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Yang–Baxter endomorphisms

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Abstract

Every unitary solution of the Yang–Baxter equation (R-matrix) in dimension d can be viewed as a unitary element of the Cuntz algebra $\mathcal{O}_d$ and as such defines an endomorphism of $\mathcal{O}_d$. These Yang–Baxter endomorphisms restrict and extend to several other C^*- and von Neumann algebras, and furthermore define a II_1 factor associated with an extremal character of the infinite braid group. This paper is devoted to a detailed study of such Yang–Baxter endomorphisms.

We discuss the relative commutants of the subfactors induced by Yang–Baxter endomorphisms, a new perspective on algebraic operations on R-matrices such as tensor products and cabling powers, the characters of the infinite braid group defined by R-matrices, and ergodicity properties. This also yields new concrete information on partial traces and spectra of R-matrices.

1. *Introduction*

This article is motivated by two circles of questions, one pertaining to the Yang–Baxter equation (YBE) and one to endomorphisms of the Cuntz algebras and related operator algebras, which are brought into contact by so-called Yang–Baxter endomorphisms. As the name suggests, these are endomorphisms of various C^*- and von Neumann algebras, as explained below, defined by unitary solutions of the YBE.

To introduce the subject, recall that the YBE is a cubic equation for an endomorphism $R \in \mathrm{End}(V \otimes V)$ of the tensor square of a vector space V, namely

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R). \tag{1.1}$$

This equation and its solutions play a prominent role in many different areas of physics and mathematics. It has its origins in statistical mechanics and quantum mechanics [**4, 65**], but is long since known to also be closely connected to braid group representations and knot theory [**41, 61**], von Neumann algebras and subfactors [**40**], and braided categories [**24, 29, 49, 60**]. Representations of quantum groups [**21, 39**] are a rich source of solutions for the YBE.

In many of these fields, one is mostly interested in the case that V is a finite-dimensional Hilbert space and R is a *unitary* solution of (1.1). Also in the present article, we will only be concerned with such *R-matrices*, henceforth always assumed to be unitary, and refer to $d := \dim V$ as the dimension of R. The set of all R-matrices of dimension d is denoted by $\mathcal{R}(d)$.

Unitary R-matrices are of great interest in several applications to quantum physics. For example, in topological quantum computation, they may serve as quantum gates [**6, 43, 56**], and in the context of integrable quantum field theories on two-dimensional Minkowski space,

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unitary solutions of a more involved YBE involving a spectral parameter play the role of two-particle collision operators [1]. Unitary solutions of (1.1), without spectral parameter, then describe the structure of short-distance scaling limits of such theories [46].

Furthermore, as will be explained further below, R-matrices give rise to certain endomorphisms of von Neumann algebras that share many structural properties with endomorphisms appearing in quantum field theories [19] with braid group statistics [26, 27, 49].

Despite this widespread interest in the YBE, only relatively little is known about its solutions, and in particular about its unitary solutions, which are very difficult to find in general. In dimension $d = 2$, all solutions are known [35] but already for $d = 3$, this is no longer the case. For special classes of solutions, see, for example, [7, 31].

The only general class of R-matrices that seems to be under good control are the *involutive* R-matrices (that is, $R^2 = 1$) which have recently been completely classified by one of us [45] up to an equivalence relation originating from algebraic quantum field theory [3]. This classification relied crucially on the fact that involutive R-matrices define extremal characters of the infinite symmetric group, a classification of which is known [59].

This situation provides one of our motivations for the present article: To develop tools that help to understand the set of R-matrices in the vastly more general non-involutive case. Although often times the braid group representations associated with an R-matrix are emphasized, these are by no means the only interesting structure attached to an R-matrix, and in this article, our focus is on certain endomorphisms and subfactors defined by R.

In order to introduce these endomorphisms, we recall some facts about the Cuntz algebras, see Section 2 for precise definitions and details. The Cuntz algebras $\mathcal{O}_d$ [16] are a family of C^*-algebras that play a prominent role in various fields — for example, in superselection theory and duality for compact groups [20], wavelets [5], and twisted cyclic cocycles in non-commutative geometry [8], to name just a very few samples from different areas.

There are two fundamental features of $\mathcal{O}_d$ that underlie the main concept of this article: First, its unitary elements $u \in \mathcal{U}(\mathcal{O}_d)$ are in bijection with its (unital, *-) endomorphisms $\lambda_u \in \mathrm{End}(\mathcal{O}_d)$ [17]. As $\mathcal{O}_d$ is a simple C^*-algebra, these are automatically injective. Second, the Cuntz algebra $\mathcal{O}_d$ can be thought of as being generated by a d-dimensional Hilbert space V, namely it contains all linear maps $V^{\otimes n} \to V^{\otimes m}$, $n, m \in \mathbb{N}_0$. In particular, there is a uniformly hyperfinite (UHF) subalgebra $\mathcal{F}_d$ isomorphic to the infinite C^*-tensor product of $\mathrm{End}\, V$.

In view of these facts, we may view an R-matrix R, which is in particular a unitary element of $\mathrm{End}(V \otimes V)$, as a unitary in $\mathcal{O}_d$ (with $d = \dim V$) and consider the corresponding endomorphism $\lambda_R \in \mathrm{End}\, \mathcal{O}_d$. They will be called *Yang–Baxter endomorphisms*, and their analysis is the main subject of this paper.

The Cuntz algebra $\mathcal{O}_d$ can be completed in a natural way to a type $\mathrm{III}_{1/d}$ factor $\mathcal{M}$, and its subalgebra $\mathcal{F}_d$ completes to a type II_1 factor $\mathcal{N} \subset \mathcal{M}$. Any endomorphism of the form λ_u with $u \in \mathcal{U}(\mathcal{F}_d)$ leaves the subalgebra $\mathcal{F}_d \subset \mathcal{O}_d$ invariant, extends to endomorphisms of their weak closures $\mathcal{M}$ and $\mathcal{N}$ (all denoted λ_u), and thus provides us with the subfactors

$$\lambda_u(\mathcal{M}) \subset \mathcal{M}, \quad \lambda_u(\mathcal{N}) \subset \mathcal{N}. \tag{1.2}$$

These and related subfactors have been studied by several researchers, often times with the aim of determining their (Jones and related) indices [2, 13, 38].

Endomorphisms of Cuntz algebras have a very rich structure with many different facets [14, 15], and Yang–Baxter endomorphisms (that is, $u = R \in \mathcal{R}(d)$) and their subfactors have further more special properties. For instance, as an additional structure present in the Yang–Baxter case, there is a von Neumann algebra $\mathcal{L}_R \subset \mathcal{N}$ generated by the braid group representation associated with R, and λ_R restricts to the canonical endomorphism φ on $\mathcal{L}_R$. We will show that $\mathcal{L}_R$ is a factor, so that any R-matrix R provides us with yet another subfactor

$$\varphi(\mathcal{L}_R) \subset \mathcal{L}_R. \tag{1.3}$$

We are thus in a situation where to any R-matrix we may associate various operator-algebraic structures, derived from their endomorphisms. On the one hand, these data provide interesting invariants of R-matrices (such as Jones indices, commuting squares, fixed point algebras, etc.) that go beyond the trivial spectral and dimension data of the R-matrix itself. On the other hand, the analysis of Yang–Baxter endomorphisms contributes to the understanding of endomorphisms of $\mathcal{O}_d$ in general, which is an area in full swing on its own.

We now give an overview of the content of the paper and the main results.

Section 2 introduces R-matrices, Cuntz algebras, and the associated von Neumann algebras $\mathcal{L}_R \subset \mathcal{N} \subset \mathcal{M}$ in detail. We recall in particular that if one takes R to be one of the most basic R-matrices, namely the tensor flip F, one obtains the canonical endomorphism $\varphi = \lambda_F \in \operatorname{End} \mathcal{O}_d$, acting as a shift on the UHF subalgebra. Drawing on the interplay of λ_R and φ, we give three different characterizations of Yang–Baxter endomorphisms (Proposition 2.3), two of which are due to Cuntz [**18**] and one of us [**12**], respectively. A notable feature is that a Yang–Baxter endomorphism is an automorphism if and only if R is a multiple of the identity (Corollary 2.4).

With the framework setup in this manner, we consider in Section 3 the three towers of relative commutants of the subfactors (1.2) (for $u = R \in \mathcal{R}(d)$) and (1.3). We give explicit characterizations of all three relative commutants. The characterizations of the relative commutants of (1.2) rely strongly on results from [**2, 13, 50**], but the characterization of the relative commutant $\mathcal{L}_{R,n} := \varphi^n(\mathcal{L}_R)' \cap \mathcal{L}_R$ (Proposition 3.4) is new: We characterize it as an intersection of $\mathcal{L}_R$ with a matrix algebra, and as the fixed point algebra of $\mathcal{L}_R^{\lambda_{\varphi^n(R)}}$, reminiscent of work of Gohm and Köstler in non-commutative probability [**30**].

The section concludes with a structural result about $\mathcal{L}_{R,n}$: For any $n \in \mathbb{N}$, the diagrams

$$\begin{array}{ccc} \mathcal{F}_d^n & \subset & \mathcal{N} \\ \cup & & \cup \\ \mathcal{L}_{R,n} & \subset & \mathcal{L}_R \end{array} \qquad \begin{array}{ccc} \varphi^n(\mathcal{N}) & \subset & \mathcal{N} \\ \cup & & \cup \\ \varphi^n(\mathcal{L}_R) & \subset & \mathcal{L}_R \end{array} \tag{1.4}$$

are commuting squares (Theorem 3.5), where $\mathcal{F}_d^n$ is the subalgebra of $\mathcal{F}_d$ isomorphic to $\operatorname{End} V^{\otimes n}$. This implies in particular that the left inverses of λ_R and φ coincide on $\mathcal{L}_R$, and is later used as a basic tool for computing braid group characters and invariants for R.

Section 4 discusses three algebraic operations on the set of all R-matrices: A tensor product, Wenzl's cabling powers [**63**], and a kind of direct sum. We relate these operations on R-matrices R to operations on the endomorphisms λ_R. In particular, the tensor product of R-matrices turns out to correspond to the tensor product of endomorphisms (on the level of the II_1 factor $\mathcal{N}$) and the cabling power $R^{(n)}$ turns out to correspond to the n-fold power λ_R^n (on $\mathcal{N}$).

In Section 5, we introduce three equivalence relations on R-matrices $R, S \in \mathcal{R}(d)$, each of which formalizes that one of their subfactors in (1.2) or (1.3) are equivalent. The equivalence relation relating to the $\mathcal{L}_R$-subfactor (1.3), denoted by $\sim$, is taken from [**45**] and shown to exactly capture the braid group character defined by R. We compare with the classification of involutive R-matrices in Section 5.1 and prove that equivalent R-matrices $R \sim S$ have similar partial traces. In this context, we also show that the left and right partial traces of an R-matrix always coincide and are normal (Theorem 5.8), which provides direct information on the R-matrices themselves.

Section 6: As a unital normal endomorphism of the type III factor $\mathcal{M}$ with finite-dimensional relative commutant, a Yang–Baxter endomorphism can be decomposed into finitely many irreducible endomorphisms of $\mathcal{M}$, unique up to inner automorphisms (that is, as sectors in quantum field theory language) [**47, 48**]. The main difficulty is that the decomposition of a Yang–Baxter endomorphism does not respect the YBE, that is, its irreducible components are no longer of Yang–Baxter form. Nonetheless, such a decomposition provides information on the underlying R-matrix; for example, we find upper and lower bounds on the minimal and Jones indices of the subfactors (1.2) in terms of spectral data of R and its partial trace

(Corollary 6.2). Another corollary is that an R-matrix whose eigenvalues are concentrated in a sufficiently small disk around 1 is necessarily the identity (Corollary 6.4). We also sketch a reduction scheme of involutive R-matrices into irreducible components.

Section 7 is about fixed points of Yang–Baxter endomorphisms. Our first result in this direction is that on the level of the type II factor $\mathcal{N}$, the relative commutant $\mathcal{L}_R' \cap \mathcal{N}$ coincides with the fixed point algebra $\mathcal{N}^{\lambda_R}$ (Proposition 7.1). Moreover, λ_R is ergodic as an endomorphism of $\mathcal{M}$ if and only if it is ergodic in restriction to $\mathcal{N}$ (Proposition 7.3). This structure enables us to obtain a clear picture of ergodicity and fixed point algebras for Yang–Baxter endomorphisms which is not known for general elements of $\operatorname{End}\mathcal{O}_d$ or $\operatorname{End}\mathcal{M}$. In particular, we give a complete characterization of ergodic Yang–Baxter endomorphisms in Theorem 7.5 in terms of a condition that only involves the adjoint action of R on $\operatorname{End} V$.

The article concludes in Section 8, devoted to an analysis of the family of all R-matrices of dimension $d = 2$. Strengthening a theorem of Dye [**22**] (building on Hietarinta's classical [**36**]), we show that $\mathcal{R}(2)$ is the disjoint union of four families that could be called trivial R-matrices, diagonal R-matrices, off-diagonal R-matrices, and a special case (see Theorem 8.1 for details). We use our previous results to analyze the properties of the corresponding endomorphisms in detail. In particular, we discuss the special case, an R-matrix that has appeared in various places in the literature (see, for example, [**11, 25, 54**]), explain why it is also special from the point of view of endomorphisms, and compute its (infinite dimensional) fixed-point algebra $\mathcal{N}^{\lambda_R}$.

2. *R-matrices and Cuntz algebras*

The algebraic structures investigated in this article are all derived from unitary solutions of the YBE, which we will refer to as *R-matrices*.

Definition 2.1. Let V be a finite dimensional Hilbert space. An *R-matrix on* V is a unitary $R : V \otimes V \to V \otimes V$ such that

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R). \tag{2.1}$$

The *dimension of* R is defined as $\dim R := \dim V$. The set of all R-matrices on Hilbert spaces of dimension $d \in \mathbb{N}$ is denoted by $\mathcal{R}(d)$, and the set of all R-matrices (of any dimension) is denoted by $\mathcal{R}$.

Many examples of R-matrices exist, but the general structure of $\mathcal{R}$ is not known. Very simple R-matrices that can be produced in any dimension are multiples of the identity, $R = q \cdot 1$ (such R-matrices will be called *trivial*), and multiples of the tensor flip, that is, $R = q \cdot F$, where $F(v \otimes w) = w \otimes v$, $v, w \in V$. Here, q lies in $\mathbb{T}$, the unit circle in the complex plane.

As is well known and will be recalled later, any $R \in \mathcal{R}$ defines representations of the braid groups. However, this is by no means the only interesting algebraic structure attached to an R-matrix, and in this article, we emphasize certain endomorphisms and subfactors defined by R. To introduce these, we have to recall some well-known facts about Cuntz algebras.

The Cuntz algebra $\mathcal{O}_d$ is the unital C^*-algebra generated by $d \geqslant 2$ isometries $S_1, \ldots, S_d$ such that $S_i^* S_j = \delta_{ij} 1$ and $\sum_{i=1}^d S_i S_i^* = 1$ [**16**]. Using standard notation for multi indices $\mu = (\mu_1, \ldots, \mu_n)$, we set $S_\mu := S_{\mu_1} \cdots S_{\mu_n}$ and refer to $|\mu| := n$ as the length of μ.

The subalgebra $\mathcal{F}_d^n := \operatorname{span}\{S_\mu S_\nu^* : |\mu| = |\nu| = n\}$ is naturally isomorphic to the n-fold tensor power $M_d^{\otimes n}$ of the full matrix algebra M_d, and we will suppress this isomorphism in our notation. In particular, we may view R-matrices $R \in \mathcal{R}(d)$ as elements of $\mathcal{F}_d^2 \subset \mathcal{O}_d$. The norm closure of the increasing family $\mathcal{F}_d^n \subset \mathcal{F}_d^{n+1} \subset \ldots$ is a UHF algebra of type d^∞ which we denote by $\mathcal{F}_d$.

An important feature of $\mathcal{O}_d$ that we will rely on throughout is that its unitary elements $u \in \mathcal{U}(\mathcal{O}_d)$ are in bijection with its (unital) endomorphisms $\lambda_u \in \operatorname{End}\mathcal{O}_d$ [**17**]. On generators, the endomorphism λ_u corresponding to $u \in \mathcal{U}(\mathcal{O}_d)$ is defined by $\lambda_u(S_i) := uS_i$ and every endomorphism of $\mathcal{O}_d$ is of this form.

Definition 2.2. A *Yang–Baxter endomorphism of* $\mathcal{O}_d$ is an endomorphism of the form λ_R, $R \in \mathcal{R}(d)$.

An important example is the so-called *canonical endomorphism* $\varphi := \lambda_F$ given by the flip $F = \sum_{i,j=1}^d S_iS_jS_i^*S_j^*$, which takes the explicit form $\varphi(x) = \sum_{i=1}^d S_ixS_i^*$, $x \in \mathcal{O}_d$. This endomorphism satisfies $S_ix = \varphi(x)S_i$ for all $x \in \mathcal{O}_d$ and $i = 1, \ldots, d$, and restricts to the one-sided shift $x \mapsto \mathrm{id}_{M_d} \otimes x$ on the infinite tensor product UHF algebra $\mathcal{F}_d \simeq M_d \otimes M_d \otimes \ldots$, which indicates its relevance for R-matrices in view of (2.1). In fact, the YBE takes the form $R\varphi(R)R = \varphi(R)R\varphi(R)$ when R is viewed as an element of $\mathcal{F}_d^2 \subset \mathcal{O}_d$.

Without further mentioning, we will often use two basic consequences of the definition of λ_u (for general unitary $u \in \mathcal{U}(\mathcal{O}_d)$) and φ: The composition law

$$\lambda_u\lambda_v = \lambda_{\lambda_u(v)u}, \quad u, v \in \mathcal{U}(\mathcal{O}_d), \tag{2.2}$$

and an explicit formula for the action of λ_u on $\mathcal{F}_d^n$: Given arbitrary unitary $u \in \mathcal{U}(\mathcal{O}_d)$ and an integer $n \geqslant 1$, we define two elements of $\mathcal{F}_d^{n+1}$,

$$u_n := u\varphi(u)\cdots\varphi^{n-1}(u), \quad {}_nu := \varphi^{n-1}(u)\cdots u = (u^*{}_n)^* \tag{2.3}$$

and see that

$$\lambda_u(x) = (\operatorname{ad} u_n)(x) \quad \text{for } x \in \mathcal{F}_d^k, \quad n \geqslant k, \tag{2.4}$$

$$\lambda_u(x) = \lim_{n\to\infty} (\operatorname{ad} u_n)(x) \quad \text{for } x \in \mathcal{F}_d. \tag{2.5}$$

The latter limit exists in the norm topology of $\mathcal{O}_d$ [**18**], and we note that for $u \in \mathcal{F}_d$, the endomorphism λ_u leaves $\mathcal{F}_d$ invariant, $\lambda_u(\mathcal{F}_d) \subset \mathcal{F}_d$.

Yang–Baxter endomorphisms can be characterized as follows. The easy proof of $(i) \Leftrightarrow (iv)$ is omitted.

Proposition 2.3. *Let $R \in \mathcal{U}(\mathcal{F}_d^2)$. The following conditions are equivalent:*

(i) *$R \in \mathcal{R}(d)$, namely $R\varphi(R)R = \varphi(R)R\varphi(R)$,*
(ii) *$\lambda_R(R) = \varphi(R)$* [**18**],
(iii) *R commutes with every element $x \in \lambda_R^2(\mathcal{O}_d)$* [**12**], *and*
(iv) *$\lambda_R^2 = \lambda_{\varphi(R)R}$.*

It is a natural question to ask whether Yang–Baxter endomorphisms can be automorphisms, that is, surjective. While it is well known that for $u \in \mathcal{F}_d^1$, the associated endomorphism λ_u is an automorphism (called quasi-free [**23**]), with inverse $\lambda_u^{-1} = \lambda_{u^*}$, the problem to recognize which endomorphisms λ_u are automorphisms is delicate in general [**15**]. For Yang–Baxter endomorphisms, the answer is, however, a straightforward consequence of Proposition 2.3(iii) [**12**] because the Cuntz algebra has trivial centre.

