon}^I \rangle, \end{aligned} \quad (6.20)$$

where we used that $(\tilde{H}_{\Lambda,\epsilon}^I - \tilde{E}_{\Lambda,\epsilon}^I) \tilde{\psi}_{\Lambda,\epsilon}^I = 0$. We can now consider the infinite volume limit of (6.20). As shown in Lemma 6.1, the infinite volume dynamics, the set of algebraic ground states and (by the uniqueness of the algebraic ground state) the GNS Hamiltonian are independent of the boundary conditions in the Ising model. Thus, by dominated convergence, the weak-* convergence of the ground states and the existence of the infinite volume dynamics τ_ϵ ,

$$\lim_{\Lambda \nearrow \Gamma} \int_0^\infty e^{-t\lambda} d\tilde{\mu}_{\Lambda,\epsilon}(\lambda) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} ds g_t(s) \omega_\epsilon \left(\sigma_0^{(1)} \tau_{\epsilon,s}(\sigma_0^{(1)}) \right) = \int_0^\infty e^{-t\lambda} d\nu_\epsilon(\lambda), \quad (6.21)$$

where

$$d\nu_\epsilon(\Lambda) = \left\langle \pi_\epsilon(\sigma_0^{(1)}) \Omega_\epsilon, d\tilde{Q}_\epsilon(\lambda) \pi_\epsilon(\sigma_0^{(1)}) \Omega_\epsilon \right\rangle, \quad (6.22)$$

and $d\tilde{Q}_\epsilon$ is the spectral measure of the GNS Hamiltonian. This shows that the moment generating functions of the measure μ_ϵ of (6.14), and that of ν_ϵ are equal in a neighbourhood of zero, and hence the measures are the same, see e.g. [10, Chapter 30]. In particular, ν_ϵ is absolutely continuous on

$[a_{0,\epsilon}, a_{1,\epsilon})$, which implies that $d\tilde{Q}_\epsilon(\lambda)$ has an absolutely continuous part throughout that interval. It remains to identify that interval with the spectral band $n = 1$ of (6.11). At $\epsilon = 0$, $\sigma_0^{(1)}$ causes a ‘spin flip’ at the origin and $\sigma_0^{(1)}\psi_{\Lambda,0} = \sigma_0^{(1)}\tilde{\psi}_{\Lambda,0}$ is an eigenstate with energy equal to 1, so that $\tilde{\mu}_{\Lambda,0}$ is a Dirac measure at $\lambda = 1$ for all Λ and so is the limit μ_0 . Since $\epsilon \mapsto \mu_\epsilon$ and $\epsilon \mapsto a_{0,\epsilon}$ are analytic, $\epsilon \mapsto [a_{0,\epsilon}, a_{1,\epsilon})$ is the band arising from the isolated point $\lambda = 1$. In particular, the interval $[a_{0,\epsilon}, a_{1,\epsilon})$ is a subset of the $n = 1$ interval given in (6.11). Finally, its non-vanishing width and the analyticity of $p \mapsto m_\epsilon(p)$ imply that $\Sigma_\epsilon(p) = m_\epsilon(-p)$ is not constant. \square

Altogether, this proves that the Ising model satisfies the standing assumptions of Section 4 for large enough transverse field, and hence the scattering theory can be constructed.

We also mention that the existence of an isolated mass shell is expected to hold in a large class of models. In fact it is proved in some sense for the strongly anisotropic antiferromagnetic spin-1/2 Heisenberg Hamiltonian

$$H_{\Lambda,\epsilon}^H = \frac{1}{2} \sum_{(x,y) \in \mathcal{E}(\Lambda)} \left(\sigma_x^{(3)} \sigma_y^{(3)} + 1 \right) + \epsilon \left(\sigma_x^{(1)} \sigma_y^{(1)} + \sigma_x^{(2)} \sigma_y^{(2)} \right), \quad (6.23)$$

whose version with periodic boundary conditions will be denoted by $\tilde{H}_{\Lambda,\epsilon}^H$. In particular, the convergence (6.14) and the properties of the measure obtained for the Ising Hamiltonian also hold for the Heisenberg model for $d \geq 2$ [62]. Moreover, in [1], it is shown that if $n \in \mathbb{N}$ indexes the eigenvalues at $\epsilon = 0$, then for all Λ large enough, the spectral projections $\tilde{P}_{\Lambda,\epsilon}^H(n)$ of $\tilde{H}_{\Lambda,\epsilon}^H$ depend analytically on ϵ for $|\epsilon|$ small enough. Furthermore, the projection $\tilde{P}_{\Lambda,\epsilon}^H(0)$ is two-dimensional, the two eigenvalues converge to each other as $\Lambda \nearrow \mathbb{Z}^d$ for fixed ϵ , while they remain uniformly isolated from the rest of the spectrum. There is however one obstruction: the ground state of the model is degenerate. Hence the analysis above would have to be adapted for the antiferromagnetic Heisenberg model. We do however believe that a similar program could be carried out to obtain a bona fide mass shell.

More generally, there is a large bulk of knowledge in the physics literature on the spectrum of spin chains with various degrees of anisotropy and additional interactions with an external magnetic field, see [53] and the many references therein. However, very few of these results have reached the level of a mathematical proof. In most cases, a variational Ansatz based on physical arguments is used to determine the dispersion relation of the mass shell, thereby yielding upper bounds on spectral gaps, while rigorous lower bounds are much harder to obtain. Further bounds on what we called here the upper gap are scarce.

In the case of the isotropic ferromagnetic model, the works [33, 40] describe the asymptotic behavior of spin waves, which are the low-lying collective excitations (see [20] for a precise statement). However, the situation is very different since the spectrum is not gapped.

7 Outlook

In this paper we have developed Haag-Ruelle scattering theory for gapped systems on a lattice, and have shown that it applies to the Ising model in a strong transversal magnetic field. Compared to the conventional quantum-mechanical scattering theory, as carried out in the framework of quantum spin systems e.g. in [33], the HR approach has two merits: First, there is no need to identify a ‘free’ Hamiltonian, since the free dynamics enters via the dispersion relation Σ in the wave packets (4.2). This is of advantage, because in the context of lattice systems identification of some free part $H_{\text{free},\Lambda}$ of a finite volume Hamiltonian H_Λ is not always apparent. Moreover, the corresponding free and interacting ground states would in general define inequivalent representations of the algebra of observables \mathfrak{A} as they differ globally. The resulting free and interacting GNS Hamiltonians H_{free}

and H , would then act in these inequivalent representations and the usual quantum mechanical formula for the wave operator $\lim_{t \rightarrow \infty} e^{itH} e^{-itH_{\text{free}}}$ would be meaningless. Second, HR theory treats all scattering channels on equal footing: Indeed, a generalization of the discussion from Section 4 to the case of several mass shells in $\text{Sp}U$ is straightforward. These two features (among others) make the HR theory particularly convenient in the case of infinite quantum systems⁷ and justify its further study in the context of spin systems on a lattice. We discuss several interesting future directions below. We will suggest many times that certain results from relativistic QFT should be adapted to the lattice setting. However, we never mean that this is automatic. A pertinent obstacle is the breakdown of the Reeh-Schlieder theorem in the context of lattice systems: This theorem states that in a vacuum representation $(\pi, \mathcal{H}, \Omega)$ of a relativistic theory we have $\overline{\pi(\mathfrak{A}(\mathcal{O}))\Omega} = \mathcal{H}$ for any open bounded region of space-time \mathcal{O} [3]. This inevitable presence of non-local correlations in the relativistic vacuum provides, at a technical level, convenient dense domains for various operators on \mathcal{H} . In contrast, for lattice systems discussed in this paper the subspaces $\pi(\mathfrak{A}(\Lambda))\Omega \subset \mathcal{H}$, $\Lambda \in \mathcal{P}_{\text{fin}}(\Gamma)$, are never dense, as they are only finite-dimensional.

7.1 Non-triviality of the S -matrix and the LSZ reduction formulae

Although Theorem 4.13 guarantees the existence and the uniqueness of the S -matrix, it does not automatically follow that it is non-trivial i.e. $S \neq I$. In other words, it is not *a priori* clear if any scattering takes place at all. In local relativistic QFT non-triviality of the S -matrix was settled in certain two-dimensional models [45, 59, 72]. Perhaps surprisingly, already in space-time of dimension three we are not aware of any local relativistic theories for which $S \neq I$ is clearly stated and proven. In the thoroughly studied $\lambda\phi_3^4$ theory the essential ingredients for such a proof seem to be available [19, 48], but the effort of putting them together remains to be made. In physical space-time construction of a local relativistic QFT with $S \neq I$ is an important open problem, which does not seem amenable to the existing methods. In contrast, in the framework of lattice models one can reasonably expect that tractable models with non-trivial S -matrix are in abundance for any $d \geq 1$. However, rigorous results of this sort are not known to us. Clarifying this point, e.g. in the Ising model of Section 6, could help to identify general properties of infinite quantum systems responsible for (non-)triviality of the S -matrix. A possible starting point of such an investigation, which in fact was accomplished in Euclidean lattice field theories [8], is to establish the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulae [46]. That is, to express the S -matrix via ground state expectation values of time-ordered products of observables. In a concrete model, these ‘Green’s functions’ should be tractable e.g. using perturbation theory.

7.2 Asymptotic completeness

The conventional property of asymptotic completeness requires that all states in the Hilbert space have an interpretation in terms of particles i.e. belong to the ranges of the wave operators. This property was established in many-body quantum mechanics for particles with quadratic dispersion relations [23, 71]. If the latter assumption is dropped, the problem of complete particle interpretation is open beyond the two-body scattering. It is therefore not a surprise that for lattice systems, whose basic excitations typically have non-quadratic dispersion relations, asymptotic completeness is rather poorly understood. An additional problem which arises in infinite systems is a possible presence of non-equivalent representations of \mathfrak{A} describing charged particles, so-called ‘charged sectors’: A configuration of charged particles whose total charge is zero gives rise to a vector in the ground state Hilbert space \mathcal{H} which is not in the ranges of the wave operators of neutral particles $W^{\text{in/out}}$. Thus in

⁷We note, however, that quantum-mechanical scattering theory can also be recast in the HR fashion [14, 69].

the presence of charged sectors the conventional asymptotic completeness relation $\text{Ran } W^{\text{in/out}} = \mathcal{H}$ cannot hold as it stands. One way to save it is to construct wave operators of charged particles as we discuss below. Another possibility is to formulate a weaker notion of asymptotic completeness in the ground state representation which is compatible with non-trivial superselection structure. In the context of local relativistic QFT, where analogous problems arise, such a notion was formulated in [29] and verified using methods from many-body quantum mechanics (e.g. propagation estimates). We are confident that the analysis from [29] could be adapted to the lattice setting and possibly pursued much further in this more tractable framework. We recall in this context that in the case of the isotropic ferromagnetic Heisenberg model⁸ a similar programme of transporting methods from quantum-mechanical scattering (propagation estimates, Mourre theory) was carried out in [33] and led to a proof of conventional asymptotic completeness of two-magnon scattering.

7.3 Scattering of charged particles

Charged particles are described by vector states in representations of \mathfrak{A} which are not equivalent to the ground state representation. To create such particles from the ground state vector Ω , which is the first step of a construction of scattering states, it is necessary to use charge-carrying fields⁹. While such fields are not elements of \mathfrak{A} , in massive relativistic theories they can be obtained from observables by some variant of what is known as the Doplicher-Haag-Roberts (DHR) construction [18, 24]. In general, charged fields have different properties than observables: First, they may not be localizable in bounded regions, but only in spacelike ‘fattening’ strings. Second, their commutation rules may not be governed by the Bose statistics - in physical space-time one obtains the familiar Bose/Fermi alternative while in lower dimensions more exotic braid group statistics corresponding to anyonic excitations is possible. In all these cases scattering theory is well understood in the abstract framework of local relativistic QFT [18, 32]. In particular, it is known that anyonic statistics implies non-triviality of the S -matrix [15]. But concrete examples satisfying all the assumptions are scarce (see however [61]).

In the setting of lattice systems the opposite situation seems to be the case: there are plenty of tractable models, containing charged [7] or even anyonic [44] excitations, but a general theory of superselection sectors and scattering is not well developed. First steps towards such a general theory for lattice systems satisfying the Lieb-Robinson bounds were taken over three decades ago in the Diplom thesis of Schmitz [70], but this direction of research seems to have been abandoned. We stress that nowadays there is strong physical motivation to revisit this subject: anyonic excitations appear in *topologically ordered* systems, which are relevant for quantum computing [74]. Thus a model independent theory of superselection sectors and scattering for anyonic systems on a lattice could prove very useful.