Corollary 2.4. *For $R \in \mathcal{R}$, λ_R is an automorphism if and only if $R \in \mathbb{C}$ is trivial.*

With the help of the canonical endomorphism φ, we may also conveniently introduce the previously mentioned braid group representations associated with $R \in \mathcal{R}(d)$. Let $B_n = \langle b_1, \ldots, b_{n-1}\rangle$ denote the braid group on n strands with its standard Artin generators b_i, and let B_∞ denote the infinite braid group, namely the inductive limit of the family $B_n \subset B_{n+1} \subset \ldots$. Given $R \in \mathcal{R}(d)$, the multiplicative extension of

$$\rho_R(b_k) := \varphi^{k-1}(R) \in \mathcal{F}_d^{k+1} \subset \mathcal{F}_d, \quad k \in \mathbb{N}, \tag{2.6}$$

is a group homomorphism $\rho_R : B_\infty \to \mathcal{U}(\mathcal{F}_d)$. We will frequently consider the C^*-algebra generated by ρ_R, namely

$$\mathcal{B}_R := C^*\{\varphi^n(R) \,:\, n \in \mathbb{N}_0\} \subset \mathcal{F}_d, \tag{2.7}$$

and the closely related C^*-algebras

$$\mathcal{A}_R := \{x \in \mathcal{O}_d \,:\, \lambda_R(x) = \varphi(x)\}, \quad \mathcal{A}_R^{(0)} := \mathcal{A}_R \cap \mathcal{F}_d. \tag{2.8}$$

LEMMA 2.5. *Let $R \in \mathcal{R}(d)$ and λ_R its corresponding Yang–Baxter endomorphism.*

(*i*) *$\mathcal{B}_R \subset \mathcal{A}_R^{(0)}$, that is,*

$$\lambda_R(x) = \varphi(x), \quad x \in \mathcal{B}_R. \tag{2.9}$$

(*ii*) *λ_R restricts to an endomorphism of $\mathcal{F}_d$, $\mathcal{A}_d$, $\mathcal{A}_d^{(0)}$, and $\mathcal{B}_R$.*

(*iii*) *For any $n \in \mathbb{N}$, one has*

$$\lambda_R^n = \lambda_{{}_nR} = \lambda_{\rho_R(b_n \cdots b_1)}, \quad n \in \mathbb{N}. \tag{2.10}$$

Proof. We first prove that λ_R restricts to $\mathcal{A}_R$, and to this end, recall that for general $u \in \mathcal{U}(\mathcal{O}_d)$, one has $\lambda_u \circ \varphi = \operatorname{ad} u \circ \varphi \circ \lambda_u$. This implies that if $x \in \mathcal{O}_d$ satisfies $\lambda_R(x) = \varphi(x)$, then

$$\lambda_R(\lambda_R(x)) = \lambda_R(\varphi(x)) = R\varphi(\lambda_R(x))R^* = R\varphi(\varphi(x))R^* = \varphi(\varphi(x)) = \varphi(\lambda_R(x)), \tag{2.11}$$

where the next to last step follows from the general fact that $\mathcal{F}_d^n$ commutes with $\varphi^n(\mathcal{O}_d)$.

This argument yields $\lambda_R(\mathcal{A}_R) \subset \mathcal{A}_R$. As $R \in \mathcal{F}_d$, we also have $\lambda_R(\mathcal{F}_d) \subset \mathcal{F}_d$ and therefore $\lambda_R(\mathcal{A}_R^{(0)}) \subset \mathcal{A}_R^{(0)}$ as well.

Regarding $\mathcal{B}_R$, the argument (2.11) can be used to prove $\lambda_R^n(R) = \varphi^n(R)$ by induction in $n \in \mathbb{N}$, the case $n = 1$ being settled by Proposition 2.3(ii). This implies, $n \in \mathbb{N}_0$,

$$\lambda_R(\varphi^n(R)) = \lambda_R^{n+1}(R) = \varphi^{n+1}(R) = \varphi(\varphi^n(R)).$$

As $\mathcal{B}_R$ is generated by $\varphi^n(R)$, $n \in \mathbb{N}_0$, we have shown both (i) and (ii).

For (iii), we note that ${}_nR = \varphi^{n-1}(R) \cdots R = \rho_R(b_n \cdots b_1)$ by definition of ${}_nR$ and ρ_R, and carry out another induction in n to show $\lambda_R^n = \lambda_{\rho_R(b_n \cdots b_1)}$. In fact, $\lambda_R^{n+1} = \lambda_R \lambda_{\rho_R(b_n \cdots b_1)} = \lambda_{\lambda_R(\varphi^{n-1}(R)\cdots R)R} = \lambda_{\varphi^n(R)\cdots R} = \lambda_{\rho_R(b_{n+1}\cdots b_1)}$. □

Any R-matrix defines several C^*-algebra inclusions, namely $\lambda_R(\mathcal{O}_d) \subset \mathcal{O}_d$, $\lambda_R(\mathcal{F}_d) \subset \mathcal{F}_d$, $\lambda_R(\mathcal{B}_R) = \varphi(\mathcal{B}_R) \subset \mathcal{B}_R$, etc. We now recall further structure that will allow us to promote these inclusions to subfactors of von Neumann algebras.

Trivial R-matrices $R = d^{-it}1$, $t \in \mathbb{R}$, define a $\frac{2\pi}{\log d}$-periodic one-parameter group of automorphisms $\sigma_t := \lambda_{d^{-it}1}$ of $\mathcal{O}_d$, and we define the spectral subspaces

$$\mathcal{O}_d^{(n)} := \{x \in \mathcal{O}_d \,:\, \sigma_t(x) = d^{-itn}x\}, \quad n \in \mathbb{Z}. \tag{2.12}$$

Sometimes, it will be more convenient to work with a rescaled version of σ, namely the (2π)-periodic gauge action $\alpha_t := \sigma_{-t/\log d} = \lambda_{e^{it}}$. One has $\mathcal{O}_d^{(0)} = \mathcal{F}_d$, and $E^0 : \mathcal{O}_d \to \mathcal{F}_d$, $E^0(x) := \frac{1}{2\pi}\int_0^{2\pi} \alpha_t(x)dt$ is a conditional expectation onto the UHF subalgebra.

Viewing $\mathcal{F}_d$ as an infinite tensor product, we have the canonical normal normalized trace state $\tau : \mathcal{F}_d \to \mathbb{C}$, and define $\omega := \tau \circ E^0$. This is a KMS state on $\mathcal{O}_d$ with modular group σ_t and we denote the von Neumann algebras generated by its GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$

$$\mathcal{M} := \pi_\omega(\mathcal{O}_d)'', \quad \mathcal{N} := \pi_\omega(\mathcal{F}_d)'' \subset \mathcal{M}. \tag{2.13}$$

It is well known that $\mathcal{M}$ is a $\mathrm{III}_{1/d}$-factor and $\mathcal{N}$ is a II_1-factor. We will use the same symbols ω, τ, and $E^0 : \mathcal{M} \to \mathcal{N}$ [**33**] for the extensions of these maps to the weak closures $\mathcal{M}$ and $\mathcal{N}$.

For our purposes, it is important to note that for any $u \in \mathcal{F}_d$ (and in particular, for any R-matrix), the corresponding endomorphism λ_u extends to a normal endomorphism of $\mathcal{M}$ leaving ω invariant [**50**]. Also here, we will use the same symbol for the extension.

To complete the picture, we also introduce the von Neumann algebra $\mathcal{L}_R$ generated by the C^*-algebra $\mathcal{B}_R$ corresponding to some R-matrix $R \in \mathcal{R}$, that is,

$$\mathcal{L}_R := \pi_\omega(\mathcal{B}_R)'' \subset \mathcal{N} \subset \mathcal{M}. \tag{2.14}$$

As an immediate consequence of (2.9), we observe $\lambda_R|_{\mathcal{L}_R} = \varphi|_{\mathcal{L}_R}$.

Further structural elements relevant for our analysis are conditional expectations and left inverses. As λ_R commutes with σ_t, Takesaki's theorem provides us with a unique ω-preserving conditional expectation $E_R : \mathcal{M} \to \lambda_R(\mathcal{M})$, which is faithful and normal and has the form

$$E_R = \lambda_R \circ \phi_R \tag{2.15}$$

with ϕ_R the corresponding ω-preserving left inverse of λ_R. Recall that $\phi_R : \mathcal{M} \to \mathcal{M}$ is a completely positive normal linear map that satisfies

$$\phi_R(\lambda_R(x)y\lambda_R(z)) = x\phi_R(y)z, \quad x, y, z \in \mathcal{M}. \tag{2.16}$$

These properties of ϕ_R and the limit formula (2.5) imply

$$\phi_R(x) = \operatorname*{w\text{-}lim}_{n\to\infty} {R_n}^* x R_n, \quad x \in \mathcal{N}. \tag{2.17}$$

As $R_n \in \mathcal{L}_R \subset \mathcal{N}$, this yields in particular

$$\phi_R(\mathcal{N}) = \mathcal{N}, \quad \phi_R(\mathcal{L}_R) = \mathcal{L}_R. \tag{2.18}$$

The left inverse ϕ_R is usually difficult to evaluate explicitly. However, in the case of the flip $R = F$, one finds $\phi_F(x) = \frac{1}{d}\sum_{k=1}^n S_k^* x S_k$, $x \in \mathcal{M}$, which restricts to the normalized partial trace on the first tensor factor on $\mathcal{N} \cong M_d \otimes M_d \otimes \ldots$, namely

$$\phi_F(a_1 \otimes a_2 \otimes a_3 \ldots) = \tau(a_1) \cdot a_2 \otimes a_3 \otimes \ldots, \quad a_i \in M_d. \tag{2.19}$$

We summarize these structures in terms of commuting squares of von Neumann algebras [**32**].

Proposition 2.6. *Let $R \in \mathcal{R}(d)$ and consider the diagram*

$$\begin{array}{ccc} \lambda_R(\mathcal{M}) & \subset & \mathcal{M} \\ \cup & & \cup \\ \lambda_R(\mathcal{N}) & \subset & \mathcal{N} \\ \cup & & \cup \\ \varphi(\mathcal{L}_R) & \subset & \mathcal{L}_R. \end{array} \tag{2.20}$$

(i) *All von Neumann algebras in the diagram are hyperfinite factors.*
(ii) *Both squares in the diagram are commuting squares.*

Proof. (i) All we need to show is that $\mathcal{L}_R$ is a factor. So let $x \in \mathcal{L}_R \cap \mathcal{L}_R'$. Then x commutes with $R_n \in \mathcal{L}_R$ for all $n \in \mathbb{N}$, and we have $\lambda_R(x) = \lim_n(\mathrm{ad}\, R_n)(x) = x$. But since λ_R restricts to φ on $\mathcal{L}_R$, we get $\varphi(x) = \lambda_R(x) = x$. The canonical endomorphism φ is well known to have only trivial fixed points, hence $x \in \mathbb{C}1$.

(ii) By Takesaki's theorem, the conditional expectation $E_R : \mathcal{M} \to \lambda_R(\mathcal{M})$ commutes with the modular group. This implies that $E_R(\mathcal{N}) \subset \mathcal{N} \cap \lambda_R(\mathcal{M}) = \lambda_R(\mathcal{N})$, that is, the upper square in the diagram is a commuting square.

Recall that for $x \in \mathcal{N}$, we have $\phi_R(x) = \text{w-lim}_n(\text{ad}\, R_n{}^*)(x)$. As $R_n \in \mathcal{L}_R$, this directly gives invariance of $\mathcal{L}_R$ under ϕ_R, and therefore $E_R(\mathcal{L}_R) \subset \lambda_R(\mathcal{L}_R) = \varphi(\mathcal{L}_R)$. This shows that the lower square in (2.20) is a commuting square. □

REMARK 2.7. As just demonstrated, any R-matrix provides us with (at least) three subfactors. Let us point out that the $\mathcal{M}$- and $\mathcal{N}$-subfactors contain only partial information about R. For example, let $R = F$ be the flip, $u \in \mathcal{U}(\mathcal{F}_d^1)$ non-trivial, and $\alpha := \lambda_u \in \operatorname{Aut} \mathcal{M}$. Then $\lambda_R \alpha = \lambda_S$ with $S = \varphi(u)F$. Diagonalizing u, it is easy to see that S is a diagonal R-matrix (cf. Def. 2.8(ii)). Moreover, λ_R and α commute, and therefore $\lambda_R^n(\mathcal{M}) = \lambda_S^n(\mathcal{M})$, $\lambda_R^n(\mathcal{N}) = \lambda_S^n(\mathcal{N})$ for all $n \in \mathbb{N}$. But despite R and S defining identical $\mathcal{M}$- and $\mathcal{N}$-subfactors, they are different from each other as R-matrices, for instance, $R^2 = 1$ and $S^2 \neq 1$.

On the other hand, the subfactors generated by the braid group representations, $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ and $\varphi(\mathcal{L}_S) \subset \mathcal{L}_S$, differ in this example. For instance, we will see later that the first one is irreducible but the second one is not.

It is a natural question to ask what the indices of the subfactors in (2.20) are. Adopting standard notation, we will write $\operatorname{Ind}_{E_R}(\lambda_R)$ for the index of $\lambda_R(\mathcal{M}) \subset \mathcal{M}$ taken with respect to the ω-invariant conditional expectation, $\operatorname{Ind}(\lambda_R)$ for the minimal index of $\lambda_R(\mathcal{M}) \subset \mathcal{M}$ [**34, 44, 47**], and $[\mathcal{N} : \lambda_R(\mathcal{N})]$, $[\mathcal{L}_R : \varphi(\mathcal{L}_R)]$ for the Jones indices [**40**] of the type II_1 subfactors $\lambda_R(\mathcal{N}) \subset \mathcal{N}$, $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, respectively.

Independently of the YBE, it is known that $\operatorname{Ind}_{E_R}(\lambda_R) = [\mathcal{N} : \lambda_R(\mathcal{N})] \leqslant d^2$ [**13, 47**], and the preceding commuting squares result implies $[\mathcal{L}_R : \varphi(\mathcal{L}_R)] \leqslant [\mathcal{N} : \lambda_R(\mathcal{N})]$ by a Pimsner–Popa inequality [**53**]. We thus have

$$[\mathcal{L}_R : \varphi(\mathcal{L}_R)] \leqslant [\mathcal{N} : \lambda_R(\mathcal{N})] = \operatorname{Ind}_{E_R}(\mathcal{M}) \leqslant d^2 < \infty. \tag{2.21}$$

New results on indices will be presented in Section 6.

We close this section by presenting a large family of simple R-matrices.

DEFINITION 2.8. (i) Let $\{p_i\}_{i=1}^N$ be a partition of unity in $\mathcal{F}_d^1$, that is, the p_i are orthogonal projections in $\mathcal{F}_d^1$ such that $p_i p_j = \delta_{ij} p_i$ and $\sum_{i=1}^N p_i = 1$. Let $c_{ij} \in \mathbb{T}$, $i, j \in \{1, \ldots, N\}$. Then

$$R := \sum_{i=1}^N c_{ii}\, p_i \varphi(p_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^N c_{ij} p_i \varphi(p_j) F \tag{2.22}$$

is an R-matrix. Such R-matrices will be referred to as *simple*.

(ii) If $R \in \mathcal{R}(d)$ is a simple R-matrix with only one-dimensional projections, that is, $\tau(p_i) = 1/d$ for all i, then there exists a unitary $u \in \mathcal{U}(\mathcal{F}_d^1)$ such that $p_i = u S_i S_i^* u^*$, and

$$R = \lambda_u(DF), \quad D = \sum_{i,j=1}^d c_{ij} S_i S_j S_j^* S_i^*. \tag{2.23}$$

Such R-matrices will be referred to as *diagonal*.

The straightforward verification of the claims made in this definition is omitted.

We will frequently use simple R-matrices as examples. Note that trivial R-matrices are simple and the flip is diagonal (choose $N = d$, $p_i = S_i S_i^*$, $c_{ij} = 1$ for all i, j). The term 'simple' should not be understood in a mathematical sense — in fact, all non-trivial simple R-matrices define reducible endomorphisms and can be decomposed into smaller R-matrices, as we shall explain later. There exist (more interesting) R-matrices that are not simple.

3. *Towers of algebras associated with R-matrices*

Having established the basic subfactors associated with R-matrices, we now turn to their analysis, in particular of their relative commutants. As the basis of our following arguments, we recall some known facts about relative commutants of localized endomorphisms (that is, endomorphisms of the form λ_u, $u \in \mathcal{F}_d$) of Cuntz algebras.

For any two endomorphisms λ, μ of $\mathcal{M}$, we write

$$(\lambda, \mu) := \{T \in \mathcal{M} \,:\, T\lambda(x) = \mu(x)T \quad \forall x \in \mathcal{M}\}$$

for the space of intertwiners from λ to μ. In particular, $(\lambda, \lambda) = \lambda(\mathcal{M})' \cap \mathcal{M}$ is the relative commutant of $\lambda(\mathcal{M}) \subset \mathcal{M}$.

For an arbitrary unitary $u \in \mathcal{U}(\mathcal{O}_d)$, one has [**50**, Proposition 2.5]

$$(\lambda_u, \lambda_u) = \{x \in \mathcal{M} \,:\, \varphi(x) = u^* x u\} = \mathcal{M}^{\mathrm{ad}\, u \circ \varphi}. \tag{3.1}$$

If, more specifically, $u \in \mathcal{U}(\mathcal{F}_d^n)$ for some $n \in \mathbb{N}$, one furthermore has [**13**, Proposition 4.2]

$$(\lambda_u, \lambda_u) = \bigoplus_{k=-n+2}^{n-2} (\lambda_u, \lambda_u)^{(k)}, \tag{3.2}$$

$$(\lambda_u, \lambda_u)^{(k)} \subset \begin{cases} (\varphi^{n-1}, \varphi^{n-1+k}) & k \geqslant 0 \\ (\varphi^{n-1-k}, \varphi^{n-1}) & k < 0 \end{cases}. \tag{3.3}$$

From this we see in particular

$$(\lambda_u, \lambda_u)^{(0)} = \lambda_u(\mathcal{M})' \cap \mathcal{N} \subset (\varphi^{n-1}, \varphi^{n-1}) = \mathcal{F}_d^{n-1}, \tag{3.4}$$

$$(\lambda_u, \lambda_u) \subset \mathcal{F}_d^1, \quad u \in \mathcal{U}(\mathcal{F}_d^2), \tag{3.5}$$

and note that (3.5) occurs in particular for R-matrices $u = R \in \mathcal{U}(\mathcal{F}_d^2)$.

Having recalled these facts, we now turn to study the subfactors given by λ_R and introduce their relative commutants, $n \in \mathbb{N}_0$,

$$\mathcal{M}_{R,n} := \lambda_R^n(\mathcal{M})' \cap \mathcal{M}, \quad \mathcal{N}_{R,n} := \lambda_R^n(\mathcal{N})' \cap \mathcal{N}, \quad \mathcal{L}_{R,n} := \varphi^n(\mathcal{L}_R)' \cap \mathcal{L}_R.$$

Thus, $\mathcal{M}_{R,n} = (\lambda_R^n, \lambda_R^n)$, but we prefer the notation $\mathcal{M}_{R,n}$ in order to distinguish the three different levels of relative commutants $\mathcal{M}_{R,n}$, $\mathcal{N}_{R,n}$, and $\mathcal{L}_{R,n}$.

We clearly have three ascending towers of algebras:

$$\begin{aligned} \mathbb{C} = \mathcal{M}_{R,0} &\subset \mathcal{M}_{R,1} \subset \cdots \subset \mathcal{M}_{R,n} \subset \mathcal{M}_{R,n+1} \subset \cdots \subset \mathcal{M}, \\ \mathbb{C} = \mathcal{N}_{R,0} &\subset \mathcal{N}_{R,1} \subset \cdots \subset \mathcal{N}_{R,n} \subset \mathcal{N}_{R,n+1} \subset \cdots \subset \mathcal{N}, \\ \mathbb{C} = \mathcal{L}_{R,0} &\subset \mathcal{L}_{R,1} \subset \cdots \subset \mathcal{L}_{R,n} \subset \mathcal{L}_{R,n+1} \subset \cdots \subset \mathcal{L}_R. \end{aligned} \tag{3.6}$$

In the following, we will derive various relations/inclusions between these algebras, and realize them as fixed point algebras for certain endomorphisms. In particular, it is not clear from the outset if there are inclusions one way or the other between $\mathcal{M}_{R,n}$, $\mathcal{N}_{R,n}$, and $\mathcal{L}_{R,n}$.