7.4 Scattering theory of anyons in Kitaev’s toric code model

The simplest example of a lattice system exhibiting topological order is Kitaev’s toric code model [44]. This model is defined as follows: On finite $\Lambda \subset \mathbb{Z}^2$ that we imagine painted as a chessboard, let Λ_B , resp. Λ_W be the subsets of black, respectively white squares, and define local Hamiltonians by

$$H_{\Lambda,0}^K = \sum_{X \in \Lambda_B} \frac{1}{2} \left(1 - \prod_{x \in X} \sigma_x^{(3)} \right) + \sum_{Y \in \Lambda_W} \frac{1}{2} \left(1 - \prod_{y \in Y} \sigma_y^{(1)} \right). \quad (7.1)$$

⁸This model is not gapped and thus outside of the scope of this paper. Nevertheless it admits a meaningful scattering theory which relies on a specific form of its Hamiltonian.

⁹The actual mathematical formulation may involve the field algebra [25] or the field bundle [24] depending on a situation.

The Hamiltonian is a sum of mutually commuting projections, and it has frustration-free ground states: for any $X \in \Lambda_B$ and $Y \in \Lambda_W$, the ground state projection $P_{\Lambda,0}^K$ is such that

$$(1 - \prod_{x \in X} \sigma_x^{(3)}) P_{\Lambda,0}^K = (1 - \prod_{y \in Y} \sigma_y^{(1)}) P_{\Lambda,0}^K = 0. \quad (7.2)$$

Imposing periodic boundary conditions, this model can be seen as living on a torus, and in fact, it can be defined on tori of higher genera. Topological order manifests itself in the ground states by a degeneracy depending on the genus g of the surface, in this particular example 4^g , while the various ground states are locally indistinguishable, see [13] for precise definitions. Excited states can be obtained by flipping a pair of either black squares or white squares. Such an excited square has a natural interpretation of an abelian anyon [44]. Note that it is only possible to create *pairs* of excitations, which can be interpreted as a form of charge conservation. To consider a single excitation, one has to go to the thermodynamic limit, where it is possible to move one excitation of the pair to infinity [55]. The anyonic nature of that ‘particle’ is another manifestation of topological order.

The first step to develop scattering theory for anyons in this model is to obtain charge-carrying fields discussed in Subsection 7.3. This may be possible by adapting to the lattice setting results from [37, 54, 65]. The second step is to check if the mass shells of the anyons satisfy the technical conditions stated in Section 4. In fact this turns out not to be the case: the excitations of the Kitaev model have flat ‘mass shells’, thus there is no propagation. A possible solution is to proceed to perturbed models of the form

$$H_{\Lambda,\epsilon}^K = H_{\Lambda,0}^K + \epsilon \sum_{X \subset \Lambda} \Phi(X), \quad (7.3)$$

where $\Phi \in \mathcal{B}_\lambda$ for some $\lambda > 0$. While the shape of the spectrum, in particular of mass shells, should be sensitive to such perturbations, the superselection structure typically is not, so we can expect $H_{\Lambda,\epsilon}^K$ to describe propagating anyons. Since $H_{\Lambda,0}^K$ belongs to a class of theories whose perturbation theory is well understood [13, 52], there are good chances to verify this claim proceeding similarly as in Section 6. In particular a band structure similar to (6.7) is available. Given mass shells and charge-carrying fields of anyons, a construction of scattering states should be performed by adapting the relativistic results from [32] to the lattice setting.

A Spectral theory of automorphism groups

A.1 Groups of isometries

We give a brief account of concepts and results in spectral analysis of automorphism groups which are used in this work. For a more extensive introduction to this subject we refer to [5] and [12, Section 3.2.3].

Let G be a locally compact abelian group. Let \widehat{G} be its dual group, that is the set of all characters $\{\lambda\}$ (i.e. continuous group homomorphisms $G \ni g \mapsto \langle \lambda, g \rangle \in S_1$). Given $f \in L^1(G)$, we define the Fourier transform of f as

$$\widehat{f}(\lambda) = \int_G \overline{\langle \lambda, g \rangle} f(g) d\mu_G(g), \quad (A.1)$$

where $d\mu_G$ is the Haar measure of G . In general \widehat{f} is an element of $C_0(\widehat{G})$. In case $\widehat{f} \in L^1(\widehat{G})$ we have the Fourier inversion formula

$$f(g) = \int_{\widehat{G}} \langle \lambda, g \rangle \widehat{f}(\lambda) d\mu_{\widehat{G}}(\lambda), \quad (A.2)$$

provided that the Haar measure of \widehat{G} is suitably normalized.

Let $G \ni g \mapsto T_g$ be a strongly continuous representation of G in a group of isometries of a complex Banach space V . Then, for any $f \in L^1(G)$, there is a unique bounded operator T_f of V defined by

$$T_f v = \int_G (T_g v) f(g) d\mu_G(g), \quad v \in V, \quad (\text{A.3})$$

where the r.h.s. is a Bochner integral.

Definition A.1. *The Arveson spectrum of T and the Arveson spectrum of v with respect to T are defined by*

$$\text{Sp} T := \{ \lambda \in \widehat{G} \mid \forall O_\lambda \exists f \in L^1(G) \text{ s.t. } \text{supp } \widehat{f} \subset O_\lambda \text{ and } T_f \neq 0 \}, \quad (\text{A.4})$$

$$\text{Sp}_v T := \{ \lambda \in \widehat{G} \mid \forall O_\lambda \exists f \in L^1(G) \text{ s.t. } \text{supp } \widehat{f} \subset O_\lambda \text{ and } T_f v \neq 0 \}, \quad (\text{A.5})$$

respectively, where O_λ denotes a neighbourhood of λ . Also, for any closed $E \subset \widehat{G}$ we define the spectral subspace

$$M^T(E) := \overline{\{v \in V \mid \text{Sp}_v T \subset E\}}, \quad (\text{A.6})$$

where the closure is taken in the weak topology.

We note that in the case of $G = \mathbb{R}^n$ the Arveson spectrum of v w.r.t. T can equivalently be defined as the support of the inverse Fourier transform of the distribution $g \mapsto T_g v$. The closure of the union of such supports (over all $v \in V$) coincides with $\text{Sp} T$. Coming back to general G , one can easily show that

$$\text{Sp}_{T_f v} T \subset \text{Sp}_v T \cap \text{supp } \widehat{f}. \quad (\text{A.7})$$

This relation allows to construct elements of V whose Arveson spectrum is contained in a prescribed set.

A.2 Groups of unitaries

As a first example fitting into the above framework let us consider a strongly continuous abelian group of unitaries $G \ni g \mapsto U(g)$ acting on a Hilbert space \mathcal{H} . In this case the SNAG theorem provides us with a spectral measure dP on \widehat{G} with values in projections on \mathcal{H} , given by

$$U(g) = \int_G \overline{\langle \lambda, g \rangle} dP(\lambda). \quad (\text{A.8})$$

$\text{Sp} U$ coincides here with the support of dP and $\text{Sp}_\Psi U$ for some $\Psi \in \mathcal{H}$, with the support of $\langle \Psi, dP(\cdot) \Psi \rangle$. Moreover,

$$M^U(E) = \text{Ran } P(E). \quad (\text{A.9})$$

Finally we remark that if $G = \mathbb{R}$ then, by the Stone theorem, $\text{Sp} U$ is simply the spectrum of the infinitesimal generator of U .

A.3 Groups of automorphisms of C^* -algebras

As a second example let us consider a strongly continuous group of automorphisms $G \ni g \mapsto \tau_g$ of a C^* -algebra \mathfrak{A} . In the case $G = \mathbb{R}$ one can show that $\text{Sp } \tau$ (resp. $\text{Sp}_A \tau$ for some $A \in \mathfrak{A}$) coincides with the operator-theoretic spectrum of the infinitesimal generator $-i\delta$ of τ on the entire Banach space \mathfrak{A} (resp. on the weakly closed subspace spanned by the orbit of A under the action of τ) [31]. Coming back to the general G , we obtain from (A.7):

$$\text{Sp}_{\tau_f(A)} \tau \subset \text{Sp}_A \tau \cap \text{supp } \widehat{f}. \quad (\text{A.10})$$

Now let $\pi : \mathfrak{A} \rightarrow B(\mathcal{H})$ be a representation of \mathfrak{A} in which τ is unitarily implemented i.e. there exists a strongly continuous group of unitaries $G \ni g \mapsto U(g)$ s.t.

$$\pi(\tau_g(A)) = U(g)\pi(A)U(g)^*, \quad g \in G. \quad (\text{A.11})$$

This group of unitaries defines a group of automorphisms $\tilde{\tau}_g(\cdot) = U(g) \cdot U(g)^*$ on $\mathcal{B}(\mathcal{H})$ and thus for any $B \in \mathcal{B}(\mathcal{H})$ we can define its Arveson spectrum¹⁰ $\text{Sp}_B \tilde{\tau}$ w.r.t $\tilde{\tau}$. It follows from Definition A.1 that

$$\text{Sp}_{\pi(A)} \tilde{\tau} \subset \text{Sp}_A \tau \quad (\text{A.12})$$

and for faithful π this inclusion is an equality. In the following theorem we summarize the relations between the spectra of τ , $\tilde{\tau}$ and U . This is a special case of Theorem 3.5 of [5]:

Theorem A.2. *Let Δ be a closed subset of \widehat{G} and $A \in \mathfrak{A}$. Then*

$$\pi(A)P(\Delta) = P(\overline{\Delta + \text{Sp}_{\pi(A)} \tilde{\tau}})\pi(A)P(\Delta). \quad (\text{A.13})$$

If π is faithful then $\text{Sp}_{\pi(A)} \tilde{\tau} = \text{Sp}_A \tau$.

To conclude this section, we point out an important difference between the spectral theory of groups and unitaries and of groups of automorphisms: in the latter case no obvious counterpart of the spectral measure (A.8) exists. Therefore, the finer structure of the Arveson spectrum of automorphism groups (e.g. its division into pure-point, absolutely continuous and singular continuous parts) remains to a large extent unexplored. Some steps to close this gap, with physical applications in mind, have been taken in [17, 28, 41].

A.4 Examples of groups and their Haar measures

In relativistic quantum field theory we have $G = \mathbb{R}^{d+1}$, $\widehat{G} = \mathbb{R}^{d+1}$, the respective Haar measures are the Lebesgue measures with normalization factors $(2\pi)^{-(d+1)/2}$, and relations (A.1), (A.2) give the usual Fourier transforms of functions on Minkowski space-time. The Arveson spectrum $\text{Sp}_A \tau$ is often called the *energy-momentum transfer* of A due to relation (A.13).

In the case of lattice systems considered in the present paper, we consider the locally compact abelian group $G = \mathbb{R} \times \Gamma$, where $\Gamma := \mathbb{Z}^d$ is the group of lattice translations. The Haar measures of Γ and G are fixed by the relations

$$\int f(x) d\mu_\Gamma(x) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} f(x), \quad f \in L^1(\Gamma), \quad (\text{A.14})$$

$$\int f(t, x) d\mu_G(t, x) = (2\pi)^{-\frac{d+1}{2}} \sum_{x \in \Gamma} \int_{\mathbb{R}} dt f(t, x), \quad f \in L^1(G). \quad (\text{A.15})$$

¹⁰The representation $g \mapsto \tilde{\tau}_g$ is usually not strongly continuous, and (A.3) can only be defined as a weak integral. By the Riesz theorem the weak integral defines a unique element of $\mathcal{B}(\mathcal{H})$ and thus the Arveson spectrum of $\tilde{\tau}$ is well defined. The closure in (A.6) should then be taken in the weak-* topology.

The dual group of G is $\widehat{G} = \mathbb{R} \times \widehat{\Gamma}$, where $\widehat{\Gamma} = S_1^d$ is the d -dimensional torus. For $p \in S_1^d$ the corresponding characters are given by

$$\langle p, x \rangle_{\Gamma} = e^{ipx}, \quad p \in \widehat{\Gamma}, \quad x \in \Gamma, \quad (\text{A.16})$$

$$\langle (E, p), (t, x) \rangle_G = e^{-iEt+ipx}, \quad (E, p) \in \widehat{G}, \quad (t, x) \in G. \quad (\text{A.17})$$

With definitions (A.14)–(A.17) relation (A.1) reproduces our formulas for the Fourier transform in (D.1), (D.2).