We begin with the relative commutants at the highest level, that is, the $\mathcal{M}_{R,n}$.

Proposition 3.1. *Let $R \in \mathcal{R}(d)$ and $n \in \mathbb{N}$. Then*

$$\mathcal{M}_{R,n} = \mathcal{M}^{\mathrm{ad}\,{}_n R \circ \varphi} = \bigoplus_{k=-n+1}^{n-1} (\mathcal{M}^{(k)})^{\mathrm{ad}\,{}_n R \circ \varphi}, \tag{3.7}$$

$$\mathcal{M}_{R,n}^{(0)} = \lambda_R^n(\mathcal{M})' \cap \mathcal{N} = (\mathcal{F}_d^n)^{\mathrm{ad}\,{}_n R \circ \varphi} = \{x \in \mathcal{F}_d^n \,:\, \varphi(x) = \lambda_{R^*}(x)\}, \tag{3.8}$$

and in particular for $n = 1$,

$$\mathcal{M}_{R,1} = \mathcal{M}_{R,1}^{(0)} = \{x \in \mathcal{F}_d^1 \,:\, \varphi(x) = R^* x R\}. \tag{3.9}$$

Proof. Recall that $\lambda_R^n = \lambda_{{}_nR}$ (2.10) and ${}_nR = \varphi^{n-1}(R)\cdots\varphi(R)R \in \mathcal{F}_d^{n+1}$. Then the two equalities in the first line immediately follow from (3.1) and (3.2).

In the second line, the first equality is the definition of $\mathcal{M}_{R,n}^{(0)}$ and the second equality follows by combining (3.1) with (3.4) and ${}_nR \in \mathcal{F}_d^{n+1}$. To get the last equality, note that for $x \in \mathcal{F}_d^n$,

$$\lambda_{R^*}(x) = \mathrm{ad}(R^*)_n(x) = \mathrm{ad}({}_nR)^*(x),$$

and therefore, $x \in (\mathcal{F}_d^n)^{\mathrm{ad}\,{}_nR\circ\varphi}$ is equivalent to $x \in \mathcal{F}_d^n$ with $\varphi(x) = \mathrm{ad}({}_nR)^*(x) = \lambda_{R^*}(x)$. The special case $n = 1$ now follows from the previous statements. □

As an example, we consider the structure of $\mathcal{M}_{R,1}$ for a class of simple R-matrices. Namely, if R is a simple R-matrix (Definition 2.8(i)) with projections $p_1, \ldots, p_N \in \mathcal{F}_d^1$ and parameters c_{ij}, $i, j \in \{1, \ldots, N\}$, such that $c_{ij} = 1$ for $i \neq j$, then one finds by a straightforward but tedious calculation that

$$\mathcal{M}_{R,1} \cong \underbrace{\mathbb{C} \oplus \ldots \oplus \mathbb{C}}_{N-m \text{ terms}} \oplus M_m, \tag{3.10}$$

where $m = |\{i \in \{1, \ldots, N\} : \tau(p_i) = 1/d,\ c_{ii} = 1\}|$. In particular, these R-matrices are reducible in the sense that $\mathcal{M}_{R,1} \neq \mathbb{C}$ unless $R \in \mathbb{C}$ $(N = 1)$.

The relative commutants $\mathcal{N}_{R,n}$ of the type II_1 factors have been characterized before. Transferred to our setting, the following result can be extracted from [**2**] and [**14**, Proposition 2.3].

PROPOSITION 3.2. *Let $R \in \mathcal{R}(d)$ and $n \in \mathbb{N}$. Then*

$$\mathcal{N}_{R,n} = \bigcap_{k \geqslant 0} (\mathrm{ad}\,{}_nR \circ \varphi)^k(\mathcal{F}_d^n) \tag{3.11}$$

is the largest subalgebra of $\mathcal{F}_d^n$ that is globally stable under $\mathrm{ad}\,{}_nR \circ \varphi$. In particular,

$$\mathcal{M}_{R,n}^{(0)} \subset \mathcal{N}_{R,n}, \quad n \in \mathbb{N}, \quad \mathcal{M}_{R,1} \subset \mathcal{N}_{R,1} \subset \mathcal{F}_d^1. \tag{3.12}$$

REMARK 3.3. Let us give an example showing that in general, $\mathcal{M}_{R,1} \neq \mathcal{N}_{R,1}$. For later use, we actually give two similar examples, both based on the flip F and a unitary $u \in \mathcal{F}_d^1$, namely

$$R := uF, \quad S := uFu^* = u\varphi(u^*)F.$$

Both R and S are R-matrices, as can be checked by direct verification of the YBE, or by realizing that they are diagonal (Definition 2.8). For $x \in \mathcal{F}_d^1$, we have

$$(\mathrm{ad}\,R \circ \varphi)(x) = RFxFR^* = uxu^*,$$
$$(\mathrm{ad}\,S \circ \varphi)(x) = SFxFS^* = u\varphi(u^*)x\varphi(u)u^* = uxu^*.$$

Thus, $\mathcal{F}_d^1$ is globally invariant under $\mathrm{ad}\,R \circ \varphi$ and $\mathrm{ad}\,S \circ \varphi$, and therefore $\mathcal{N}_{R,1} = \mathcal{N}_{S,1} = \mathcal{F}_d^1$. But for $u \notin \mathbb{C}$, the above formula shows that not every $x \in \mathcal{F}_d^1$ is a fixed point of $\mathrm{ad}\,R \circ \varphi$ or $\mathrm{ad}\,S \circ \varphi$, that is, $\mathcal{M}_{R,1} = \mathcal{M}_{S,1}$ is a proper subalgebra of $\mathcal{F}_d^1$.

We now move on to the relative commutants $\mathcal{L}_{R,n}$ on the level of the von Neumann algebra $\mathcal{L}_R$ generated by the B_∞-representation ρ_R. In this representation, R represents the first generator $b_1 \in B_\infty$; in particular, $\mathcal{L}_R = \mathcal{L}_{R^*}$.

The following result contains a fixed point characterization $\mathcal{L}_{R,n} = \mathcal{L}_R^{\lambda_{\varphi^n(R)}}$ which is similar to the work of Gohm and Köstler [**30**], where analogues of $\lambda_{\varphi^n(R)}$ are called ‘partial shifts.’

PROPOSITION 3.4. *Let $R \in \mathcal{R}(d)$ and $n \in \mathbb{N}_0$. Then*

(i) $\mathcal{L}_{R,n} = \mathcal{F}_d^n \cap \mathcal{L}_R = \mathcal{L}_R^{\lambda_{\varphi^n(R)}} = \mathcal{M}_{R,n} \cap \mathcal{L}_R$, *and all these algebras are invariant under exchanging R and R^*.*

(ii) $C^*(\rho_R(B_n)) \subset \mathcal{L}_{R,n}$, $n \geqslant 1$.

Proof. (i) We will demonstrate the inclusions

$$\mathcal{L}_{R,n} \overset{\text{(i)}}{\subset} \mathcal{L}_R^{\lambda_{\varphi^n(R)}} \overset{\text{(ii)}}{\subset} \mathcal{M}_{R^*,n} \cap \mathcal{L}_R \overset{\text{(iii)}}{\subset} \mathcal{F}_d^n \cap \mathcal{L}_R \overset{\text{(iv)}}{\subset} \mathcal{L}_{R,n}.$$

Note the appearance of R^* instead of R in the third algebra. Nonetheless, this chain of inclusions implies the claimed equalities because we have $\mathcal{L}_R = \mathcal{L}_{R^*}$ and may thus run through the chain of inclusions once more with R and R^* interchanged, realizing that all algebras are invariant under replacing R with R^*.

To begin with, we note that $\lambda_{\varphi^n(R)}(x)$, $x \in \mathcal{N}$, can be written as

$$\begin{aligned}\lambda_{\varphi^n(R)}(x) &= \lim_{k\to\infty} \varphi^n(R)\cdots\varphi^{n+k}(R)x\varphi^{n+k}(R^*)\cdots\varphi^n(R^*) \\ &= \varphi^{n-1}(R^*)\cdots R^*\lambda_R(x)R\cdots\varphi^{n-1}(R) = {}_n(R^*)\lambda_R(x)_n(R^*)^*.\end{aligned}$$

The first line shows that any $x \in \varphi^n(\mathcal{L}_R)'$ is fixed by $\lambda_{\varphi^n(R)}$, that is, we have inclusion (i). Any $x \in \mathcal{L}_R$ satisfies $\lambda_R(x) = \varphi(x)$, and thus, the above calculation yields

$$\mathcal{L}_R^{\lambda_{\varphi^n(R)}} \subset \{x \in \mathcal{L}_R : x = (\mathrm{ad}\,_n(R^*) \circ \varphi)(x)\} = \mathcal{M}_{R^*,n} \cap \mathcal{L}_R,$$

where we have used Proposition 3.1. This shows the inclusion (ii).

As $\mathcal{L}_R \subset \mathcal{N}$, we also have $\mathcal{M}_{R^*,n} \cap \mathcal{L}_R = \mathcal{M}_{R^*,n}^{(0)} \cap \mathcal{L}_R \subset \mathcal{F}_d^n \cap \mathcal{L}_R$ by Proposition 3.1, showing inclusion (iii). Inclusion (iv) is evident because $\mathcal{F}_d^n$ and $\varphi^n(\mathcal{L}_R)$ commute in $\mathcal{N}$.

(ii) By definition of ρ_R, we have $C^*(\rho_R(B_n)) \subset \mathcal{F}_d^n \cap \mathcal{L}_R = \mathcal{L}_{R,n}$. □

We have seen that the relative commutants satisfy

$$\mathcal{L}_{R,n} \subset \mathcal{M}_{R,n}^{(0)} \subset \mathcal{N}_{R,n} \subset \mathcal{F}_d^n, \quad n \in \mathbb{N}. \tag{3.13}$$

We furthermore note that λ_R and ϕ_R act on these three towers according to

$$\lambda_R(\mathcal{M}_{R,n}) \subset \mathcal{M}_{R,n+1}, \quad \lambda_R(\mathcal{N}_{R,n}) \subset \mathcal{N}_{R,n+1}, \quad \lambda_R(\mathcal{L}_{R,n}) \subset \mathcal{L}_{R,n+1}, \tag{3.14}$$

$$\phi_R(\mathcal{M}_{R,n+1}) = \mathcal{M}_{R,n}, \quad \phi_R(\mathcal{N}_{R,n+1}) = \mathcal{N}_{R,n}, \quad \phi_R(\mathcal{L}_{R,n+1}) = \mathcal{L}_{R,n}, \tag{3.15}$$

and that R fits into these algebras via

$$R \in \mathcal{L}_{R,2} \subset \mathcal{M}_{R,2}^{(0)} \subset \mathcal{M}_{R,2} \cap \mathcal{N}_{R,2}, \tag{3.16}$$

$$\phi_R(R) \in \mathcal{L}_{R,1} \subset \mathcal{M}_{R,1} \subset \mathcal{N}_{R,1}. \tag{3.17}$$

This also implies that the inclusion $C^*(\rho_R(B_n)) \subset \mathcal{L}_{R,n}$ in Proposition 3.4(ii) is proper in general. For example, for $n = 1$, the group B_n is trivial, that is, $C^*(\rho_R(B_1)) = \mathbb{C}$, but $\mathcal{L}_{R,1}$ contains $\phi_R(R)$ which is non-trivial in general.

Our main results concerning the relative positions of $\varphi^n(\mathcal{L}_R)$ and $\mathcal{L}_{R,n}$ in $\mathcal{N}$ are stated in the following theorem. The τ-preserving conditional expectation $\mathcal{N} \to \mathcal{F}_d^n$ will be denoted by E_n.

THEOREM 3.5. *Let $R \in \mathcal{R}$ and $n \in \mathbb{N}$. Then the squares*

$$\begin{array}{ccc} \mathcal{F}_d^n & \subset & \mathcal{N} \\ \cup & & \cup \\ \mathcal{L}_{R,n} & \subset & \mathcal{L}_R \end{array} \qquad \begin{array}{ccc} \varphi^n(\mathcal{N}) & \subset & \mathcal{N} \\ \cup & & \cup \\ \varphi^n(\mathcal{L}_R) & \subset & \mathcal{L}_R \end{array} \tag{3.18}$$

commute, that is, $E_n(\mathcal{L}_R) = \mathcal{L}_{R,n}$ *and* $\phi_R(x) = \phi_F(x)$, $x \in \mathcal{L}_R$.

The proof splits naturally into two parts, one for each diagram. The proof for the left diagram is given below. The proof for the right diagram requires more work and is best done after more structure has been introduced. It is therefore postponed to Section 5 (p. 21).

Proof (first half). Let $H_{R,n}$ denote the τ-preserving conditional expectation of $\mathcal{N}^{\lambda_{\varphi^n(R)}} \subset \mathcal{N}$. As $\mathcal{L}_R \subset \mathcal{N}$ is invariant under $\lambda_{\varphi^n(R)}$ by Proposition 3.4, the map $H_{R,n}$ restricts to the τ-preserving conditional expectation from $\mathcal{L}_R$ onto $\mathcal{L}^{\lambda_{\varphi^n(R)}} = \mathcal{L}_{R,n} = \mathcal{F}_d^n \cap \mathcal{L}_R$.

Given $x \in \mathcal{L}_R$, we want to show that $H_{R,n}(x)$ coincides with $E_n(x)$. Indeed, both $H_{R,n}(x)$ and $E_n(x)$ lie in $\mathcal{F}_d^n$, so we only have to show $\tau(yH_{R,n}(x)) = \tau(yE_n(x))$ for all $y \in \mathcal{F}_d^n$. But $\mathcal{F}_d^n$ is clearly contained in the fixed point algebra $\mathcal{N}^{\lambda_{\varphi^n(R)}}$. Thus, for $x \in \mathcal{L}_R$, $y \in \mathcal{F}_d^n$,

$$\tau(yH_{R,n}(x)) = \tau(H_{R,n}(yx)) = \tau(yx) = \tau(E_n(yx)) = \tau(yE_n(x)).$$

This shows $E_n(x) = H_{R,n}(x) \in \mathcal{L}_{R,n}$, which is equivalent to the left square commuting. $\square$

So far, we have concentrated on the 'horizontal inclusions' in (3.6). The 'vertical inclusions' $\mathcal{L}_R \subset \mathcal{N}$, $\mathcal{L}_R \subset \mathcal{M}$, are closely connected to fixed points of λ_R and will be discussed in Section 7.

4. *Algebraic operations on $\mathcal{R}$*

Although the structure of the set $\mathcal{R}(d)$ of all R-matrices of dimension d is not known, a number of symmetries of $\mathcal{R}(d)$ are known. For example, $R \mapsto R^*$, $R \mapsto c \cdot R$, $c \in \mathbb{T}$, $R \mapsto (u \otimes u)R(u \otimes u)^*$, $u \in \mathcal{U}(\mathcal{F}_d^1)$, and $R \mapsto FRF$ with the flip $F \in \mathcal{R}(d)$, are bijections of $\mathcal{R}(d)$[†].

However, it is often more interesting to consider algebraic operations that exist only on $\mathcal{R} = \bigcup_d \mathcal{R}(d)$ and do not preserve the spaces $\mathcal{R}(d)$ of R-matrices of fixed dimension d. In this section, we will discuss three such structures: A tensor product $R \boxtimes S$ (with $\dim(R \boxtimes S) = \dim R \cdot \dim S$), Wenzl's cabling powers $R^{(n)}$ (with $\dim(R^{(n)}) = (\dim R)^n$), and a sum operation $R \boxplus S$ (with $\dim(R \boxplus S) = \dim R + \dim S$).

On the level of R-matrices, all these operations are known. What is new in our approach is that we relate them to natural operations on the corresponding Yang–Baxter endomorphisms.

In the following, the dimension d will be explicitly indicated in our notation, that is, we write $\mathcal{N}_d$ for the infinite tensor product of matrix algebras M_d, and τ_d, φ_d for its canonical trace and shift, $F_d \in \mathcal{U}(\mathcal{F}_d^2)$ for the flip in dimension d, etc.

4.1. *Tensor products of R-matrices*

Let $R \in \mathcal{R}(d) \subset \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$, $\tilde{R} \in \mathcal{R}(\tilde{d}) \subset \mathrm{End}(\mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^{\tilde{d}})$ be R-matrices. The tensor product of R, $\tilde{R}$ is defined as

$$R \boxtimes \tilde{R} := F_{23}(R \otimes \tilde{R})F_{23} \in \mathrm{End}((\mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}}) \otimes (\mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}})), \tag{4.1}$$

where $F_{23} : \mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}} \to \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^{\tilde{d}}$ is the flip unitary exchanging the two middle factors. Evidently $R \boxtimes \tilde{R}$ is a unitary R-matrix of dimension $d\tilde{d}$, that is, $R \boxtimes \tilde{R} \in \mathcal{R}(d\tilde{d})$. We will refer to $R \boxtimes \tilde{R}$ as the *tensor product* of R and $\tilde{R}$ (although it slightly differs from the actual tensor product $R \otimes \tilde{R}$). It is also clear that $(R \boxtimes \tilde{R})^* = R^* \boxtimes \tilde{R}^*$, and that if both R and $\tilde{R}$ are involutive, then so is $R \boxtimes \tilde{R}$.

From the point of view of the Cuntz algebras, we may consider $R \in \mathcal{F}_d^2$, $S \in \mathcal{F}_{\tilde{d}}^2$, and $R \boxtimes \tilde{R} \in \mathcal{F}_{d\tilde{d}}^2$. The following discussion will give us a precise relation between the associated subfactors.

Let $\mathcal{O}_d$ and $\mathcal{O}_{\tilde{d}}$ be Cuntz algebras with canonical generators S_i, $1 \leqslant i \leqslant d$ and $\tilde{S}_j$, $1 \leqslant j \leqslant \tilde{d}$, respectively. Namely, all the S_i and $\tilde{S}_j$ are isometries such that $\sum_{i=1}^d S_i S_i^* = 1$,

[†]The maps $R \mapsto (u \otimes u)R(u \otimes u)^*$ and $R \mapsto FRF$ will be discussed in more detail in Section 5.

$\sum_{j=1}^{\tilde d}\tilde S_j\tilde S_j^*=1$, and $\mathcal{O}_d=C^*(S_1,\dots,S_d)$, $\mathcal{O}_{\tilde d}=C^*(\tilde S_1,\dots,\tilde S_{\tilde d})$. The tensor product C^*-algebra $\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}$ is generated by the elements $S_i\otimes 1$ and $1\otimes\tilde S_j$, $1\leqslant i\leqslant d$, $1\leqslant j\leqslant\tilde d$.† In general, $\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}$ is not a Cuntz algebra‡.

Consider also the Cuntz algebra $\mathcal{O}_{d\tilde d}$, with canonical generating isometries U_{ij}, $1\leqslant i\leqslant d, 1\leqslant j\leqslant\tilde d$ such that $\sum_{i,j}U_{ij}U_{ij}^*=1$. Since, for every $1\leqslant i\leqslant d$ and $1\leqslant j\leqslant\tilde d$, $S_i\otimes\tilde S_j$ is an isometry in $\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}$ and, moreover, $\sum_{i,j}S_i\otimes\tilde S_j(S_i\otimes\tilde S_j)^*=(\sum_i S_iS_i^*)\otimes(\sum_j\tilde S_j\tilde S_j^*)=1\otimes 1$, there is an injective $*$-homomorphism

$$\iota_{d,\tilde d}:\mathcal{O}_{d\tilde d}\to\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}\tag{4.2}$$

such that $\iota_{d,\tilde d}(U_{ij})=S_i\otimes\tilde S_j$.

In order to simplify the notation, in the sequel, we will often drop the symbol $\iota_{d,\tilde d}$ and identify accordingly U_{ij} with $S_i\otimes\tilde S_j$. All in all, we have thus identified a copy of $\mathcal{O}_{d\tilde d}$ inside $\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}$, as the C^*-subalgebra of the tensor product generated by the isometries $S_i\otimes\tilde S_j$. Moreover, it is not difficult to see that $\mathcal{O}_{d\tilde d}=(\mathcal{O}_d\otimes\mathcal{O}_{\tilde d})^\beta$, where β denotes the 2π-periodic 'twisted' $\mathbb{R}$-action $\beta^t:=\alpha_d^t\otimes\alpha_{\tilde d}^{-t}=\lambda_{e^{it}1_d}\otimes\lambda_{e^{-it}1_{\tilde d}}$[**9, 52**], and there exists a faithful conditional expectation $\mathcal{O}_d\otimes\mathcal{O}_{\tilde d}\to\mathcal{O}_{d\tilde d}$ obtained by averaging β.

Under the identification of $\mathcal{O}_{d\tilde d}$ with $(\mathcal{O}_d\otimes\mathcal{O}_{\tilde d})^\beta$, there are coherent identifications of $\mathcal{F}^n_{d\tilde d}$ with $\mathcal{F}^n_d\otimes\mathcal{F}^n_{\tilde d}$, $n\in\mathbb{N}$, such that

$$\begin{aligned}&U_{i_1j_1}U_{i_2j_2}\cdots U_{i_nj_n}U^*_{i'_nj'_n}\cdots U^*_{i'_2j'_2}U^*_{i'_1j'_1}\\&\quad=(S_{i_1}\otimes\tilde S_{j_1})(S_{i_2}\otimes\tilde S_{j_2})\cdots(S_{i_n}\otimes\tilde S_{j_n})(S_{i'_n}\otimes\tilde S_{j'_n})^*\cdots(S_{i'_2}\otimes\tilde S_{j'_2})^*(S_{i'_1}\otimes\tilde S_{j'_1})^*\\&\quad=(S_{i_1}S_{i_2}\cdots S_{i_n}S^*_{i'_n}\cdots S^*_{i'_2}S^*_{i'_1})\otimes(\tilde S_{j_1}\tilde S_{j_2}\cdots\tilde S_{j_n}\tilde S^*_{j'_n}\cdots\tilde S^*_{j'_2}\tilde S^*_{j'_1}),\end{aligned}$$

and thus of $\mathcal{F}_{d\tilde d}=\mathcal{O}_{d\tilde d}^{\alpha_{d\tilde d}}$ with $\mathcal{F}_d\otimes\mathcal{F}_{\tilde d}=\mathcal{O}_d^{\alpha_d}\otimes\mathcal{O}_{\tilde d}^{\alpha_{\tilde d}}$.

For the following lemma, the YBE is not needed.

Lemma 4.1. (i) *Let $R\in\mathcal{U}(\mathcal{O}_d)$ and $\tilde R\in\mathcal{U}(\mathcal{O}_{\tilde d})$. Then $\lambda_R\otimes\lambda_{\tilde R}\in\mathrm{End}(\mathcal{O}_d\otimes\mathcal{O}_{\tilde d})$ restricts to an endomorphism of $\mathcal{O}_{d\tilde d}$ if and only if $R\in\mathcal{F}_d$ and $\tilde R\in\mathcal{F}_{\tilde d}$.*

(ii) *Let $R\in\mathcal{U}(\mathcal{F}^2_d)$, $\tilde R\in\mathcal{U}(\mathcal{F}^2_{\tilde d})$. Then $\iota_{d,\tilde d}(R\boxtimes\tilde R)=R\otimes\tilde R$, and*

$$(\lambda_R\otimes\lambda_{\tilde R})|_{\mathcal{O}_{d\tilde d}}=\lambda_{R\boxtimes\tilde R}.\tag{4.3}$$

Proof. (i) On generators, the endomorphism $\lambda_R\otimes\lambda_{\tilde R}\in\mathrm{End}(\mathcal{O}_d\otimes\mathcal{O}_{\tilde d})$ acts according to $(\lambda_R\otimes\lambda_{\tilde R})(S_i\otimes\tilde S_j)=(R\otimes\tilde R)(S_i\otimes\tilde S_j)$ for all i, j. Thus, $\lambda_R\otimes\lambda_{\tilde R}$ restricts to $\mathcal{O}_{d\tilde d}$, that is $(\lambda_R\otimes\lambda_{\tilde R})(\mathcal{O}_{d\tilde d})\subset\mathcal{O}_{d\tilde d}$, precisely when $R\otimes\tilde R\in\mathcal{O}_{d\tilde d}$, that is, precisely when $\alpha_d^t(R)\otimes\alpha_{\tilde d}^{-t}(\tilde R)=R\otimes\tilde R$ for all $t\in\mathbb{R}$. This latter condition is satisfied if and only if both R and $\tilde R$ are eigenvectors for α_d and $\alpha_{\tilde d}$, respectively, that is, $R\in\mathcal{O}_d^{(n)}$, $\tilde R\in\mathcal{O}_{\tilde d}^{(n)}$ for some $n\in\mathbb{Z}$. But this is easily seen to be in conflict with the KMS condition for ω if $n\neq 0$. Thus, $R\in\mathcal{O}_d^{(0)}=\mathcal{F}_d$, $\tilde R\in\mathcal{O}_{\tilde d}^{(0)}=\mathcal{F}_{\tilde d}$.

(ii) Note that the matrix elements of $R\boxtimes\tilde R$ are $(R\boxtimes\tilde R)^{(\alpha i)(\beta j)}_{(\gamma k)(\delta l)}=R^{\alpha\beta}_{\gamma\delta}\tilde R^{ij}_{kl}$, where $\alpha,\beta,\gamma,\delta\in\{1,\dots,d\}$ and $i,j,k,l\in\{1,\dots,\tilde d\}$. Thus,

$$R\boxtimes\tilde R=\sum R^{\alpha\beta}_{\gamma\delta}\tilde R^{ij}_{kl}U_{\alpha i}U_{\beta j}U^*_{\delta l}U^*_{\gamma k}\overset{\iota_{d\tilde d}}{\longmapsto}\sum R^{\alpha\beta}_{\gamma\delta}\tilde R^{ij}_{kl}S_\alpha S_\beta S^*_\delta S^*_\gamma\otimes S_iS_jS^*_lS^*_k=R\otimes\tilde R,$$

and the calculation in (i) shows $(\lambda_R\otimes\lambda_{\tilde R})|_{\mathcal{O}_{d\tilde d}}=\lambda_{R\boxtimes\tilde R}$. $\square$

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†Since $\mathcal{O}_d$ is nuclear, there is no ambiguity on the choice of the cross-norm on the algebraic tensor product.

‡However, it is well known that $\mathcal{O}_2\otimes\mathcal{O}_d\simeq\mathcal{O}_2$, for all $d\geqslant 2$, although none of these isomorphisms has been concretely exhibited.

Let us look at two special cases, the identity $1_d \in \mathcal{O}_d$ and the flip $F_d \in \mathcal{O}_d$. Then $1_d \boxtimes 1_{\tilde{d}} = 1_{d\tilde{d}}$ and $F_d \boxtimes F_{\tilde{d}} = F_{d\tilde{d}}$. For the canonical 2π-periodic actions of $\mathbb{R}$, this implies that $\lambda_{e^{it}1_d} \otimes \lambda_{e^{it}1_{\tilde{d}}} \in \mathrm{Aut}(\mathcal{O}_d \otimes \mathcal{O}_{\tilde{d}})$ restricts to $\lambda_{e^{2it}1_{d\tilde{d}}}$ on $\mathcal{O}_{d\tilde{d}}$, and for the canonical shifts, this implies that $\varphi_d \otimes \varphi_{\tilde{d}}$ restricts to $\varphi_{d\tilde{d}}$. Indeed, for all i and j,

$$\begin{aligned}\varphi_{d\tilde{d}}(S_i \otimes \tilde{S}_j) &= \sum_{i',j'} (S_{i'} \otimes \tilde{S}_{j'})(S_i \otimes \tilde{S}_j)(S_{i'} \otimes \tilde{S}_{j'})^* \\ &= \left(\sum_{i'} S_{i'} S_i S_{i'}^*\right) \otimes \left(\sum_{j'} \tilde{S}_{j'} \tilde{S}_j \tilde{S}_{j'}^*\right) = \varphi_d(S_i) \otimes \varphi_{\tilde{d}}(\tilde{S}_j).\end{aligned}$$

Notice that the index of $\varphi_d(\mathcal{N}_d) \subset \mathcal{N}_d$ is d^2, so that in this example, we see immediately that the index of the endomorphism associated to the tensor product $F_d \boxtimes F_{\tilde{d}}$ is the product of the indices of the endomorphisms given by F_d and $F_{\tilde{d}}$.

This is an instance of a general fact. Since $\mathcal{F}_{d\tilde{d}}$ is identified with $\mathcal{F}_d \otimes \mathcal{F}_{\tilde{d}}$, the same holds on the level of the weak closures, and

$$\lambda_{R\boxtimes\tilde{R}}(\mathcal{N}_{d\tilde{d}}) = (\lambda_R \otimes \lambda_{\tilde{R}})(\mathcal{N}_d \otimes \mathcal{N}_{\tilde{d}}) = \lambda_R(\mathcal{N}_d) \otimes \lambda_{\tilde{R}}(\mathcal{N}_{\tilde{d}}). \tag{4.4}$$

From here we readily get the multiplicativity of the Jones index under the tensor product.

THEOREM 4.2. *Let $R \in \mathcal{U}(\mathcal{F}_d^2)$, $\tilde{R} \in \mathcal{U}(\mathcal{F}_{\tilde{d}}^2)$. Then the Jones indices of the type II_1 subfactors associated to R, $\tilde{R}$, and $R \boxtimes \tilde{R}$ are related by*

$$[\mathcal{N}_{d\tilde{d}} : \lambda_{R\boxtimes\tilde{R}}(\mathcal{N}_{d\tilde{d}})] = [\mathcal{N}_d : \lambda_R(\mathcal{N}_d)] \cdot [\mathcal{N}_{\tilde{d}} : \lambda_{\tilde{R}}(\mathcal{N}_{\tilde{d}})] \ . \tag{4.5}$$

Since this result applies in particular to R-matrices, we see that the subset of the positive real line $\mathbb{R}_+$ of all Jones indices arising from unitary solutions of the YBE (in any dimension) is closed under taking ordinary products.

The next result is about the relative commutants given by the tensor product.

PROPOSITION 4.3. *Let $R \in \mathcal{R}(d)$, $\tilde{R} \in \mathcal{R}(\tilde{d})$. Then*

$$\mathcal{M}_{R,1} \otimes \mathcal{M}_{\tilde{R},1} \subseteq \mathcal{M}_{R\boxtimes\tilde{R},1} \subseteq \mathcal{N}_{R\boxtimes\tilde{R},1} = \mathcal{N}_{R,1} \otimes \mathcal{N}_{\tilde{R},1}. \tag{4.6}$$

Proof. On the one hand,

$$\begin{aligned}\mathcal{M}_{R,1} \otimes \mathcal{M}_{\tilde{R},1} &= \{x \in \mathcal{F}_d^1 \ : \ \lambda_{R^*}(x) = \varphi_d(x)\} \otimes \{y \in \mathcal{F}_{\tilde{d}}^1 \ : \ \lambda_{\tilde{R}^*}(y) = \varphi_{\tilde{d}}(y)\} \\ &\subseteq \{T \in \mathcal{F}_d^1 \otimes \mathcal{F}_{\tilde{d}}^1 \ : \ (\lambda_{R^*} \otimes \lambda_{\tilde{R}^*})(T) = (\varphi_d \otimes \varphi_{\tilde{d}})(T)\} \\ &= \{T \in \mathcal{F}_{d\tilde{d}}^1 \ : \ \lambda_{(R\boxtimes\tilde{R})^*}(T) = \varphi_{d\tilde{d}}(T)\} = \mathcal{M}_{R\boxtimes\tilde{R},1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{M}_{R\boxtimes\tilde{R},1} \subseteq \mathcal{N}_{R\boxtimes\tilde{R},1} &= \{T \in \mathcal{F}_{d\tilde{d}}^1 \ : \ [T, \lambda_{R\boxtimes\tilde{R}}(x)] = 0, \ x \in \mathcal{F}_{d\tilde{d}} \ \} \\ &= \{T \in \mathcal{F}_d^1 \otimes \mathcal{F}_{\tilde{d}}^1 \ : \ [T, (\lambda_R \otimes \lambda_{\tilde{R}})(x)] = 0, \ x \in \mathcal{F}_d \otimes \mathcal{F}_{\tilde{d}}\} \\ &= \big(\lambda_R(\mathcal{F}_d) \otimes \lambda_{\tilde{R}}(\mathcal{F}_{\tilde{d}})\big)' \cap \big(\mathcal{F}_d^1 \otimes \mathcal{F}_{\tilde{d}}^1\big) \\ &= \big(\lambda_R(\mathcal{N}_d)' \otimes \lambda_{\tilde{R}}(\mathcal{N}_{\tilde{d}})'\big) \cap (\mathcal{F}_d^1 \otimes \mathcal{F}_{\tilde{d}}^1) = \mathcal{N}_{R,1} \otimes \mathcal{N}_{\tilde{R},1}.\end{aligned}$$

□

FIGURE 1. *Illustration of* ${}_3R_3 = \varphi^2(R)\varphi(R)R \cdot \varphi^3(R)\varphi^2(R)\varphi(R) \cdot \varphi^4(R)\varphi^3(R)\varphi^2(R)$.

4.2. *Cabling powers of R-matrices*

The second algebraic operations on $\mathcal{R}$ that we want to discuss are cabling powers [**55, 63**]. Given $d, n \in \mathbb{N}$, we define 'cabling maps' between type II$_1$-factors, $c_n : \mathcal{N}_d \to \mathcal{N}_{d^n}$, such that $c_n(\bigotimes_{i=1}^{nm} M_d) = \bigotimes_{i=1}^{m} M_{d^n}$ for all $m \geqslant 1$, by linear and weakly continuous extension from algebraic tensor products,

$$c_n\left(\bigotimes_{i=1}^{nm} x_i\right) := \left(\bigotimes_{i=1}^{n} x_i\right) \otimes \left(\bigotimes_{i=n+1}^{2n} x_i\right) \otimes \ldots \otimes \left(\bigotimes_{i=(m-1)n+1}^{nm} x_i\right), \quad x_i \in M_d. \tag{4.7}$$

It follows that c_n is an isomorphism with the properties

$$c_n(1) = 1, \quad \tau_{d^n} \circ c_n = \tau_d, \quad \varphi_{d^n} \circ c_n = c_n \circ \varphi_d^n, \quad c_n(\mathcal{F}_d^{kn}) = \mathcal{F}_{d^n}^k, \quad k \in \mathbb{N}.$$

To define the nth cabling power of $R \in \mathcal{R}(d)$, we also introduce

$$\begin{aligned} {}_nR_n &:= ({}_nR)_n = {}_nR \cdots \varphi^{n-1}({}_nR) \\ &= \varphi^{n-1}(R) \cdots R \cdot \varphi^n(R) \cdots \varphi(R) \cdots \varphi^{2n-2}(R) \cdots \varphi^{n-1}(R) = {}_n(R_n). \end{aligned}$$

Note that ${}_nR_n$ is a unitary in $\mathcal{F}_d^{2n}$ which satisfies $({}_nR_n)^* = {}_n(R^*)_n$. For low n, we have ${}_1R_1 = R$ and ${}_2R_2 = \varphi(R)R\varphi^2(R)\varphi(R)$. A graphical illustration of ${}_3R_3$ is given in Figure 1.

Wenzl's cabling powers of an R-matrix take in our setting the following form.

DEFINITION 4.4. Let $R \in \mathcal{R}(d)$ and $n \in \mathbb{N}$. The *nth cabling power of R is*

$$R^{(n)} := c_n({}_nR_n) \in \mathcal{U}(\mathcal{F}_{d^n}^2) \tag{4.8}$$

$R^{(n)}$ is an R-matrix in $\mathcal{R}(d^n)$, and $(R^{(n)})^* = (R^*)^{(n)}$.

The proof that $R^{(n)} \in \mathcal{R}(d^n)$ can be found in [**63**].

We now show that at least on the level of the type II factor $\mathcal{N}$, cabling powers of R-matrices correspond to ordinary powers of their corresponding Yang–Baxter endomorphisms.

PROPOSITION 4.5. *Let $R \in \mathcal{R}(d)$ and $n \in \mathbb{N}$. Then*

$$(c_n^{-1}\lambda_{R^{(n)}}c_n)(x) = \lambda_R^n(x), \quad x \in \mathcal{N}_d. \tag{4.9}$$

In particular,

$$[\mathcal{N}_{d^n} : \lambda_{R^{(n)}}(\mathcal{N}_{d^n})] = [\mathcal{N}_d : \lambda_R(\mathcal{N}_d)]^n. \tag{4.10}$$

Proof. We calculate, $k \in \mathbb{N}$,

$$\begin{aligned} c_n^{-1}((R^{(n)})_k) &= c_n^{-1}(R^{(n)} \cdots \varphi_{d^n}^{k-1}(R^{(n)})) = {}_nR_n \cdot \varphi_d^n({}_nR_n) \cdots \varphi_d^{n(k-1)}({}_nR_n) \\ &= {}_nR \cdots \varphi^{n-1}({}_nR) \cdot \varphi^n({}_nR) \cdots \varphi^{2n-1}({}_nR) \cdot \ldots \cdot \varphi^{n(k-1)}({}_nR) \cdots \varphi^{nk-1}({}_nR). = ({}_nR)_{kn}. \end{aligned}$$

Hence, for any $x \in \mathcal{N}_d$,

$$(c_n^{-1}\lambda_{R^{(n)}}c_n)(x) = \lim_{k\to\infty} \operatorname{ad} c_n^{-1}((R^{(n)})_k)(x) = \lim_{k\to\infty} \operatorname{ad}(({}_nR)_{nk})(x) = \lambda_{{}_nR}(x) = \lambda_R^n(x).$$

As all the subfactors $\lambda_R^{k+1}(\mathcal{N}) \subset \lambda_R^k(\mathcal{N})$, $k \in \mathbb{N}_0$, are isomorphic, this implies (4.10). □

REMARK 4.6. Let $R \notin \mathbb{C}$ be non-trivial, and recall that λ_R^n is reducible for $n \geqslant 2$ in the sense that $\mathcal{M}_{R,n} \neq \mathbb{C}$; namely $R \in \mathcal{M}_{R,2} \subset \mathcal{N}_{R,2}$. Thus, Proposition 4.5 immediately implies that $\lambda_{R^{(n)}}$ is reducible as an endomorphism of $\mathcal{N}_{d^n}$. This remains true on the level of the III_{1/d^n}-factor because $c_n(R) \in \mathcal{M}_{R^{(n)},1}$.

The identity and the flip reproduce themselves under taking cabling powers, that is, $1_d^{(n)} = 1_{d^n}$ and $F_d^{(n)} = F_{d^n}$. For later reference, we note that this implies in particular

$$\varphi_{d^n} = \lambda_{F^{(n)}} \in \operatorname{End}\mathcal{N}_{d^n}, \quad \phi_{F^{(n)}} = c_n \circ \phi_F^n \circ c_n^{-1}. \tag{4.11}$$

4.3. *Sums of R-matrices*

The third operation on $\mathcal{R}$ that we want to discuss is additive on dimension. Given $R \in \mathcal{R}(d)$, $\tilde{R} \in \mathcal{R}(\tilde{d})$, we define $R \boxplus \tilde{R} \in \operatorname{End}((\mathbb{C}^d \oplus \mathbb{C}^{\tilde{d}}) \otimes (\mathbb{C}^d \oplus \mathbb{C}^{\tilde{d}}))$ by [**45**]

$$R \boxplus \tilde{R} := R \oplus \tilde{R} \oplus F \quad \text{on} \tag{4.12}$$

$$(\mathbb{C}^d \oplus \mathbb{C}^{\tilde{d}}) \otimes (\mathbb{C}^d \oplus \mathbb{C}^{\tilde{d}}) = (\mathbb{C}^d \otimes \mathbb{C}^d) \oplus (\mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^{\tilde{d}}) \oplus ((\mathbb{C}^d \otimes \mathbb{C}^{\tilde{d}}) \oplus (\mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^d)).$$

In other words, $R \boxplus \tilde{R}$ acts as R on $\mathbb{C}^d \otimes \mathbb{C}^d$, as $\tilde{R}$ on $\mathbb{C}^{\tilde{d}} \otimes \mathbb{C}^{\tilde{d}}$, and as the flip on the mixed tensors involving factors from both, $\mathbb{C}^d$ and $\mathbb{C}^{\tilde{d}}$.