To construct the Haar measure on \widehat{G} and $\widehat{\Gamma}$ we proceed as follows: Let us consider two distinct points $p_{\pm} \in S_1$ and open intervals I_{\pm} identified with $S_1 \setminus \{p_{\pm}\}$ via

$$I_{\pm} \ni p^1 \mapsto e^{ip^1} \in S_1. \quad (\text{A.18})$$

For example, we can choose $I_+ = (0, 2\pi)$, $I_- = (-\pi, \pi)$. Now let $\varphi_{\pm} \in C^{\infty}(S_1)$ be a smooth partition of unity on S_1 s.t. $\text{supp } \varphi_{\pm} \subset I_{\pm}$. We can express the Haar measure on S_1 by the relation

$$\int_{S_1} f(p^1) d\mu_{S_1}(p^1) = (2\pi)^{-\frac{1}{2}} \sum_{i \in \{\pm\}} \int_{I_i} \varphi_i(p^1) f(p^1) dp^1, \quad f \in L^1(S_1). \quad (\text{A.19})$$

By the uniqueness of the Haar measure (up to normalization) the r.h.s. above does not depend on the details of the construction. Thus the Haar measure on the torus $\widehat{\Gamma} = S_1^d$ can be expressed as

$$\int_{\widehat{\Gamma}} f(p) d\mu_{\widehat{\Gamma}}(p) = (2\pi)^{-\frac{d}{2}} \sum_{i_1, \dots, i_d \in \{\pm\}} \int_{I_{i_1} \times \dots \times I_{i_d}} \varphi_{i_1}(p^1) \dots \varphi_{i_d}(p^d) f(p^1, \dots, p^d) dp, \quad f \in L^1(\widehat{\Gamma}), \quad (\text{A.20})$$

where $dp = dp^1 \dots dp^d$. This careful construction of $d\mu_{\widehat{\Gamma}}$ is important in the proof of Proposition 4.3. Otherwise, due to the close relation between $d\mu_{\widehat{\Gamma}}$ and the Lebesgue measure we use the short-hand notation

$$\int_{\widehat{\Gamma}} f(p) d\mu_{\widehat{\Gamma}}(p) =: (2\pi)^{-\frac{d}{2}} \int_{\widehat{\Gamma}} f(p) dp, \quad f \in L^1(\widehat{\Gamma}). \quad (\text{A.21})$$

In case f is supported on a subset of $\widehat{\Gamma}$ which is diffeomorphic with \mathbb{R}^d , (A.21) makes sense also literally. Using this short-hand notation, the Haar measure on \widehat{G} is given by

$$\int_{\widehat{G}} f(E, p) d\mu_{\widehat{G}}(E, p) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R} \times \widehat{\Gamma}} f(E, p) dE dp, \quad f \in L^1(\widehat{G}). \quad (\text{A.22})$$

Given expressions (A.21), (A.22), relation (A.2) reproduces formulas (D.4), (D.5) for the inverse Fourier transforms.

Finally, we note that partial derivatives of a function $f \in C^{\infty}(\Delta)$, where Δ is an open subset of $\widehat{\Gamma}$, are independent of a chosen parametrization of the torus, an observation which is implicitly used e.g. in Definition 4.1.

B The energy-momentum transfer lemma

Here we present a proof of the energy-momentum transfer relation, equation (3.15), in the context of lattice systems. This is a special case of Theorem 3.5 of [5], but for the benefit of the reader we here adapt an elementary argument given in the context of relativistic QFT in [26].

Lemma B.1. *Let $A \in \mathfrak{A}$ and τ be the group of space-time translation automorphisms, and U its implementing group of unitaries as in Subsection 3.1. Write P for the associated spectral measure defined in (3.7). We then have the energy-momentum transfer relation*

$$\pi(A)P(\Delta) = P(\overline{\Delta + \text{Sp}_A\tau})\pi(A)P(\Delta) \quad (\text{B.1})$$

for any Borel subset $\Delta \subset \mathbb{R} \times \widehat{\Gamma}$.

Proof. By countable additivity of the spectral measure, we can assume without loss of generality that Δ is bounded. Now for any $g \in S(\mathbb{R} \times \Gamma)$ let

$$P(g) := (2\pi)^{-\frac{d+1}{2}} \sum_{x \in \Gamma} \int_{\mathbb{R}} dt U(t, x) g(t, x) \quad (\text{B.2})$$

which has the following properties: Firstly, $P(g) = P(K)P(g)$ for any closed $K \supset \text{supp}(\widehat{g})$. Secondly, $P(\Delta) = P(g)P(\Delta)$ for any function g such that $\widehat{g} \upharpoonright_{\Delta} = 1$.

Now, let $f, g \in S(\mathbb{R} \times \Gamma)$. We have that

$$\begin{aligned} \pi(\tau_f(A))P(g) &= (2\pi)^{-(d+1)} \sum_{x, y \in \Gamma} \int_{\mathbb{R}^2} dt ds f(t, x) g(s, y) U(t, x) \pi(A) U(t-s, x-y)^* \\ &= (2\pi)^{-(d+1)} \sum_{z, y \in \Gamma} \int_{\mathbb{R}^2} dr ds f(r+s, z+y) g(s, y) U(s, y) \pi(\tau_{(r,z)}(A)) \\ &= (2\pi)^{-\frac{d+1}{2}} \sum_{z \in \Gamma} \int_{\mathbb{R}} dr P(h_{(r,z)}) \pi(\tau_{(r,z)}(A)), \end{aligned} \quad (\text{B.3})$$

where the function $h_{(r,z)}(s, y) = f(r+s, z+y)g(s, y)$ clearly satisfies

$$\text{supp}(\widehat{h_{(r,z)}}) \subset \text{supp}(\widehat{f}) + \text{supp}(\widehat{g}). \quad (\text{B.4})$$

Hence

$$\pi(\tau_f(A))P(g) = P(\text{supp}(\widehat{f}) + \text{supp}(\widehat{g}))\pi(\tau_f(A))P(g). \quad (\text{B.5})$$

Let now $\varphi \in C^\infty(\mathbb{R} \times \widehat{\Gamma})$ be bounded (as well as all its derivatives) and such that $\varphi \upharpoonright_{\text{Sp}_A\tau} = 1$ and $\varphi \upharpoonright_{(\text{Sp}_A\tau)_\epsilon^c} = 0$, where for any $\Omega \subset \mathbb{R} \times \widehat{\Gamma}$ and $\epsilon > 0$, we define $\Omega_\epsilon := \{x \in \mathbb{R} \times \widehat{\Gamma} : \text{dist}(x, \Omega) \leq \epsilon\}$, and Ω_ϵ^c is its complement. Such φ can be obtained by convoluting the characteristic function of $(\text{Sp}_A\tau)_{\epsilon/2}$ with a suitable function from $C_0^\infty(\mathbb{R} \times \widehat{\Gamma})$. This yields a decomposition $\widehat{f} = \widehat{f}_1 + \widehat{f}_2 = \varphi\widehat{f} + (1-\varphi)\widehat{f}$, where both f_1, f_2 are Schwartz functions, and further

$$\tau_f(A) = \tau_{f_1}(A) + \tau_{f_2}(A). \quad (\text{B.6})$$

By definition of the Arveson spectrum, $\tau_{f_2}(A) = 0$.