If $R, \tilde{R}$ are R-matrices, then so is $R \boxplus \tilde{R}$ [**45**]. We also mention that we clearly have $(R \boxplus \tilde{R})^* = R^* \boxplus \tilde{R}^*$, and $F_d \boxplus F_{\tilde{d}} = F_{d+\tilde{d}}$. The identity is, however, not preserved under this sum. For example, we have $1_1 \boxplus 1_1 = F_2$.

Given $R \in \mathcal{R}(d)$, $\tilde{R} \in \mathcal{R}(\tilde{d})$, we get an endomorphism $\lambda_{R\boxplus\tilde{R}} \in \operatorname{End}(\mathcal{O}_{d+\tilde{d}})$. We currently have no detailed picture of $\lambda_{R\boxplus\tilde{R}}$. However, it is clear that $\lambda_{R\boxplus\tilde{R}}$ is always reducible, as follows from the following result.

PROPOSITION 4.7. *Let $R \in \mathcal{R}(d), \tilde{R} \in \mathcal{R}(\tilde{d})$. Then*

$$\mathcal{M}_{R,1} \oplus \mathcal{M}_{\tilde{R},1} \subset \mathcal{M}_{R\boxplus\tilde{R},1}; \tag{4.13}$$

in particular, $\lambda_{R\boxplus\tilde{R}}$ is always reducible. The inclusion (4.13) is proper in general. We also have

$$\phi_{R\boxplus\tilde{R}}(R \boxplus \tilde{R}) = \tfrac{d}{d+\tilde{d}}\,\phi_R(R) \oplus \tfrac{\tilde{d}}{d+\tilde{d}}\,\phi_{\tilde{R}}(\tilde{R}). \tag{4.14}$$

Proof. Let $x \in \mathcal{M}_{R,1} \subset \mathcal{F}_d^1$ and $\tilde{x} \in \mathcal{M}_{\tilde{R},1} \subset \mathcal{F}_{\tilde{d}}^1$, that is, $R^*xR = \varphi_d(x)$ and $\tilde{R}^*\tilde{x}\tilde{R} = \varphi_{\tilde{d}}(\tilde{x})$. We may view $\mathcal{F}_{d+\tilde{d}}^1$ as $\operatorname{End}(\mathbb{C}^d \oplus \mathbb{C}^{\tilde{d}})$, and define $p := 1 \oplus 0$, $p^\perp := 1 - p = 0 \oplus 1$ to be the orthogonal projections onto the two summands. Then

$$\begin{aligned}(R^* \boxplus \tilde{R}^*)(x \oplus \tilde{x})(R \boxplus \tilde{R}) &= (R^* \boxplus \tilde{R}^*)(pxp + p^\perp\tilde{x}p^\perp)\varphi_{d+\tilde{d}}(p+p^\perp)(R \boxplus \tilde{R})\\ &= p\varphi_d(p)R^*xRp\varphi_d(p) + p^\perp\varphi_{\tilde{d}}(p^\perp)\tilde{R}^*\tilde{x}\tilde{R}p^\perp\varphi_{\tilde{d}}(p^\perp)\\ &\quad + \varphi_d(pxp)p^\perp + \varphi_{\tilde{d}}(p^\perp\tilde{x}p^\perp)p\end{aligned}$$

$$= p\varphi_d(pxp) + p^\perp\varphi_{\tilde d}(p^\perp\tilde x p^\perp) + \varphi_d(pxp)p^\perp + \varphi_{\tilde d}(p^\perp\tilde x p^\perp)p$$
$$= \varphi_d(pxp) + \varphi_{\tilde d}(p^\perp\tilde x p^\perp) = \varphi_{d+\tilde d}(x\oplus\tilde x).$$

This proves $x\oplus\tilde x\in\mathcal{M}_{R\boxplus\tilde R,1}$.

The second statement follows from Theorem 3.5: For each R-matrix $R\in\mathcal{R}(d)$, we have $\phi_R(R)=\phi_F(R)$ with $F\in\mathcal{R}(d)$ the flip, that is, $\phi_R(R)$ coincides with the normalized left partial trace of R. The claim then follows from the fact that the non-normalized partial trace maps $\boxplus$ sums to direct sums [**45**, Lemma 4.2(iv)]. □

Remark 4.8. The sum operation $\boxplus$ allows us to write down many examples of R-matrices and is the concept behind the definition of simple R-matrices (Definition 2.8). Namely, we can start from trivial R-matrices $R=c\cdot 1_d\in\mathcal{R}(d)$, $c\in\mathbb{T}$, and build non-trivial ones by summation, that is,

$$R = c_1 1_{d_1}\boxplus c_2 1_{d_2}\boxplus\cdots\boxplus c_N 1_{d_N}\in\mathcal{R}(d_1+\cdots+d_N),\quad c_1,\ldots,c_N\in\mathbb{T}.$$

Note that we may describe such R-matrices equivalently as follows: There is a partition of unity in $\mathcal{F}_d^1$, that is, pairwise orthogonal projections $p_1,\ldots,p_N\in\mathcal{F}_d^1$ such that $p_1+\cdots+p_N=1$. To each projection p_i, we have associated a phase factor $c_i\in\mathbb{T}$. Then

$$R=\sum_{i=1}^N c_i\,(p_i\otimes p_i) + F\sum_{\substack{i,j=1\\ i\neq j}}^N (p_i\otimes p_j), \tag{4.15}$$

which we realize to be a special form of simple R-matrix (Definition 2.8). The more general form (2.22) is obtained by a slightly more general form of sum $\boxplus$, involving the parameters c_{ij}, $i\neq j$.

5. *Equivalences of R-matrices*

In the last section, we related natural operations on R-matrices to operations on their endomorphisms. Conversely, one can start from a natural operation/relation on endomorphisms and relate it to structure on the level of the underlying R-matrices. The most obvious operation, namely composition of endomorphisms, does, however, not preserve the YBE, that is, the product of two Yang–Baxter endomorphisms is usually not Yang–Baxter. Instead, we will consider equivalence relations given by conjugation with automorphisms, and define corresponding equivalence relations on $\mathcal{R}(d)$.

Definition 5.1. Let $R,S\in\mathcal{R}(d)$.

(i) R,S are *$\mathcal{M}$-equivalent* if and only if there exists an automorphism $\alpha\in\operatorname{Aut}\mathcal{M}$ such that $\lambda_R=\alpha\circ\lambda_S\circ\alpha^{-1}$, and we write $R\approxeq S$ in this case.

(ii) R,S are *$\mathcal{N}$-equivalent* if and only if there exists an automorphism $\beta\in\operatorname{Aut}\mathcal{N}$ such that $\lambda_R|_{\mathcal{N}}=\beta\circ\lambda_S|_{\mathcal{N}}\circ\beta^{-1}$, and we write $R\approx S$ in this case.

(iii) R,S are *equivalent* if and only if there exists an isomorphism $\gamma_{R,S}:\mathcal{L}_R\to\mathcal{L}_S$ such that $\gamma_{R,S}(R)=S$ and $\varphi(\gamma_{R,S}(x))=\gamma_{R,S}(\varphi(x))$ for all $x\in\mathcal{L}_R$, and we write $R\sim S$ in this case.

(iv) R,S have *equivalent representations* if and only if for each $n\in\mathbb{N}$, the representations $\rho_R^{(n)}$ and $\rho_S^{(n)}$ of the braid group B_n on n strands are unitarily equivalent.

It is clear that the subfactors $\lambda_R(\mathcal{M})\subset\mathcal{M}$, $\lambda_R(\mathcal{N})\subset\mathcal{N}$, and $\varphi(\mathcal{L}_R)\subset\mathcal{L}_R$ are equivalent to $\lambda_S(\mathcal{M})\subset\mathcal{M}$, $\lambda_S(\mathcal{N})\subset\mathcal{N}$, and $\varphi(\mathcal{L}_S)\subset\mathcal{L}_S)$ if $R\approxeq S$, $R\approx S$ and $R\sim S$, respectively. It is also clear that the relations $\approxeq$, $\approx$, $\sim$ are different from each other.

The last equivalence relation (equivalence of representations) was originally introduced in [3] and played a prominent role in the classification of involutive R-matrices [45]. It essentially captures the *character* of an R-matrix, defined as the positive definite normalized class function

$$\tau_R : B_\infty \to \mathbb{C}, \quad \tau_R := \tau \circ \rho_R. \tag{5.1}$$

Equivalence of representations turns out to be the same as equivalence ($\sim$):

PROPOSITION 5.2. *Let $R, S \in \mathcal{R}(d)$. The following are equivalent.*

(i) *R and S have equivalent representations.*
(ii) *$R \sim S$.*
(iii) *R and S have the same character $\tau_R = \tau_S$.*

Proof. (i) $\Rightarrow$ (ii) If R and S have equivalent representations, there exist unitaries $Y_n \in \mathcal{U}(\mathcal{F}_d^n)$ such that $Y_n \varphi^k(R) Y_n^* = \varphi^k(S)$, $k \in \{0, 1, \ldots, n-2\}$. This implies that for any $x \in \rho_R(\mathbb{C}B_\infty)$,

$$\gamma_{R,S}(x) := \lim_{n\to\infty} Y_n x Y_n^* \tag{5.2}$$

exists, and the so-defined map $\gamma_{R,S}$ is an isomorphism $\rho_R(\mathbb{C}B_\infty) \to \rho_S(\mathbb{C}B_\infty)$ with $\gamma_{R,S}(\varphi^k(R)) = \varphi^k(S)$, $k \in \mathbb{N}_0$. Obviously $\gamma_{R,S}$ preserves τ and extends to an isomorphism $\mathcal{L}_R \to \mathcal{L}_S$ (denoted by the same symbol).

It remains to show $\varphi(\gamma_{R,S}(x)) = \gamma_{R,S}(\varphi(x))$ for all $x \in \mathcal{L}_R$. Indeed,

$$\begin{aligned}\gamma_{R,S}(\varphi(x)) = \gamma_{R,S}(\lambda_R(x)) &= \operatorname*{w-lim}_{n\to\infty} \gamma_{R,S}((\mathrm{ad}\, R_n)(x)) \\ &= \operatorname*{w-lim}_{n\to\infty} (\mathrm{ad}\, S_n)(\gamma_{R,S}(x)) = \lambda_S(\gamma_{R,S}(x)) = \varphi(\gamma_{R,S}(x)).\end{aligned}$$

Hence $R \sim S$.

(ii) $\Rightarrow$ (iii) Let $R \sim S$. From the definition of this equivalence relation, we have an isomorphism $\gamma_{R,S} : \mathcal{L}_R \to \mathcal{L}_S$ such that $\gamma_{R,S} \circ \rho_R = \rho_S$, and the uniqueness of the trace implies that $\gamma_{R,S}$ preserves τ. Hence, for any $b \in B_\infty$,

$$\tau_S(b) = \tau(\rho_S(b)) = \tau(\gamma_{R,S}(\rho_R(b))) = \tau(\rho_R(b)) = \tau_R(b).$$

(iii) $\Rightarrow$ (i) Let R, S have coinciding characters $\tau_R = \tau_S$, and pick $n \in \mathbb{N}$, $x \in \mathbb{C}B_n$. Then

$$\tau(\rho_R^{(n)}(x)^* \rho_R^{(n)}(x)) = \tau_R(x^* x) = \tau_S(x^* x) = \tau(\rho_S^{(n)}(x)^* \rho_S^{(n)}(x)),$$

and the faithfulness of τ yields $\ker \rho_R^{(n)} = \ker \rho_S^{(n)}$. So $\alpha : \rho_R^{(n)}(\mathbb{C}B_n) \to \rho_S^{(n)}(\mathbb{C}B_n)$, $\rho_R^{(n)}(x) \mapsto \rho_S^{(n)}(x)$, is an isomorphism of finite-dimensional C^*-algebras. Furthermore, equality of characters $\tau_R = \tau_S$ implies $\tau \circ \alpha = \tau$ on $\rho_R^{(n)}(\mathbb{C}B_n)$.

But a trace-preserving isomorphism of finite-dimensional C^*-algebras represented on Hilbert spaces of the same dimension is always implemented by a unitary between these Hilbert spaces, that is, there exists a unitary $Y_n \in \mathcal{F}_d^n$ such that $Y_n \rho_R(x) Y_n^{-1} = \rho_S(x)$, $x \in \mathbb{C}B_n$. This shows that R and S have equivalent representations. $\square$

We mention as an aside that we may view τ_R as a state on $\mathbb{C}B_\infty$, and that the von Neumann algebra generated by the GNS construction of $(\mathbb{C}B_\infty, \tau_R)$ is naturally isomorphic to the factor $\mathcal{L}_R$. Thus, we see that τ_R is an *extremal* (or indecomposable) character, that is, an extreme point in the convex set of positive normalized class functions, generalizing a result from [45] to non-involutive R-matrices.

In general, the character equivalence relation $\sim$ does not imply the 'higher' equivalences $\approx$, $\approxeq$, but sometimes $\gamma_{R,S} : \mathcal{L}_R \to \mathcal{L}_S$ extends to appropriate automorphisms of $\mathcal{N}$ or $\mathcal{M}$. In the following, we discuss three example scenarios that we will subsequently refer to as 'type 1–3.'

Type 1. Let $R \in \mathcal{R}(d)$ and $u \in \mathcal{U}(\mathcal{F}_d^1)$. Then $S := u\varphi(u)R\varphi(u)^*u^* = \lambda_u(R) \in \mathcal{R}(d)$ and $R \sim S$. One can choose the intertwiners as $Y_n := u_n$, and easily verifies that λ_u is an automorphism satisfying $\lambda_S = \lambda_u \circ \lambda_R \circ \lambda_u^{-1}$. Since λ_u leaves $\mathcal{N}$ invariant, we have $R \approxeq S$ and $R \approx S$ in this case, with the isomorphisms $\alpha, \beta, \gamma_{R,S}$ from the various equivalence relations all being given by (restrictions of) λ_u.

Type 2. Let $R \in \mathcal{R}(d)$ and $u \in \mathcal{U}(\mathcal{F}_d^1)$ such that $\lambda_u(R) = R$ (that is, R commutes with $u\varphi(u)$). Then $S := \varphi(u)R\varphi(u)^* \in \mathcal{R}(d)$ and $R \sim S$. One can choose the intertwiners as $Y_n := u\varphi(u^2)\cdots\varphi^{n-1}(u^n)$. Hence in this case, $\gamma_{R,S}$ is given by

$$\Lambda_u := \lim_{n\to\infty} \mathrm{ad}(u\varphi(u^2)\cdots\varphi^{n-1}(u^n)), \tag{5.3}$$

which trivially exists as an automorphism of $\bigcup_n \mathcal{F}_d^n \subset \mathcal{N}$ and extends to $\mathcal{N}$. Clearly Λ_u restricts to an isomorphism $\mathcal{L}_R \to \mathcal{L}_S$ matching the representations ρ_R and $\rho_S = \Lambda_u \circ \rho_R$. For $x \in \mathcal{F}_d^n$, we therefore have

$$\Lambda_u(\lambda_R(x)) = \Lambda_u(R_n x R_n{}^*) = S_n \Lambda_u(x) S_n{}^* = \lambda_S(\Lambda_u(x)).$$

Hence, in this case, we also have $R \approx S$. Note that since $\varphi(u)R\varphi(u)^* = u^*Ru$, so exchanging u with u^* we also have the $\mathcal{N}$-equivalence $R \sim uRu^*$, with isomorphism Λ_{u^*}.

We give an example to show that Λ_u does in general not extend to $\mathcal{M}$, that is, to an $\mathcal{M}$-equivalence $R \approxeq S$.

EXAMPLE 5.3. Let $u \in \mathcal{F}_d^1$ and $R := uFu^*$. Since the flip F commutes with $u\varphi(u)$, we have $R \approx F$, and now show $R \not\approxeq F$. In fact, if we had $R \approxeq F$, then the type *III* subfactors given by R and F would be equivalent, and in particular their relative commutants $\mathcal{M}_{R,1}$ and $\mathcal{M}_{F,1}$ would have the same dimension. Recalling $\mathcal{M}_{R,1} = \{x \in \mathcal{F}_d^1 : \varphi(x) = R^*xR\}$ (3.9), we have $\mathcal{M}_{F,1} = \mathcal{F}_d^1$. But as shown in Remark 3.3, $\mathcal{M}_{R,1} = \mathcal{M}_{S,1} \neq \mathcal{F}_d^1$ if $u \notin \mathbb{C}$. Hence $R \not\approxeq F$.

Type 3. The third type of equivalence is given by an R-matrix R and its 'flipped' version FRF, where F is the flip [**45**]. The corresponding intertwiners are best described in terms of the so-called *fundamental braids* $\Delta_n \in B_n$ [**28**], defined recursively by

$$\Delta_1 := e, \quad \Delta_2 := b_1, \quad \Delta_{n+1} := b_1 \cdots b_n \cdot \Delta_n. \tag{5.4}$$

The fundamental braids satisfy [**42**]

$$\Delta_n b_k = b_{n-k}\Delta_n, \quad k \in \{1, \ldots, n-1\}. \tag{5.5}$$

Moreover, Δ_n^2 generates the center of B_n. In particular, $\Delta_n b \Delta_n^{-1} = \Delta_n^{-1} b \Delta_n$ for $b \in B_n$.

LEMMA 5.4. *Let $R \in \mathcal{R}(d)$. Then $FRF \in \mathcal{R}(d)$ and $R \sim FRF$, and the intertwiners can be chosen as*

$$Y_n := \rho_{FRF}(\Delta_n)\rho_F(\Delta_n), \quad n \in \mathbb{N}. \tag{5.6}$$

Proof. We skip the straightforward proof of $FRF \in \mathcal{R}(d)$.

The representative $\rho_F(\Delta_n) \in \mathrm{End}((\mathbb{C}^d)^{\otimes n})$ of the fundamental braid given by the involutive R-matrix F acts by total inversion permutation of the n tensor factors. In view of the tensor product structure of the representation ρ_R,

$$\rho_F(\Delta_n)\varphi^{k-1}(R)\rho_F(\Delta_n)^{-1} = \varphi^{n-k-1}(FRF), \quad k \in \{1, \ldots, n-1\}. \tag{5.7}$$

Using (5.5), this implies

$$\begin{aligned} Y_n \rho_R(b_k) Y_n^{-1} &= \rho_{FRF}(\Delta_n)\rho_F(\Delta_n)\varphi^{k-1}(R)\rho_F(\Delta_n)^{-1}\rho_{FRF}(\Delta_n)^{-1} \\ &= \rho_{FRF}(\Delta_n)\rho_{FRF}(b_{n-k})\rho_{FRF}(\Delta_n)^{-1} \\ &= \rho_{FRF}(b_k). \end{aligned}$$

As $b_1, \ldots, b_{n-1}$ generate B_n, this establishes the intertwiner property of Y_n. □

We add a lemma that concerns the isomorphism $\gamma_{R,FRF} : \mathcal{L}_R \to \mathcal{L}_{FRF}$, which extends to an algebra closely related to the C^*-algebra $\mathcal{A}_R^{(0)}$ introduced in (2.8).

LEMMA 5.5. *Let $R \in \mathcal{R}(d)$, $n \in \mathbb{N}$, and $x \in \mathcal{F}_d^n$ such that $\varphi(x) = \lambda_R(x)$ (this is satisfied in particular by any $x \in \mathcal{L}_{R,n}$). Then*

$$Y_m x Y_m^* = Y_n x Y_n^*, \quad m \geqslant n, \tag{5.8}$$

where Y_m is the intertwiner (5.6). In particular, $\gamma_{R,FRF} = \lim_m \operatorname{ad} Y_m$ extends to such elements x, and $\gamma_{R,FRF}(x) = Y_n x Y_n^*$ for all $x \in \mathcal{L}_{R,n}$.

Proof. To prove this lemma, we first establish a recursion relation for the intertwiners Y_m. We claim

$$Y_{m+1} = Y_m \cdot \rho_F(b_1 \cdots b_m)^{-1} \rho_R(b_1 \cdots b_m), \quad m \in \mathbb{N}. \tag{5.9}$$

To show this, recall that we already know the identity

$$\rho_F(\Delta_m)\rho_R(b)\rho_F(\Delta_m)^{-1} = \rho_{FRF}(\Delta_m b \Delta_m^{-1}), \quad b \in B_m;$$

this was shown in the proof of Lemma 5.4. Thus, we may rewrite the intertwiners as $Y_m = \rho_{FRF}(\Delta_m)\rho_F(\Delta_m) = \rho_F(\Delta_m)\rho_R(\Delta_m)$.

We furthermore note that $\rho_R(\Delta_m) \in \mathcal{L}_{R,m}$, and therefore

$$\begin{aligned} \operatorname{ad} \rho_R(b_1 \cdots b_m)[\rho_R(\Delta_m)] &= \lambda_R(\rho_R(\Delta_m)) = \varphi(\rho_R(\Delta_m)) \\ &= \operatorname{ad}(\rho_F(b_1 \cdots b_m))[(\rho_R(\Delta_m))]. \end{aligned}$$

Moreover, since $F^2 = 1$, we have $\rho_F(\Delta_m) = \rho_F(\Delta_m^{-1})$. Together with the recursion relation $\Delta_{m+1} = b_1 \cdots b_m \Delta_m$, this gives

$$\begin{aligned} Y_{m+1} &= \rho_F(\Delta_{m+1}^{-1})\rho_R(\Delta_{m+1}) \\ &= \rho_F(\Delta_m^{-1})\rho_F(b_1 \cdots b_m)^{-1}\rho_R(b_1 \cdots b_m)\rho_R(\Delta_m) \\ &= \rho_F(\Delta_m^{-1})\rho_F(b_1 \cdots b_m)^{-1}\varphi(\rho_R(\Delta_m))\rho_R(b_1 \cdots b_m) \\ &= \rho_F(\Delta_m^{-1})\rho_R(\Delta_m)\rho_F(b_1 \cdots b_m)^{-1}\rho_R(b_1 \cdots b_m) \\ &= Y_m \cdot \rho_F(b_1 \cdots b_m)^{-1}\rho_R(b_1 \cdots b_m), \end{aligned}$$

proving (5.9).

Now let $x \in \mathcal{F}_d^n$ such that $\varphi(x) = \lambda_R(x)$. Then $\operatorname{ad}(\rho_R(b_1 \cdots b_m))[x] = \lambda_R(x) = \varphi(x) = \operatorname{ad}(\rho_F(b_1 \cdots b_m))[x]$ for any $m \geqslant n$, and therefore $\operatorname{ad}(Y_{n+1})(x) = \operatorname{ad}(Y_n)(x)$. Clearly, this implies $\operatorname{ad} Y_m(x) = (\operatorname{ad} Y_n)(x)$ for all $m \geqslant n$.

The isomorphism $\gamma_{R,FRF}$ is defined by the limit formula $\lim_m \operatorname{ad} Y_m$ on $\rho_R(\mathbb{C}B_\infty)$ and showed that it uniquely extends to an isomorphism $\mathcal{L}_R \to \mathcal{L}_S$. Thus, as $\lim_m(\operatorname{ad} Y_m)(x)$ exists and equals $Y_n x Y_n^*$ for $x \in \mathcal{F}_d^n$ as in the statement of the lemma, we find $\gamma_{R,FRF}(x) = Y_n x Y_n^*$ as claimed. □

Let us emphasize that in general, it is not known whether the $\sim$ equivalence class of an R-matrix is exhausted by the three cases listed above. Furthermore, in general, the equivalences $R \approxeq S$ or $R \approx S$ do not imply $R \sim S$ (for example, $R \approx -R$ for all R, but usually $R \not\sim -R$.)

Making use of the type 3 intertwiners, we can now also give the postponed second part of the proof of Theorem 3.5.

Proof of Theorem 3.5 (second half). Let $R \in \mathcal{R}$ and $S := FRF$. We want to show that $\mathcal{L}_R$ is invariant under ϕ_F. As a preparation, we first show, $n \in \mathbb{N}$,

$$\operatorname{ad} \rho_F(\Delta_n)(\mathcal{L}_{R,n}) = \mathcal{L}_{S,n}. \tag{5.10}$$

In fact, we know from Lemma 5.5 that the intertwiner isomorphism $\gamma_{R,S}$ coincides with $\operatorname{ad} Y_n$ on $\mathcal{L}_{R,n}$, with the intertwiners $Y_n = \rho_S(\Delta_n)\rho_F(\Delta_n)$ (5.6). Thus,

$$\operatorname{ad} \rho_F(\Delta_n)(\mathcal{L}_{R,n}) = \operatorname{ad} \rho_S(\Delta_n)^{-1}(\operatorname{ad} Y_n(\mathcal{L}_{R,n})) = \operatorname{ad} \rho_S(\Delta_n)^{-1}(\mathcal{L}_{S,n}) = \mathcal{L}_{S,n},$$

where the last step follows from $\operatorname{ad} \rho_S(\Delta_n)^{-1}$ being an inner automorphism of $\mathcal{L}_{S,n}$.

Now let $x \in \mathcal{L}_{R,n+1}$, $n \in \mathbb{N}_0$. As $\phi_F(x)$ acts by tracing out the first tensor factor of x (see (2.19)), and $E_n(x)$ acts by tracing out the $(n+1)$st tensor factor of x, we have

$$\phi_F(x) = E_n({F_n}^* x F_n) = E_n(\rho_F(b_1 \cdots b_n)^{-1} x \rho_F(b_1 \cdots b_n)). \tag{5.11}$$

Using the recursion relation $\Delta_{n+1} = b_1 \cdots b_n \cdot \Delta_n$ for the fundamental braids and $\rho_F(\Delta_n) \in \mathcal{F}_d^n$, we have

$$\phi_F(x) = E_n(\operatorname{ad} \rho_F(\Delta_n \Delta_{n+1}^{-1})(x)) = \operatorname{ad} \rho_F(\Delta_n)\big[E_n(\operatorname{ad} \rho_F(\Delta_{n+1}^{-1})(x))\big].$$

In this formula, $\operatorname{ad} \rho_F(\Delta_{n+1}^{-1})(x) \in \mathcal{L}_{S,n+1}$ by (5.10) (note $\rho_F(\Delta_{n+1}^{-1}) = \rho_F(\Delta_{n+1})$), and thus $E_n(\operatorname{ad} \rho_F(\Delta_{n+1}^{-1})(x)) \in \mathcal{L}_{S,n}$ by the first part of Theorem 3.5. If we now apply (5.10) once more, with the roles of R and S exchanged, we arrive at $\phi_F(x) \in \mathcal{L}_{R,n}$.

Proceeding to general $x \in \mathcal{L}_R$, we have $E_n(x) \in \mathcal{L}_{R,n}$ and $E_n(x) \to x$ weakly as $n \to \infty$. As we have just shown $\phi_F(E_n(x)) \in \mathcal{L}_R$ for all $n \in \mathbb{N}$ and ϕ_F is normal, it follows that $\phi_F(x) \in \mathcal{L}_R$.

The uniqueness of the τ-preserving conditional expectation $E_R = \lambda_R \circ \phi_R$ of $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ now implies that for any $x \in \mathcal{L}_R$,

$$\varphi(\phi_R(x)) = E_R(x) = E_F(x) = \varphi(\phi_F(x)),$$

and thus, $\phi_R(x) = \phi_F(x)$. This shows that the right diagram in (3.18) is a commuting square for $n = 1$, and the case $n > 1$ follows by composing several isomorphic commuting squares. $\square$

Applications of Theorem 3.5 will appear in the next section.

We now describe a situation in which $R \approxeq S$ does imply $R \sim S$.

Proposition 5.6. (i) *Let $R, w \in \mathcal{U}(\mathcal{O}_d)$ such that $\alpha^{-1} := \lambda_w \in \operatorname{Aut} \mathcal{M}$. Then*

$$\alpha \circ \lambda_R \circ \alpha^{-1} = \lambda_{\alpha(R)} \iff w \in \mathcal{O}_d^{\lambda_{\varphi(R)}}. \tag{5.12}$$

(ii) *In the same situation as in* (i)*, assume in addition that $R \in \mathcal{R}(d)$ and $S := \alpha(R) \in \mathcal{F}_d^2$. Then $S \in \mathcal{R}(d)$ and $S \sim R$.*

Proof. (i) We write $\alpha = \lambda_v$ and compute

$$\alpha \lambda_R \alpha^{-1} = \lambda_v \lambda_R \lambda_w = \lambda_v \lambda_{\lambda_R(w)R} = \lambda_{\lambda_v(\lambda_R(w)R)v},$$

which coincides with $\lambda_{\alpha(R)} = \lambda_{\lambda_v(R)}$ if and only if $\lambda_v(\lambda_R(w)R)v = \lambda_v(R)$. Applying λ_w to both sides of this equation and observing that $\lambda_w \lambda_v = \mathrm{id}$ implies $\lambda_w(v) = w^*$, we see that

$\alpha \circ \lambda_R \circ \alpha^{-1} = \lambda_{\alpha(R)}$ is equivalent to

$$w = (\mathrm{ad}\, R^* \circ \lambda_R)(w) = \lambda_{\varphi(R)}(w), \tag{5.13}$$

that is, $w \in \mathcal{O}_d^{\lambda_{\varphi(R)}}$.

(ii) We now assume that $R \in \mathcal{R}(d)$ is an R-matrix, and set $S := \alpha(R)$. Then, $n \in \mathbb{N}_0$,

$$\alpha(\varphi^n(R)) = (\alpha\lambda_R^n\alpha^{-1})(S) = \lambda_S^n(S).$$

In particular, $\varphi(\alpha(R)) = \alpha(\varphi(R))$, which immediately implies $\varphi(S)S\varphi(S) = S\varphi(S)S$. Since $S \in \mathcal{F}_d^2$ as well, S is also an R-matrix. Thus, $\lambda_S^n(S) = \varphi^n(S)$, that is, we have $\alpha(\varphi^n(R)) = \varphi^n(S)$, which shows that α restricts to an isomorphism $\mathcal{L}_R \to \mathcal{L}_S$ such that $\varphi(\alpha(x)) = \alpha(\varphi(x))$ for all $x \in \mathcal{L}_R$. This verifies the definition of $R \sim S$. □

We thus see that the enhanced form of $\approx$ equivalence spelled out in (5.12) is parameterized by the fixed points of $\lambda_{\varphi(R)}$. The appearance of fixed points warrants a more systematic look at fixed points and ergodicity of Yang–Baxter endomorphisms. This is done in Section 7.

5.1. *Equivalent R-matrices and braid group characters*

While a classification of all R-matrices seems out of reach, a more accessible (though still challenging) question is to classify all Yang–Baxter *characters*, that is, all traces τ_R, $R \in \mathcal{R}$, on B_∞. This amounts to classifying R-matrices up to the equivalence relation $\sim$.

In order to explain how our results can contribute to this problem, it is instructive to compare this situation with the special case of *involutive* R-matrices (that is, $R^2 = 1$, equivalently $R = R^*$) which has been studied before. Note that for involutive R-matrices, τ_R can be viewed as a character of the infinite symmetric group S_∞ rather than the infinite braid group.

In preparation for the following, we define *R-matrices of normal form* to be special simple R-matrices (Definition 2.8) with parameters $c_{ij} = 1$ for $i \neq j$ and $\varepsilon_i := c_{ii} \in \{+1, -1\}$ for all i. That is, normal form R-matrices are given by a partition of unity $p_1, \ldots, p_N$ in $\mathcal{F}_d^1$ and signs $\varepsilon_1, \ldots, \varepsilon_N$ such that

$$R = \sum_{i=1}^{N} \varepsilon_i\, p_i\varphi(p_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{N} p_i\varphi(p_j)F = \boxplus_{i=1}^{N} \varepsilon_i 1_{d_i}, \tag{5.14}$$

where $d_i = d\tau(p_i)$ are the dimensions of the projections p_i. These normal forms can be described by a pair of Young diagrams with d boxes in total.

Theorem 5.7 [45].

(i) *Let $R, S \in \mathcal{R}(d)$ be involutive. Then $R \sim S$ if and only if $\phi_R(R) \cong \phi_S(S)$ are similar, that is, $\phi_R(R) = u\phi_S(S)u^*$ for some $u \in \mathcal{U}(\mathcal{F}_d^1)$.*

(ii) *Each involutive R is equivalent to a unique R-matrix of normal form.*

(iii) *Let R be an R-matrix of normal form, with projections $p_1, \ldots, p_N$ and signs $\varepsilon_1, \ldots, \varepsilon_N$. Define the rational numbers*

$$\alpha_i := \tau(p_i), \quad \varepsilon_i = +1, \tag{5.15}$$

$$\beta_j := \tau(p_j), \quad \varepsilon_j = -1. \tag{5.16}$$

Then the character $\tau_R(\sigma)$, $\sigma \in S_\infty$, takes the following form: If the disjoint cycle decomposition of σ is given by m_n cycles of length n, $n \in \mathbb{N}$, then

$$\tau_R(\sigma) = \prod_n \left(\sum_i \alpha_i^n + (-1)^{n+1} \sum_j \beta_j^n \right)^{m_n}. \tag{5.17}$$

Furthermore, the signed parameters α_i, $-\beta_j$ are exactly the eigenvalues of $\phi_R(R)$.

The proofs of these facts rely crucially on the fact that ρ_R factors through the infinite symmetric group. In particular, (i) a parameterization of all extremal characters of S_∞ is known from the work of Thoma [**59**] (in terms of the *Thoma parameters* α_i, β_j (5.15)), (ii) S_∞ allows for a disjoint cycle decomposition, and (iii) for involutive R-matrices, $\phi_R(R)$ is selfadjoint.

The results of Theorem 5.7 do not carry over to general (not necessarily involutive) R-matrices. However, certain aspects can be generalized, which is the content of the following theorem.

THEOREM 5.8. *Let $R, S \in \mathcal{R}(d)$.*

(i) *$\phi_R(R) = \phi_F(R) = \phi_F(FRF)$ is a normal element of $\mathcal{F}_d^1$ with norm $\|\phi_R(R)\| \leqslant 1$. In particular, R has identical left and right partial traces*[†]*.*

(ii) *$\tau(R\varphi(R)\cdots\varphi^{n-1}(R)) = \tau(\phi_R(R)^n)$, $n \in \mathbb{N}_0$.*

(iii) *If $R \sim S$, then $\phi_R(R) \cong \phi_S(S)$ (unitary similarity).*

Proof. (i) By Theorem 3.5, we know $\phi_F(x) = \phi_R(x)$ for all $x \in \mathcal{L}_R$, so in particular $\phi_F(R) = \phi_R(R)$. We also know that $E_1(R) = \phi_F(FRF) \in \mathcal{L}_{R,1}$. Given arbitrary $y \in \mathcal{F}_d^1$, we compute

$$\tau(y\phi_F(FRF)) = \tau(\varphi(y)FRF) = \tau(yR) = \tau(\lambda_R(y)R) = \tau(y\phi_R(R)),$$

which shows $\phi_F(FRF) = \phi_R(R)$.

In general, left inverses/partial traces do not preserve normality, but in our situation, we can show that $\phi_R(R)$ is always normal, that is, $\phi_R(R)\phi_R(R)^* = \phi_R(R)^*\phi_R(R)$. Since $\phi_R(R) \in \mathcal{F}_d^1$, it is enough to compare traces against arbitrary elements $x \in \mathcal{F}_d^1$.

In the following computation, we use the property (2.16) of ϕ_R and $\tau \circ \phi_R = \tau$, the fact that $\lambda_R = \operatorname{ad} R$ on $\mathcal{F}_d^1$, and $\lambda_R(R^*) = \varphi(R^*)$. This yields

$$\begin{aligned}\tau(x\phi_R(R)\phi_R(R)^*) &= \tau(\lambda_R(x\phi_R(R))R^*) = \tau(x\phi_R(R)R^*) = \tau(\lambda_R(x)R\varphi(R^*)) \\ &= \tau(Rx\varphi(R^*)) = \tau(xR\varphi(R^*)).\end{aligned}$$

On the other hand, using $\phi_R(R) = \phi_F(FRF)$ and $\phi_R(R) = \phi_F(R) = \phi_{R^*}(R)$ (this follows because $R \in \mathcal{L}_R = \mathcal{L}_{R^*}$), we find

$$\begin{aligned}\tau(x\phi_R(R)^*\phi_R(R)) &= \tau(x\phi_F(FR^*F)\phi_R(R)) = \tau(\varphi(x)FR^*F\varphi(\phi_R(R))) = \tau(xR^*\phi_R(R)) \\ &= \tau(xR^*\phi_{R^*}(R)) = \tau(\lambda_{R^*}(x)\varphi(R^*)R) = \tau(xR\varphi(R^*)),\end{aligned}$$

which coincides with the previous result. This proves that $\phi_R(R)$ is normal. The norm estimate is a standard property of the conditional expectation $E_R = \lambda_R\phi_R$.

(ii) For $k, m \in \mathbb{N}_0$, define

$$t_{k,m} := \tau(\varphi^k(R)\varphi^{k-1}(R)\cdots R \cdot \phi_R(R)^m). \tag{5.18}$$

We will prove $t_{k,m} = t_{k+1,m-1}$, which implies the claim as $t_{n,0} = t_{0,n}$.
As before, we use the four facts (i) $x\phi_R(y) = \phi_R(\lambda_R(x)y)$, (ii) $\lambda_R(a) = \varphi(a)$ for $a \in \mathcal{L}_R$, (iii) $\tau \circ \phi_R = \tau$, (iv) $\lambda_R(\phi_R(R)) = R\phi_R(R)R^*$, and compute

$$\begin{aligned}t_{k,m} &= \tau(\varphi^k(R)\cdots R\phi_R(R)^{m-1} \cdot \phi_R(R)) = \tau\big(\phi_R\big(\lambda_R\big(\varphi^k(R)\cdots R \cdot \phi_R(R)^{m-1}\big)R\big)\big) \\ &= \tau\big(\varphi^{k+1}(R)\cdots\varphi(R) \cdot R\phi_R(R)^{m-1}R^*R\big) = t_{k+1,m-1}.\end{aligned}$$

(iii) Let $R \sim S$, that is, $\tau_R = \tau_S$. Then part (ii) implies that $\phi_R(R)^n$ and $\phi_S(S)^n$ have the same trace for any $n \in \mathbb{N}_0$. Thus $\phi_R(R)$ and $\phi_S(S)$ have the same characteristic polynomial, and as they are normal by part (i), it follows that $\phi_R(R)$ and $\phi_S(S)$ are unitarily equivalent. □

[†] In matrix notation, $\phi_F(R) = d^{-1}(\mathrm{Tr} \otimes \mathrm{id})(R)$ and $\phi_F(FRF) = d^{-1}(\mathrm{id} \otimes \mathrm{Tr})(R)$ are the normalized left and right partial traces of R.

REMARK 5.9. (i) This theorem states in particular that the spectrum of the (left or right) partial trace of an R-matrix is an invariant for $\sim$. Since any normal matrix can be diagonalized by conjugation with a unitary, we also see that given $R \in \mathcal{R}(d)$, there exists $u \in \mathcal{U}(\mathcal{F}_d^1)$ such that $\lambda_u(R) \sim R$ ('type 1,' see p. 18) and $\lambda_u(R)$ has diagonal partial traces.

(ii) While it is known in the setting of involutive R-matrices that $R \sim S$ is *equivalent* to $\phi_R(R) \cong \phi_S(S)$, the implication $\Longleftarrow$ does *not* hold in general. In fact, it is not difficult to construct unitary R-matrices R, S such that $\phi_R(R) = \phi_S(S)$ (and $R \cong S$), but, for example, $\tau(R^2\varphi(R)) \neq \tau(S^2\varphi(S))$, that is, $R \not\sim S$.

(iii) In the involutive case, it is furthermore known that $\phi_R(R)$ is always invertible. We currently do not know whether this remains true in the general case.

(iv) Specializing to an involutive R, Theorem 5.8 recovers Thoma's formula (5.17) for cycles.