Let K be a compact set such that $K \cap (\Delta + \text{Sp}_A\tau) = \emptyset$, and consider $g \in S(\mathbb{R} \times \Gamma)$ such that $\widehat{g} \upharpoonright_{\Delta} = 1$ and $\widehat{g} \upharpoonright_{\Delta_\epsilon^c} = 0$. Then,

$$\begin{aligned} P(K)\pi(\tau_f(A))P(\Delta) &= P(K)\pi(\tau_{f_1}(A))P(g)P(\Delta) \\ &= P(K)P(\text{supp}(\widehat{f}_1) + \text{supp}(\widehat{g}))\pi(\tau_{f_1}(A))P(g)P(\Delta). \end{aligned} \quad (\text{B.7})$$

Since K is disjoint from the compact set $\overline{\Delta + \text{Sp}_A\tau}$, there is ϵ small enough such that

$$\text{dist}\left(K, \text{supp}(\widehat{f}_1) + \text{supp}(\widehat{g})\right) \geq \text{dist}\left(K, (\text{Sp}_A\tau)_\epsilon + \Delta_\epsilon\right) > 0, \quad (\text{B.8})$$

and hence $P(K)P(\text{supp}(\widehat{f}_1) + \text{supp}(\widehat{g})) = 0$. Therefore, $P(K)\pi(\tau_f(A))P(\Delta) = 0$.

Finally, let $(f_n)_{n \in \mathbb{N}}$ be defined by

$$f_n(t, x) = \widetilde{f}_n(t)\delta_0(x), \quad (\text{B.9})$$

where $\widetilde{f}_n(t) = (2\pi)^{\frac{d+1}{2}}(4\pi n^{-1})^{-\frac{1}{2}}e^{-nt^2/4}$ is a sequence converging to the Dirac δ (multiplied by $(2\pi)^{\frac{d+1}{2}}$), and δ_0 is Kronecker's delta on \mathbb{Z}^d . For any $\psi, \psi' \in \mathcal{H}$,

$$\langle \psi', \pi(\tau_{f_n}(A))\psi \rangle = (2\pi)^{-\frac{d+1}{2}} \sum_{x \in \Gamma} \int_{\mathbb{R}} dt \langle U(t, x)\psi', \pi(A)U(t, x)\psi \rangle f_n(t, x) \rightarrow \langle \psi', \pi(A)\psi \rangle, \quad (\text{B.10})$$

by the strong continuity of $(t, \cdot) \mapsto U(t, \cdot)$, $U(0, 0) = 1$ and the dominated convergence theorem. Hence, for Δ, K as above,

$$P(K)\pi(A)P(\Delta) = \text{w-}\lim_n P(K)\pi(\tau_{f_n}(A))P(\Delta) = 0. \quad (\text{B.11})$$

The restriction of K being compact can be lifted by considering a countable partition of the complement of $\overline{\Delta + \text{Sp}_A \tau}$ into bounded sets, so that the statement above extends to any K such that $K \cap (\overline{\Delta + \text{Sp}_A \tau}) = \emptyset$. It follows that

$$\pi(A)P(\Delta)\mathcal{H} \subset \left(P \left((\mathbb{R} \times \widehat{\Gamma}) \setminus (\overline{\Delta + \text{Sp}_A \tau}) \right) \mathcal{H} \right)^\perp = P(\overline{\Delta + \text{Sp}_A \tau})\mathcal{H}, \quad (\text{B.12})$$

i.e. $\pi(A)P(\Delta) = P(\overline{\Delta + \text{Sp}_A \tau})\pi(A)P(\Delta)$. Since the reverse inclusion is trivial, this proves the result. \square

C Proof of Theorem 3.10

In this section we give a proof that space-time translations leave the algebra $\mathfrak{A}_{\text{a-loc}}$ invariant, and that the same is true if we smear against Schwartz functions. To this end, we first recall Lemma 3.2 of [6]:

Lemma C.1. *If \mathcal{H}_1 and \mathcal{H}_2 are Hilbert spaces, and if $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is such that*

$$\|[A, 1 \otimes B]\| < \epsilon \|B\|$$

for all $B \in \mathcal{B}(\mathcal{H}_2)$, then for any normal state ω on $\mathcal{B}(\mathcal{H}_2)$, the map $\Pi_\omega = \text{id} \otimes \omega$ satisfies the bound

$$\|\Pi_\omega(A) - A\| \leq 2\epsilon.$$

The key step in our proof of Theorem 3.10 is the following consequence of the Lieb-Robinson bound:

Proposition C.2. *For any $Y \in \mathcal{P}_{\text{fin}}(\Gamma)$, $A \in \mathfrak{A}(Y)$, $\delta > 0$ and $t \in \mathbb{R}$ there exists $A_{t,\delta} \in \mathfrak{A}(Y^{(v\lambda|t|+\delta)})$ such that*

$$\|\tau_t(A) - A_{t,\delta}\| \leq C(A, \lambda)e^{-\lambda\delta}, \quad (\text{C.1})$$

and $\lambda > 0$ is as in Theorem 2.1. Moreover, the dependence $t \mapsto A_{t,\delta}$ is piecewise continuous w.r.t. the norm topology.

Proof. In order to apply Lemma C.1 in the present case, let ρ be an arbitrary state on \mathfrak{A} . It is necessarily locally normal with density matrices ρ_Λ , that is $\rho(A) = \text{Tr}(\rho_\Lambda A)$ for $A \in \mathfrak{A}(\Lambda)$. For any finite $X \subset \Lambda \subset \Gamma$, we consider the projection $\Pi_{\Lambda, X} : \mathfrak{A}(\Lambda) \rightarrow \mathfrak{A}(X) \otimes 1_{\Lambda \setminus X} \simeq \mathfrak{A}(X)$ onto the subalgebra $\mathfrak{A}(X)$ given by $\Pi_{\Lambda, X} := \Pi_{\rho_{\Lambda \setminus X}} = \text{id}_{\mathfrak{A}(X)} \otimes \rho_{\Lambda \setminus X}$. It satisfies the consistency condition $\Pi_{\Lambda, X}(A) = \Pi_{\Lambda', X}(A)$ whenever $A \in \mathfrak{A}(\Lambda)$ with $X \subset \Lambda \subset \Lambda' \subset \Gamma$. We can therefore define $\Pi_X : \mathfrak{A} \rightarrow \mathfrak{A}(X)$ as follows: for any $A \in \mathfrak{A}_{\text{loc}}$, for any Λ such that $A \in \mathfrak{A}(\Lambda)$ we let $\Pi_X(A) = \Pi_{\Lambda, X}(A)$. Since, moreover, $\|\Pi_{\Lambda, X}(A)\| \leq \|A\|$, the densely defined linear map Π_X can be extended to a bounded operator on the whole algebra \mathfrak{A} with the same bound.

Let now $A \in \mathfrak{A}(Y)$, $t \in \mathbb{R}$, and $Y \subset \Lambda \subset \Lambda'$. Since $\tau_t^\Lambda(A) \in \mathfrak{A}(\Lambda) \subset \mathfrak{A}(\Lambda')$, we have for any $r > 0$ such that $Y^r \subset \Lambda$,

$$\left\| \Pi_{Y^r}(\tau_t^\Lambda(A)) - \Pi_{Y^r}(\tau_t^{\Lambda'}(A)) \right\| = \left\| \Pi_{\Lambda', Y^r} \left(\tau_t^\Lambda(A) - \tau_t^{\Lambda'}(A) \right) \right\| \leq \left\| \tau_t^\Lambda(A) - \tau_t^{\Lambda'}(A) \right\|. \quad (\text{C.2})$$

Hence,

$$A'_{t,r} := \lim_{\Lambda \nearrow \Gamma} \Pi_{Y^r}(\tau_t^\Lambda(A)) \quad (\text{C.3})$$

exists and $A'_{t,r} \in \mathfrak{A}(Y^r)$. Now we can write

$$\left\| \tau_t(A) - A'_{t,r} \right\| \leq \left\| \tau_t(A) - \tau_t^\Lambda(A) \right\| + \left\| \tau_t^\Lambda(A) - \Pi_{Y^r}(\tau_t^\Lambda(A)) \right\| + \left\| \Pi_{Y^r}(\tau_t^\Lambda(A)) - A'_{t,r} \right\|, \quad (\text{C.4})$$

keeping in mind that r is restricted by $Y^r \subset \Lambda$. Let us now eliminate this restriction: The Lieb-Robinson bound (2.14) yields

$$\left\| [\tau_t^\Lambda(A), 1 \otimes B] \right\| \leq \frac{2\|A\|\|B\|}{C_\lambda} |Y| \|F\| e^{-\lambda(r-v_\lambda|t|)} \quad (\text{C.5})$$

for any $B \in \mathfrak{A}(\Lambda \setminus Y^r)$. Therefore Lemma C.1 gives

$$\left\| \Pi_{Y^r}(\tau_t^\Lambda(A)) - \tau_t^\Lambda(A) \right\| \leq \frac{4\|A\|}{C_\lambda} |Y| \|F\| e^{-\lambda(r-v_\lambda|t|)}, \quad (\text{C.6})$$

uniformly in Λ . We apply this bound to the second term on the r.h.s of (C.4) and then take the limit $\Lambda \nearrow \Gamma$ which eliminates the remaining two terms and the restriction on r . Thus we are left with

$$\left\| \tau_t(A) - A'_{t,r} \right\| \leq \frac{4\|A\|}{C_\lambda} |Y| \|F\| e^{-\lambda(r-v_\lambda|t|)}, \quad (\text{C.7})$$

where $r > 0$ is arbitrary. Now for any $\delta > 0, t \in \mathbb{R}$ we set r equal to $r(\delta, t) = v_\lambda|t| + \delta$ and define $A_{\delta,t} := A'_{t,r(\delta,t)}$. This concludes the proof of (C.1).

The piecewise continuity of $t \mapsto A_{\delta,t}$, follows from two observations: First, $r \mapsto A'_{t,r}$, given by (C.3), is a step function for fixed t (since $r \mapsto Y^r$ depends on r in a discrete manner). Second, $t \mapsto A'_{t,r}$ is continuous in norm for fixed r , which follows by replacing $\tau_t^{\Lambda'}(A)$ with $\tau_t^\Lambda(A)$ in (C.2). \square

Now we are ready to show that by smearing a local observable with a Schwartz class function we obtain an almost local observable.

Lemma C.3. *Let $\Lambda \in \mathcal{P}_{\text{fin}}(\Lambda)$, $A \in \mathfrak{A}(\Lambda)$ and $f \in S(\mathbb{R} \times \Gamma)$. Then $\tau_f(A) \in \mathfrak{A}_{\text{a-loc}}$.*

Proof. The fact that $\tau_f(A) \in \mathfrak{A}$ follows from the strong continuity of τ and properties of Bochner integrals, or from the approximation procedure below. Let us set

$$A_{(r)} := \int_{|t| \leq r} dt \sum_{x \in \Gamma, |x| \leq r} \tau_{(t,x)}(A) f(t, x). \quad (\text{C.8})$$

Since $f \in S(\mathbb{R} \times \Gamma)$, we have

$$\tau_f(A) = A_{(r)} + O(r^{-\infty}). \quad (\text{C.9})$$

Now we fix some $\delta > 0$, depending on r , which will be specified later. Proposition C.2 gives observables $A_{t,\delta} \in \mathfrak{A}(\Lambda^{r\delta(t)})$, with $r_\delta(t) := v_\lambda t + \delta$, such that

$$\|\tau_t(A) - A_{t,\delta}\| \leq C(A, \lambda)e^{-\lambda\delta}. \quad (\text{C.10})$$

We note that $A_{t,\delta} \in \mathfrak{A}(\Lambda^{r\delta(r)})$, since $|t| \leq r$ in (C.8). Estimate (C.10) gives

$$A_{(r)} = \int_{|t| \leq r} dt \sum_{x \in \Gamma, |x| \leq r} \tau_x(A_{t,\delta})f(t, x) + O(e^{-\lambda\delta}), \quad (\text{C.11})$$

where we exploited the fact that the function $t \mapsto A_{t,\delta}$ is piecewise continuous in norm and thus Bochner integrable (see Proposition C.2). We denote by A_r the integral on the r.h.s. of (C.11) and note that $A_r \in \mathfrak{A}(\Lambda^{v_\lambda r + \delta + r})$. We set $\delta = r$ and obtain from (C.11)

$$A_{(r)} = A_r + O(r^{-\infty}). \quad (\text{C.12})$$

Noting that $\Lambda^{(v_\lambda+2)r} \subset Z^{2(v_\lambda+2)r}$ for $r \geq R_0$, where R_0 depends on Λ , modifying A_r for $r \leq R_0$ and rescaling r we conclude the proof. \square

Proof of Theorem 3.10: Let $A' \in \mathfrak{A}_{\text{a-loc}}$ i.e. we have that there exist $A'_r \in \mathfrak{A}(Z^r)$ s.t. $A' - A'_r = O(r^{-\infty})$.