6. *Irreducibility, reduction, and index*

In the following, we will call an R-matrix R *irreducible* if and only if λ_R is irreducible as an endomorphism of $\mathcal{M}$, that is, if and only if $\mathcal{M}_{R,1} = \lambda_R(\mathcal{M})' \cap \mathcal{M} = \mathbb{C}$. This does not necessarily mean that λ_R is irreducible as an endomorphism of $\mathcal{N}$: In view of (3.13),

$$\mathcal{L}_{R,1} \subset \mathcal{M}_{R,1} \subset \mathcal{N}_{R,1} \subset \mathcal{F}_d^1, \quad R \in \mathcal{R}(d), \tag{6.1}$$

and in general, the relative commutants $\mathcal{L}_{R,1}$, $\mathcal{M}_{R,1}$, and $\mathcal{N}_{R,1}$ are all different from each other. It is therefore conceivable that there exist R-matrices such that, for instance, λ_R is irreducible but $\lambda_R|_{\mathcal{N}}$ is not, or that $\lambda_R|_{\mathcal{L}_R}$ is irreducible but λ_R is not[†]. Our notion of irreducibility always refers to $\lambda_R \in \operatorname{End}\mathcal{M}$, and we will explicitly indicate whenever we consider λ_R as an endomorphism of $\mathcal{N}$ or $\mathcal{L}_R$ by restriction.

A Yang–Baxter endomorphism λ_R is a unital normal endomorphism of the type III factor $\mathcal{M}$ with finite-dimensional relative commutant $\mathcal{M}_{R,1} \subset \mathcal{F}_d^1$ (6.1). We may therefore decompose it into finitely many irreducible endomorphisms of $\mathcal{M}$, unique up to inner automorphisms (that is, as sectors). In the following, we will rely on results of R. Longo, see [**47, 48**] for the original articles and [**37**] for a summary, to obtain information about λ_R and the minimal index $\operatorname{Ind}(\lambda_R)$.

By a *partition of unity* in $\mathcal{M}_{R,n}$ (for some $n \in \mathbb{N}$), we will mean a family $\{p_i\}_{i=1}^{d_1} \subset \mathcal{M}_{R,1}$ of orthogonal projections such that $p_i p_j = \delta_{ij} p_i$ and $\sum_{i=1}^{d_1} p_i = 1$. Note that since $\mathcal{M}_{R,n}$ is finite-dimensional, there always exist finite partitions of unity by minimal projections.

Square brackets $[\lambda]$ denote the sector of λ, that is, $[\lambda] = \{\operatorname{ad} u \circ \lambda : u \in \mathcal{U}(\mathcal{M})\}$. The following statement is a consequence of well-known facts, adapted to our context.

PROPOSITION 6.1. *Let $R \in \mathcal{R}$, $n \in \mathbb{N}$, and $\{p_{n,i}\}_{i=1}^{d_n}$ a partition of unity in $\mathcal{M}_{R,n}$. Then there exist isometries $v_{n,i} \in \mathcal{M}$ such that as sectors*

$$[\lambda_R^n] = \bigoplus_{i=1}^{d_n} [\mu_{n,i}], \quad \mu_{n,i}(\cdot) = v_{n,i}^* \lambda_R^n(\cdot) v_{n,i}. \tag{6.2}$$

The minimal index of λ_R is bounded below by

$$d_n^{2/n} \leqslant \operatorname{Ind} \lambda_R. \tag{6.3}$$

In case $v_{n,i} \in \mathcal{O}_d$, we have $\mu_{n,i} = \lambda_{u_{n,i}}$ with $u_{n,i} = v_{n,i}^* \cdot {}_nR\,\varphi(v_{n,i})$.

These estimates give concrete index bounds when applied to spectral decompositions.

[†] An example for the latter situation is given by $R = F$.

COROLLARY 6.2. *Let $R \in \mathcal{R}(d)$ and consider the spectra $\sigma(R)$ of R and $\sigma(\phi_R(R))$ of $\phi_R(R)$. Denoting cardinality by $|\cdot|$, we have*

$$|\sigma(R)| \leqslant \operatorname{Ind}\lambda_R, \quad |\sigma(\phi_R(R))|^2 \leqslant \operatorname{Ind}\lambda_R. \tag{6.4}$$

Proof. The R-matrix R is a unitary in $\mathcal{M}_{R,2}$ (Proposition 2.3(iii)); hence, its spectral projections define a partition of unity of $d_2 = |\sigma(R)|$ many projections in $\mathcal{M}_{R,2}$. For the second bound, we recall that $\phi_R(R)$ is a normal element in $\mathcal{M}_{R,1}$ (Theorem 5.8(i)), hence its spectral projections define a partition of unity of $d_1 = |\sigma(\phi_R(R))|$ many projections in $\mathcal{M}_{R,1}$. □

As an example, we describe the decomposition of λ_R for diagonal R-matrices.

LEMMA 6.3. *Let $R = DF \in \mathcal{R}(d)$ be a diagonal R-matrix, $D = \sum_{i,j} c_{ij} S_i S_j S_j^* S_i^*$. Then*

$$\lambda_R = \lambda_u \circ \sum_{i=1}^{d} S_i \lambda_{u_i}(\cdot) S_i^* \circ \lambda_u^{-1}, \quad u_i := \sum_{j=1}^{d} c_{ij} S_j S_j^* \in \mathcal{U}(\mathcal{F}_d^1) \tag{6.5}$$

decomposes into a sum of d quasifree automorphisms. In particular,

$$[\mathcal{N} : \lambda_R(\mathcal{N})] = \operatorname{Ind}_{E_R}(\mathcal{M}) = d^2. \tag{6.6}$$

Proof. Let $i, j \in \{1, \ldots, d\}$. It is clear that λ_{u_i} is an automorphism, with $\lambda_{u_i}(S_j) = c_{ij} S_j$ and $\lambda_{u_i}(S_j^*) = \overline{c_{ij}} S_j^*$. One computes $\lambda_R(S_j) S_i = c_{ij} S_i S_j$ and $\lambda_R(S_j^*) S_i = \overline{c_{ij}} S_i S_j^*$. Hence $S_i \lambda_{u_i}(x) = \lambda_R(x) S_i$ whenever $x = S_j$ or $x = S_j^*$. This implies (6.5).

Since each automorphism has dimension 1, it follows that the minimal index is $\operatorname{Ind}(\lambda_R) = d^2$. Since $\operatorname{Ind}(\lambda_R) \leqslant \operatorname{Ind}_{E_R}(\lambda_R) = [\mathcal{N} : \lambda_R(\mathcal{N})] \leqslant d^2$, (6.6) follows. □

Similar to diagonal ones, also general simple nontrivial R-matrices are reducible. Irreducible R-matrices do exist (and are, in fact, likely to be the most interesting ones), but a general overview over irreducible R-matrices is currently not known. In Section 8, we will see an example.

Regarding *upper* bounds on the index, we have the completely general bound $[\mathcal{N} : \lambda_R(\mathcal{N})] \leqslant d^2$ on the Jones index [**13**] (and hence on the minimal index). In the special case that $\phi_R(R) = \tau(R)1 \neq 0$, then it was also shown in [**13**] that $[\mathcal{N} : \lambda_R(\mathcal{N})] \leqslant |\tau(R)|^{-2}$. More generally, if $\phi_R(R)$ is invertible† but not necessarily scalar, then

$$[\mathcal{N} : \lambda_R(\mathcal{N})] \leqslant \|\phi_R(R)^{-1}\|^4. \tag{6.7}$$

This bound is not necessarily sharper than the general bound d^2, but has an interesting consequence for R-matrices that we record here, following [**13**, Corollary 5.5]. It states that the spectrum of a non-trivial R-matrix cannot be concentrated in a disc of radius less than the universal bound $1 - 2^{-1/4} \approx 0.159$ (this value is probably not optimal).

COROLLARY 6.4. *Let $R \in \mathcal{R}$ and $\mu \in \mathbb{T}$ such that $\|R - \mu\| < 1 - 2^{-1/4}$. Then R is trivial.*

Proof. Passing from R to $\mu^{-1} R \in \mathcal{R}$ we may assume $\mu = 1$ without loss of generality.

By assumption, $\|\phi_R(R) - 1\| \leqslant \|R - 1\| < 1 - 2^{-1/4} < 1$. Hence $\phi_R(R)$ is invertible, and the inverse satisfies $\|\phi_R(R)^{-1}\| \leqslant (1 - \|R - 1\|)^{-1} < 2^{1/4}$. Thus (6.7) implies $[\mathcal{N} : \lambda_R(\mathcal{N})] < 2$, that is, $[\mathcal{N} : \lambda_R(\mathcal{N})] = 1$ and λ_R is an automorphism. This is only possible for trivial R (Corollary 2.4). □

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†For involutive R-matrices, $\phi_R(R)$ is known to be invertible [**45**]. We currently have no proof (but also no counterexample) that this property remains true for general $R \in \mathcal{R}$.

Remark 6.5. Akemann showed in [2] that if the inclusion diagram

$$\begin{array}{ccc} \mathcal{F}_d^1 & \subset & \mathcal{F}_d^2 \\ \cup & & \cup \\ \lambda_R(\mathcal{N}) \cap \mathcal{F}_d^1 & \subset & \lambda_R(\mathcal{N}) \cap \mathcal{F}_d^2 \end{array} \tag{6.8}$$

is a commuting square, then the index $[\mathcal{N} : \lambda_R(\mathcal{N})]$ is an *integer*.

We remark here that one can show that for arbitrary $R \in \mathcal{R}$,

$$\mathcal{F}_d^1 \cap \lambda_R(\mathcal{N}) = (\mathcal{F}_d^1)^{\lambda_R}.$$

With the results of the next section, it is then easy to check that if λ_R is *ergodic* (that is, $\mathcal{N}^{\lambda_R} = \mathbb{C}$), then (6.8) commutes and hence $[\mathcal{N} : \lambda_R(\mathcal{N})] \in \mathbb{N}$. However, the square does *not* commute for general R-matrices. Any simple R-matrix containing a projection of dimension greater than 1 is a counterexample.

Presently, it is unknown whether $[\mathcal{N} : \lambda_R(\mathcal{N})]$ is integer[†] for any $R \in \mathcal{R}$, and whether $\{[\mathcal{N} : \lambda_R(\mathcal{N})] : R \in \mathcal{R}\} = \mathbb{N}$.

Our considerations so far show that the decomposition of a Yang-Baxter endomorphism into irreducible endomorphisms does not preserve the YBE. This can for example be seen from the decomposition of the endomorphism of a diagonal R-matrix (6.5) which yields non-trivial automorphisms λ_{U_i}.

In the context of Yang–Baxter endomorphisms, one rather wants to consider a different reduction scheme that does preserve the YBE and works directly on the level of the R-matrix. While the general form of such a reduction is the subject of ongoing research, we sketch here how it works for the special class of involutive R-matrices.

So let $R \in \mathcal{R}(d)$ be involutive (that is, $R = R^*$) and reducible, namely there exist non-trivial projections $p \in \mathcal{M}_{R,1} \subset \mathcal{F}_d^1$. Then $R^*(p \otimes 1)R = (1 \otimes p)$ (cf. (3.9)). Taking into account that involutive R-matrices are selfadjoint and unitary, it is then easy to see that R commutes with the projections $p \otimes p$ and $p^\perp \otimes p^\perp$, whereas $R(p \otimes p^\perp)R = p^\perp \otimes p$. Denoting the base space of R by V, it follows that R can be restricted to two involutive R-matrices $S : pV \otimes pV \to pV \otimes pV$ and $T : p^\perp V \otimes p^\perp V \to p^\perp V \otimes p^\perp V$, and restricts to a unitary $U : pV \otimes p^\perp V \to p^\perp V \otimes pV$. By adapting the arguments in [45, Proposition 4.4], one can show that as far as the character of R is concerned, one may replace U by the flip F. Namely, one has the equivalence $R \sim S \boxplus T$.

As S and T are also involutive R-matrices, this scheme can be applied repeatedly, yielding $R \sim R^1 \boxplus \cdots \boxplus R^n$, where the $R^i \in \mathcal{R}(d_i)$ are involutive irreducible R-matrices (the superscript is just a label, not a power). Involutive irreducible R-matrices are of the form $R^i = \pm 1_{d_i}$ or $R^i \sim \pm F_{d_i}$ [45]. Decomposing also the flip parts according to $F_{d_i} = 1_1 \boxplus \cdots \boxplus 1_1$ (d_i terms) then yields

$$R \sim \bigboxplus_{k=1}^{N} \varepsilon_k 1_{D_k}, \quad \varepsilon_k \in \{\pm 1\}, \quad \sum_{k=1}^{N} D_k = d.$$

This is the normal form found in [45], on which we now have a new perspective from the endomorphism picture. The above argument also identifies two simplifying features of the involutive case: On the one hand, every involutive R is completely reducible in the sense explained, and on the other hand, there exist only very few irreducible involutive R-matrices.

An investigation of these properties for general R-matrices is left to a future work.

[†] It is known, however, that $[\mathcal{L}_R : \varphi(\mathcal{L}_R)]$ is typically *not* integer [58, 64].

7. *Ergodicity and fixed points*

Fixed point subalgebras of automorphisms and endomorphisms of $\mathcal{O}_d$ have not been investigated systematically but in few cases. For instance, $\mathcal{O}_d^\varphi = \mathbb{C}$, but there exists an order two quasi-free automorphism λ_f of $\mathcal{O}_2$, $f = S_1S_2^* + S_2S_1^*$, such that ${\mathcal{O}_2}^{\lambda_f} \simeq \mathcal{O}_2$ [10]. More interestingly, $\mathcal{O}_2^{\lambda_{-1}} \simeq \mathcal{O}_4$, as it is the C^*-subalgebra of $\mathcal{O}_2$ generated by S_iS_j, $1 \leqslant i, j \leqslant 2$. This example is the fixed point algebra of the R-matrix $R = -1 \in \mathcal{R}(2)$.

In this section, we discuss fixed point algebras of Yang–Baxter endomorphisms λ_R at the level of the C^*-algebras $\mathcal{O}_d$, $\mathcal{F}_d$ and the von Neumann algebras $\mathcal{M}$, $\mathcal{N}$. What is special in the Yang–Baxter context is that fixed point algebras of λ_R are closely related to the relative commutants $\mathcal{L}_R \subset \mathcal{N}$, $\mathcal{L}_R \subset \mathcal{M}$, as we demonstrate now.

Proposition 7.1. *Let $R \in \mathcal{R}(d)$.*

(i) $\mathcal{M}^{\lambda_R} \subset \bigcap_{n\geqslant 1} \lambda_R^n(\mathcal{M}) \subset \mathcal{L}_R' \cap \mathcal{M}$.

(ii) $\mathcal{N}^{\lambda_R} = \bigcap_{n\geqslant 1} \lambda_R^n(\mathcal{N}) = \mathcal{L}_R' \cap \mathcal{N}$.

(iii) *Let $i, j \in \{1, \ldots, d\}$. Then $S_i^*\mathcal{M}^{\lambda_R}S_j \subset \mathcal{M}^{\lambda_R}$ and $S_i^*\mathcal{N}^{\lambda_R}S_j \subset \mathcal{N}^{\lambda_R}$.*

Proof. (i) The first inclusion is trivial. For the second one, let $x \in \bigcap_{n\geqslant 1} \lambda_R^n(\mathcal{M})$ and $m \in \mathbb{N}_0$. Then $x = \lambda_R^{m+2}(y)$ for some $y \in \mathcal{M}$, and taking into account that $R \in \mathcal{M}_{R,2} = (\lambda_R^2, \lambda_R^2)$, we find

$$\varphi^m(R)x = \lambda_R^m(R)\lambda_R^{m+2}(y) = \lambda_R^m(R\lambda_R^2(y)) = \lambda_R^m(\lambda_R^2(y)R) = x\varphi^m(R).$$

Since m was arbitrary, this implies $x \in \mathcal{L}_R' \cap \mathcal{M}$.

(ii) Exactly as in part (i) we have the two '$\subset$' inclusions, and it remains to show $\mathcal{L}_R' \cap \mathcal{N} \subset \mathcal{N}^{\lambda_R}$. Let $x \in \mathcal{L}_R' \cap \mathcal{N}$, that is, $[x, \varphi^n(R)] = 0$ for all $n \in \mathbb{N}_0$. Then

$$\lambda_R(x) = \lim_{n\to\infty} R \cdots \varphi^n(R)x\varphi^n(R)^* \cdots R^* = x,$$

that is, $x \in \mathcal{N}^{\lambda_R}$.

(iii) Let $x \in \mathcal{M}^{\lambda_R}$. Taking into account that x commutes with R by part (i), we have

$$\lambda_R(S_i^*xS_j) = S_i^*R^*\lambda_R(x)RS_j = S_i^*R^*xRS_j = S_i^*xS_j. \qquad \square$$

Remark 7.2. (i) In standard terminology, an endomorphism λ of a von Neumann algebra $\mathcal{N}$ is called *ergodic* if $\mathcal{N}^\lambda = \mathbb{C}$ and a *shift* if $\bigcap_{n\geqslant 1} \lambda^n(\mathcal{N}) = \mathbb{C}$. We have thus shown that that $\lambda_R|_\mathcal{N}$ is ergodic if and only if $\lambda_R|_\mathcal{N}$ is a shift. Furthermore, Proposition 7.1(ii) shows that $\lambda_R|_\mathcal{N}$ is ergodic if and only if $\mathcal{L}_R \subset \mathcal{N}$ is irreducible.

(ii) We will later discuss an example with $\dim \mathcal{N}^{\lambda_R} = \infty$, that is, in particular $[\mathcal{N} : \mathcal{L}_R] = \infty$.

(iii) All statements of this proposition hold without changes on the level of the C^*-algebras, that is,

$$\mathcal{O}_d^{\lambda_R} \subset \bigcap_{n\geqslant 1} \lambda_R^n(\mathcal{O}_d) \subset \mathcal{B}_R' \cap \mathcal{O}_d \text{ and } \mathcal{F}_d^{\lambda_R} = \bigcap_{n\geqslant 1} \lambda_R^n(\mathcal{F}_d) = \mathcal{B}_R' \cap \mathcal{F}_d.$$

It is currently not clear if one has equalities in Proposition 7.1(i), or if $\mathcal{M}^{\lambda_R} \subset \mathcal{N}^{\lambda_R}$ for all non-trivial R. We next show that at least ergodicity of λ_R can be decided on the level of $\mathcal{N}$.

For this and following results, we will make use of a (von Neumann version of) family of linear maps $E^n : \mathcal{M} \to \mathcal{N}$, $n \in \mathbb{Z}$, introduced in [16], namely ($n \geqslant 0$)

$$E^n(x) = \int_\mathbb{T} \alpha_z(xS_1^{*n}), \quad E^{-n}(x) = \int_\mathbb{T} \alpha_z(S_1^n x), \tag{7.1}$$

where $\alpha_z = \lambda_{z\cdot 1}$ are the gauge automorphisms, integration is over the circle $z \in \mathbb{T}$ with respect to $\frac{dz}{2\pi i z}$, and the choice of S_1 as a reference generator is by convention. We also introduce the closely related spectral components $x^{(n)} \in \mathcal{M}^{(n)}$ of x as

$$x^{(n)} := \int \alpha_z(x) z^{-n} = \begin{cases} E^n(x) S_1^n & n \geqslant 0 \\ S_1^{*-n} E^n(x) & n < 0 \end{cases}. \tag{7.2}$$

Recall that $x = 0$ is equivalent to $x^{(n)} = 0$ for all $n \in \mathbb{Z}$ [**33, 57**]. Moreover, we clearly have $(x^*)^{(n)} = (x^{(-n)})^*$ for all $x \in \mathcal{M}$ and all $n \in \mathbb{Z}$.

For any unitary $U \in \mathcal{U}(\mathcal{F}_d)$, the endomorphism λ_U commutes with the gauge action, so that the fixed point algebra $\mathcal{M}^{\lambda_U}$ is globally $\mathbb{T}$-invariant and for any $x \in \mathcal{M}^{\lambda_U}$, also all its spectral components $x^{(n)}$ are fixed points of λ_U. This applies in particular to R-matrices $R \in \mathcal{U}(\mathcal{F}_d^2)$.