To show (a), we note that for $(t, x) \in \mathbb{R} \times \Gamma$

$$\tau_{(t,x)}(A') - \tau_{(t,x)}(A'_r) = O(r^{-\infty}). \quad (\text{C.13})$$

Now by Proposition C.2, we obtain $A_r \in \mathfrak{A}((Z^r + x)^r)$ s.t.

$$\tau_{(t,x)}(A'_r) - A_r = O(r^{-\infty}). \quad (\text{C.14})$$

Using that $(Z^r + x)^r = \{x\}^{2r} \subset Z^{3r}$ for r sufficiently large, we conclude part (a) as in the last step of the proof of Lemma C.3.

To show (b) we note that for $f \in S(\mathbb{R} \times \Gamma)$ we have

$$\|\tau_f(A') - \tau_f(A'_r)\| \leq \|A' - A'_r\| \|f\|_1 \quad (\text{C.15})$$

and therefore $\tau_f(A') - \tau_f(A'_r) = O(r^{-\infty})$. Now by Lemma C.3 we obtain $A_r \in \mathfrak{A}((Z^r)^r)$ s.t.

$$\tau_f(A'_r) - A_r = O(r^{-\infty}). \quad (\text{C.16})$$

Using that $(Z^r)^r \subset Z^{2r}$, we complete the proof in the case of $f \in S(\mathbb{R} \times \Gamma)$. The remaining cases follow similarly. \square

D Notations

For the convenience of the readers we here collect the main notations that we use and at the same time fix our normalisation constants for the Fourier transform. Note that we use a Minkowski sign convention, in that the time and spatial components carry an opposite sign in the Fourier transform.

1. For $f \in L^1(\mathbb{R} \times \Gamma)$, $g \in L^1(\Gamma)$, $h \in L^1(\mathbb{R})$ we define the following norms and Fourier transforms

$$\|f\|_1 := (2\pi)^{-\frac{(d+1)}{2}} \sum_{x \in \Gamma} \int_{\mathbb{R}} dt |f(t, x)|, \quad \widehat{f}(E, p) := (2\pi)^{-\frac{(d+1)}{2}} \sum_{x \in \Gamma} \int_{\mathbb{R}} dt e^{iEt - ipx} f(t, x), \quad (\text{D.1})$$

$$\|g\|_1 := (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} |g(x)|, \quad \widehat{g}(p) := (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} e^{-ipx} g(x), \quad (\text{D.2})$$

$$\|h\|_1 := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} dt |h(t)|, \quad \widehat{h}(E) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} dt e^{iEt} h(t). \quad (\text{D.3})$$

2. For $\widehat{f} \in L^1(\mathbb{R} \times \widehat{\Gamma})$, $\widehat{g} \in L^1(\widehat{\Gamma})$, $\widehat{h} \in L^1(\mathbb{R})$ we write

$$\|\widehat{f}\|_1 := (2\pi)^{-\frac{(d+1)}{2}} \int_{\mathbb{R} \times \widehat{\Gamma}} dEdp |\widehat{f}(E, p)|, \quad f(t, x) = (2\pi)^{-\frac{(d+1)}{2}} \int_{\mathbb{R} \times \widehat{\Gamma}} dEdp e^{-iEt + ipx} \widehat{f}(E, p), \quad (\text{D.4})$$

$$\|\widehat{g}\|_1 := (2\pi)^{-\frac{d}{2}} \int_{\widehat{\Gamma}} dp |\widehat{g}(p)|, \quad g(x) = (2\pi)^{-\frac{d}{2}} \int_{\widehat{\Gamma}} dp e^{ipx} \widehat{g}(p), \quad (\text{D.5})$$

$$\|\widehat{h}\|_1 := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} dE |\widehat{h}(E)|, \quad h(t) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} dE e^{-iEt} \widehat{h}(E). \quad (\text{D.6})$$

3. We write $\langle x \rangle := \sqrt{1 + x^2}$.

4. We say that $g \in S(\Gamma)$ if $g(x) = O(\langle x \rangle^{-\infty})$. Equivalently, $g \in S(\Gamma)$ if $\widehat{g} \in C^\infty(\widehat{\Gamma})$.

5. We say that $f \in S(\mathbb{R} \times \Gamma)$, if $f \in C^\infty(\mathbb{R} \times \Gamma)$ and $\partial_t^n f(t, x) = O(\langle t \rangle + \langle x \rangle)^{-\infty}$ for any $n \in \mathbb{N}$. Equivalently, $f \in S(\mathbb{R} \times \Gamma)$ if $\widehat{f} \in C^\infty(\mathbb{R} \times \widehat{\Gamma})$ and $\partial_p^\alpha \partial_E^n \widehat{f}(E, p) = O(\langle E \rangle^{-\infty})$, uniformly in $p \in \widehat{\Gamma}$, for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$.

6. Given $B \in \mathcal{B}(\mathcal{H})$ we set $B(t, x) := U(t, x)BU(t, x)^*$, $B(x) := B(0, x)$, $B_t := B(t, 0)$.

7. Given $B \in \mathcal{B}(\mathcal{H})$, $f \in L^1(\mathbb{R} \times \Gamma)$ and $g \in L^1(\Gamma)$ we write

$$B(f) := (2\pi)^{-\frac{(d+1)}{2}} \sum_{x \in \Gamma} \int dt B(t, x) f(t, x), \quad B(g) := (2\pi)^{-\frac{d}{2}} \sum_{x \in \Gamma} B(x) g(x). \quad (\text{D.7})$$

Thus for $A \in \mathfrak{A}$ we have $\pi(A)(f) = \pi(\tau_f(A))$, $\pi(A)(g) = \pi(\tau_g^{(d)}(A))$.

8. $O(x)$ denotes a term such that $\|O(x)\| \leq C|x|$ in some specified norms. $O(x^{-\infty})$ denotes a term which is $O(x^{-n})$ for any $n \in \mathbb{N}$.

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