Proposition 7.3. *Let $U \in \mathcal{U}(\mathcal{F}_d)$. If $\mathcal{F}_d^{\lambda_U} = \mathbb{C}$ then $\mathcal{O}_d^{\lambda_U} = \mathbb{C}$; if $\mathcal{N}^{\lambda_U} = \mathbb{C}$ then $\mathcal{M}^{\lambda_U} = \mathbb{C}$.*

Proof. If $x \in \mathcal{O}_d^{\lambda_U}$ was nontrivial, it would not lie in $\mathcal{F}_d$ and would have a nonzero spectral component. Without loss of generality, we may then assume that $x^{(n)} \neq 0$ for some $n > 0$, and as remarked above, $x^{(n)} \in \mathcal{O}_d^{\lambda_U}$. Now, both $x^{(n)}(x^{(n)})^*$ and $(x^{(n)})^* x^{(n)}$ are fixed points in $\mathcal{F}_d$ and thus positive scalars, say μ and ν. It follows immediately that ν must be equal to μ, and thus $x^{(n)}$ is a multiple of a unitary. However, it is easy to see that this is in conflict with the KMS condition (recall that $\lambda_{d^{-it}1}$ is the modular group with respect to the state $\omega = \tau \circ E^0$).

The proof for the von Neumann algebras $\mathcal{M}$, $\mathcal{N}$ is identical. □

Proposition 7.3 implies that λ_R is ergodic if and only if $\lambda_R|_{\mathcal{N}}$ is ergodic. In this case, we will simply say that $R \in \mathcal{R}$ is ergodic.

Remark 7.4. (i) It is clear that the equivalence relations $R \approxeq S$ and $R \approx S$ (Definition 5.1) provide automorphisms of $\mathcal{M}$ and $\mathcal{N}$ that identify the fixed point algebras of λ_R and λ_S. In particular, the 'type 1' and 'type 2' cases of $\sim$ equivalences (see p. 5) preserve ergodicity.

(ii) R is ergodic if and only if R^* is ergodic because

$$\mathcal{N}^{\lambda_{R^*}} = \mathcal{L}'_{R^*} \cap \mathcal{N} = \mathcal{L}'_R \cap \mathcal{N} = \mathcal{N}^{\lambda_R}. \tag{7.3}$$

We now turn to an explicit characterization of ergodicity. Let $H_R : \mathcal{N} \to \mathcal{N}^{\lambda_R}$ denote the unique τ-preserving conditional expectation onto the fixed point algebra. As λ_R preserves τ, the ergodic theorem allows us to write H_R as

$$H_R(x) = \operatorname*{s\text{-}lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_R^k(x), \quad x \in \mathcal{N}. \tag{7.4}$$

Also recall that E_n denotes the τ-preserving conditional expectation $\mathcal{N} \to \mathcal{F}_d^n$, which acts by tracing out all tensor factors except the first n (in particular, $E_0 = \tau$). We will refer to the condition in part (i) of the next theorem as 'the ergodicity condition' in the following.

Theorem 7.5. *Let $R \in \mathcal{R}(d)$. The following are equivalent.*

(i) *$E_1(RxR^*) = \tau(x)$ for all $x \in \mathcal{F}_d^1$.*
(ii) *$E_n(\varphi^{n-1}(R) x \varphi^{n-1}(R^*)) = E_{n-1}(x)$ for all $n \in \mathbb{N}, x \in \mathcal{F}_d^n$.*
(iii) *$H_R(x) = \tau(x)$ for all $x \in \mathcal{F}_d^1$.*
(iv) *R is ergodic.*

If R is ergodic, then so are all its cabling powers $R^{(n)}$, $n \in \mathbb{N}$.

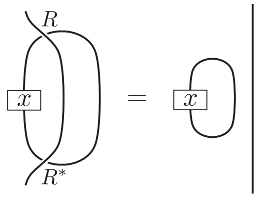

FIGURE 2. *The ergodicity condition in graphical notation. Note that this is trivially satisfied for $R = F$, and trivially violated for $R = 1$.*

REMARK 7.6. (i) In matrix notation, the ergodicity condition reads as follows: Let $(e_k)_{k=1}^d$ be the standard basis of $\mathbb{C}^d$, and let $R_{kl}^{ij} := \langle e_i \otimes e_j, R(e_k \otimes e_l)\rangle$. Then the ergodicity condition is equivalent to

$$\sum_{n,m=1}^{d} R_{kn}^{im}\overline{R_{ln}^{jm}} = \delta_j^i\, \delta_l^k \quad i,j,k,l \in \{1,\dots,d\}, \tag{7.5}$$

as can be seen by choosing x as the matrix unit $\mathsf{e}_{kl} \in M_d$. In the special case of involutive R-matrices equivalent to the flip, Wassermann gave a proof of an analogue of Theorem 7.5 already in [**62**], also based on the condition (7.5).

(ii) The ergodicity condition is best understood in graphical notation Noting that E_1 acts as a normalized right partial trace on $\mathcal{F}_d^2$, we have a diagrammatic representation as in Figure 2.

(iii) The ergodicity condition also appears in [**13**], where it was shown to imply that the left inverse ϕ_R is localized in the sense that for any $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $\phi_R(\mathcal{F}_d^n) \subset \mathcal{F}_d^k$.

Proof. (i)⇒(ii) We give a proof by induction in n, the case $n = 1$ being equivalent to (i). For the induction step, note that the definition of E_n implies $S_i^* E_n(\cdot) S_j = E_{n-1}(S_i^* \cdot S_j)$ for any i, j. Thus, we have, $i, j \in \{1,\dots,d\}$, $x \in \mathcal{F}_d^{n+1}$,

$$S_i^* E_{n+1}(\varphi^n(R)x\varphi^n(R^*))S_j = E_n(S_i^*\varphi^n(R)x\varphi^n(R^*)S_j) = E_n(\varphi^{n-1}(R)S_i^* x S_j \varphi^{n-1}(R^*)).$$

As $S_i^* x S_j \in \mathcal{F}_d^n$, this simplifies by induction assumption to $E_{n-1}(S_i^* x S_j) = S_i^* E_n(x) S_j$. Since i, j were arbitrary, this finishes the proof.

(ii)⇒(iii) Let $x \in \mathcal{F}_d^1$, $n \in \mathbb{N}$, and $y \in \mathcal{F}_d^n$. Noting that $\varphi^{k-1}(R)$ commutes with y for $k - 1 \geqslant n$, we calculate

$$\begin{aligned}
\tau(yH_R(x)) &= \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \tau(y\lambda_R^k(x)) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \tau(y\varphi^{k-1}(R)\cdots RxR^*\cdots\varphi^{k-1}(R^*)) \\
&= \lim_{m\to\infty} \frac{1}{m} \left\{ \sum_{k=0}^{n} \tau(y\, {}_kRx({}_kR)^*) + \sum_{k=n+1}^{m-1} \tau(y\varphi^{n-1}(R)\cdots RxR^*\cdots\varphi^{n-1}(R^*)) \right\} \\
&= \tau(y\varphi^{n-1}(R)\cdots RxR^*\cdots\varphi^{n-1}(R^*)).
\end{aligned} \tag{7.6}$$

We now insert E_n into the trace and use (ii) iteratively to arrive at

$$\begin{aligned}
\tau(yH_R(x)) &= \tau(yE_n(\varphi^{n-1}(R)\cdots RxR^*\cdots\varphi^{n-1}(R)^*)) \\
&= \tau(yE_{n-1}(\varphi^{n-2}(R)\cdots RxR^*\cdots\varphi^{n-2}(R)^*)) = \tau(yE_0(x)) = \tau(y)\tau(x).
\end{aligned}$$

As n was arbitrary and τ is faithful, this implies $H_R(x) = \tau(x)$, that is, we have shown (iii).

(iii)⇒(iv) To amplify (iii) to ergodicity, we will use the cabling maps c_n and cabling powers $R^{(n)}$, $n \in \mathbb{N}$. The first step is to realize that if R satisfies the ergodicity condition, then so does

$R^{(n)}$, that is,

$$E_{d^n,1}(R^{(n)}c_n(x)(R^{(n)})^*) = \tau(x), \quad x \in \mathcal{F}_d^n.$$

Applying c_n^{-1}, this condition is seen to be equivalent to

$$E_n({}_nR_n \cdot x \cdot {}_n(R^*)_n) = \tau(x), \quad x \in \mathcal{F}_d^n,$$

which can be proven by induction in n with the help of the ergodicity condition for R, expressed as in (ii) (and is obvious in graphical notation).

Let $n \in \mathbb{N}$ and $x \in \mathcal{F}_d^n$. Then $c_n(x) \in \mathcal{F}_{d^n}^1$, and since $R^{(n)}$ satisfies (i) and thus also (iii), we have $H_{R^{(n)}}(c_n(x)) = \tau(c_n(x)) = \tau(x)$ and therefore

$$\tau(x) = (c_n^{-1} \circ H_{R^{(n)}} \circ c_n)(x), \quad x \in \mathcal{F}_d^n. \tag{7.7}$$

We now recall that $c_n^{-1} \circ \lambda_{R^{(n)}} \circ c_n = \lambda_R^n$ as endomorphisms of $\mathcal{N}_d$ (4.9). Expressing $H_{R^{(n)}}$ as an ergodic mean as in (7.4), we then see that $H_{R,n} := c_n^{-1} \circ H_{R^{(n)}} \circ c_n$ is the τ-preserving conditional expectation from $\mathcal{N}_d$ onto its fixed point subalgebra $\mathcal{N}_d^{\lambda_R^n}$.

Equation (7.7) states that $H_{R,n}$ acts as the trace on $\mathcal{F}_d^n$. As clearly $\mathcal{N}_d^{\lambda_R} \subset \mathcal{N}_d^{\lambda_R^n}$, also the conditional expectation H_R acts as the trace on $\mathcal{F}_d^n$. In other words, $\tau(yH_R(x)) = \tau(y)\tau(x)$ for all $y \in \mathcal{N}_d$ and all x in the algebraic infinite tensor product $\bigcup_n \mathcal{F}_d^n$. By continuity, this extends to $\tau(yH_R(x)) = \tau(y)\tau(x)$ for all $x, y \in \mathcal{N}_d$, which is equivalent to ergodicity, $H_R = \tau$, by the faithfulness of τ.

(iv)⇒(i) Let $x \in \mathcal{F}_d^1$. According to the calculation (7.6) in the proof of (ii)⇒(iii), specialized to $n = 1$, we have for all $y \in \mathcal{F}_d^1$

$$\tau(yH_R(x)) = \tau(yRxR^*) = \tau(yE_1(RxR^*)).$$

If λ_R is ergodic, we have $H_R(x) = \tau(x)$. As $E_1(RxR^*)$ is an element of $\mathcal{F}_d^1$, and $y \in \mathcal{F}_d^1$ was arbitrary, we see that $E_1(RxR^*) = \tau(x)$, that is, (i) holds. □

As an application of Theorem 7.5, we show that diagonal R-matrices (Def. 2.8) are ergodic.

Corollary 7.7. *Diagonal R-matrices are ergodic.*

Proof. A diagonal R-matrix is of the form $R = \lambda_u(S)$ with $u \in \mathcal{U}(\mathcal{D}_d^1)$ and $S \in \mathcal{R}(d)$ of the form $S_{kl}^{ij} = c_{lk}\delta_l^i\delta_k^j$, $i,j,k,l \in \{1,\dots,d\}$ with parameters $c_{lk} \in \mathbb{T}$. It is straightforward to verify the ergodicity condition (7.5) for S. Since $R \approxeq S$ (type 1), R is ergodic as well. □

Remark 7.8. Any non-trivial fixed point $x = \lambda_R(x) = RxR^* \in \mathcal{F}_d^1$ satisfies $E_1(RxR^*) = x$ and therefore violates the ergodicity condition. Conversely, if some $x \in \mathcal{F}_d^1$ violates the ergodicity condition, then the argument in the proof (iv)⇒(i) of Theorem 7.5 shows that $H_R(x) \neq \tau(x)$. That is, we have a non-trivial fixed point $H_R(x) \in \mathcal{N}^{\lambda_R}$ in this case. However, typically $H_R(x)$ will not lie in $\mathcal{F}_d^1$ or even $\mathcal{F}_d$, but only in its weak closure $\mathcal{N}$.

Thus, one might expect that the condition that λ_R has no non-trivial fixed points in $\mathcal{F}_d^1$,

$$\mathbb{C} \stackrel{!}{=} (\mathcal{F}_d^1)^{\lambda_R} = \{x \in \mathcal{F}_d^1 \,:\, RxR^* = x\}, \tag{7.8}$$

is strictly weaker than the ergodicity condition. We will prove this later by an example.

So far, we have not ruled out completely the possibility that $\mathcal{O}_d^{\lambda_R} \not\subset \mathcal{F}_d$. The next result shows that at least there are no *algebraic* fixed points outside $\mathcal{F}_d$ if R is non-trivial. It also shows that (7.8) captures precisely the absence of non-trivial algebraic fixed points.

Here and in the following, we write ${}^0\mathcal{O}_d \subset \mathcal{O}_d$ for the algebraic part of $\mathcal{O}_d$, that is, the unital *-algebra of polynomials in the generators $S_1, \dots, S_d$ and their adjoints, and

${}^0\mathcal{F}_d := {}^0\mathcal{O}_d \cap \mathcal{F}_d = \bigcup_{n\in\mathbb{N}} \mathcal{F}_d^n = {}^0\mathcal{N}$ for the algebraic part of $\mathcal{F}_d$. We also use the shorthand notations ${}^0\mathcal{O}_d^{\lambda_R} := {}^0\mathcal{O}_d \cap \mathcal{O}_d^{\lambda_R}$ and ${}^0\mathcal{F}_d^{\lambda_R} := {}^0\mathcal{F}_d \cap \mathcal{F}_d^{\lambda_R}$.

PROPOSITION 7.9. *Let $R \in \mathcal{R}(d)$.*

(i) *If $R \notin \mathbb{C}$, then all algebraic fixed points of λ_R are contained in $\mathcal{F}_d$, that is, ${}^0\mathcal{O}_d^{\lambda_R} = {}^0\mathcal{F}_d^{\lambda_R}$.*
(ii) *${}^0\mathcal{F}_d^{\lambda_R} = \mathbb{C}$ if and only if $(\mathcal{F}_d^1)^{\lambda_R} = \mathbb{C}$.*

Proof. (i) Let $x \in {}^0\mathcal{O}_d$ be an algebraic fixed point of λ_R that is not contained in $\mathcal{F}_d$, without loss of generality assumed to be selfadjoint. As $x \notin \mathcal{F}_d = \mathcal{O}_d^{(0)}$, it has a non-zero spectral component $x^{(n)}$, $n > 0$, which also lies in ${}^0\mathcal{O}_d^{\lambda_R}$. We may therefore express it as $x^{(n)} = E^n(x)S_1^n$ with $E^n(x) \in \mathcal{F}_d^k$ for some $k \in \mathbb{N}_0$. Then, for all multi indices α, β of length $|\alpha| = |\beta| = k$, we have $t_{\alpha,\beta} := S_\alpha^* E^n(x) S_\beta \in \mathbb{C}$.

Now define $T := S_\alpha^* x^{(n)} S_\beta = S_\alpha^* E^n(x) S_1^n S_\beta$ where we have chosen α, β such that $T \neq 0$; this is possible because $x^{(n)} \neq 0$. By virtue of Proposition 7.1(iii), T is a fixed point. Furthermore, T can be expressed as

$$T = S_\alpha^* E^n(x) S_1^n S_\beta = \sum_{\gamma:|\gamma|=k} S_\alpha^* E^n(x) S_\gamma \, S_\gamma^* S_1^n S_\beta = \sum_{\gamma:|\gamma|=k} t_{\alpha,\gamma} S_\gamma^* S_1^n S_\beta.$$

As the multi-indices β and γ have the same length k for all terms in the sum, we see that T is a linear combination of products of n generators $S_{i_1} \cdots S_{i_n}$. In particular, T is a (non-zero) multiple of an isometry.
To conclude the proof, note that as a consequence of R being an element of $\mathcal{F}_d^2$, and in view of the form of T, we have $(T^*)^2 R T^2 \in \mathbb{C}$. But as a fixed point, T commutes with R (cf. Proposition 7.1(i)). Therefore,

$$\mathbb{C} \ni (T^*)^2 R T^2 = (T^*)^2 T^2 R,$$

and as $(T^*)^2 T^2$ is a non-zero scalar, the triviality of R follows.

(ii) The implication $\Rightarrow$ is trivial. For the reverse implication, let $x \in (\mathcal{F}_d^k)^{\lambda_R}$ for some $k \in \mathbb{N}$. Then, by Proposition 7.1(iii), $S_{i_1}^* \cdots S_{i_{k-1}}^* x S_{j_{k-1}} \cdots S_{j_1} \in (\mathcal{F}_d^1)^{\lambda_R} = \mathbb{C}$ for all i_l, j_l. Thus,

$$x = \sum_{l=1}^{k-1} \sum_{i_l,j_l=1}^{d} S_{i_{k-1}} \cdots S_{i_1} \left(S_{i_1}^* \cdots S_{i_{k-1}}^* x S_{j_{k-1}} \cdots S_{j_1} \right) S_{j_1}^* \cdots S_{j_{k-1}}^* \in \mathcal{F}_d^{k-1},$$

and inductively it follows that $x \in (\mathcal{F}_d^1)^{\lambda_R} = \mathbb{C}$. □

We now compare the ergodicity condition and the condition $(\mathcal{F}_d^1)^{\lambda_R} = \mathbb{C}$ in more detail. It turns out that they have quite different behavior with respect to taking box sums.

LEMMA 7.10. *Let $R, S \in \mathcal{R}$.*

(i) *$R \boxplus S$ satisfies the ergodicity condition if and only if both R and S do.*
(ii) *$\lambda_{R\boxplus S}$ has no non-trivial algebraic fixed points.*

Proof. (i) Let us view $R \in \mathcal{R}(d) \subset \mathrm{End}(V \otimes V)$, $S \in \mathcal{R}(d') \subset \mathrm{End}(W \otimes W)$ with $\dim V = d$, $\dim W = d'$, and pick orthonormal bases $\{e_i : i = 1, \dots, d\}$ of V and $\{f_j : j = 1, \dots, d'\}$ of W. We denote the orthogonal projection from $V \oplus W$ onto V and W by p and $p^\perp$, respectively.

Recall that E_1 acts as the normalized right partial trace on $\mathrm{End}((V \oplus W) \otimes (V \oplus W))$. Writing $U := R \boxplus S$ as a shorthand, we have, $x \in \mathrm{End}(V \oplus W)$,

$$(d+d')\langle e_i, E_1(UxU^*)e_j\rangle = \sum_{k=1}^{d} \langle e_i \otimes e_k, UxU^*(e_j \otimes e_k)\rangle + \sum_{l=1}^{d'} \langle e_i \otimes f_l, UxU^*(e_j \otimes f_l)\rangle$$

$$= \sum_{k=1}^{d} \langle e_i \otimes e_k, RpxpR^*(e_j \otimes e_k)\rangle + \delta^i_j \sum_{l=1}^{d'} \langle f_l, p^\perp x p^\perp f_l\rangle.$$

The ergodicity condition demands that for every x, this equals

$$(d+d')\langle e_i, \tau(x)e_j\rangle = \delta^i_j \sum_{k=1}^{d} \langle e_k, pxpe_k\rangle + \delta^i_j \sum_{l=1}^{d'} \langle f_l, p^\perp x p^\perp f_l\rangle.$$

Comparing the expressions, we see that the ergodicity condition for $R \boxplus S$ implies the ergodicity condition for R. Analogously, one shows that ergodicity of S is necessary for ergodicity of $R \boxplus S$. To check that this is sufficient, we also have to consider the 'mixed' expectation values of $E_1(UxU^*)$ between vectors in V and W, namely $\langle e_i, E_1(UxU^*)f_j\rangle$. But since $R \boxplus S$ acts as the flip on mixed tensors, it follows that these necessarily vanish, in agreement with the ergodicity condition. Hence ergodicity of R and S is also sufficient for ergodicity of $R \boxplus S$.

(ii) We need to show that the only $x \in \mathrm{End}(V \oplus W)$ commuting with $U = R \boxplus S$ are multiples of the identity (cf. Proposition 7.9). We have

$$UxU^*(p \otimes p) = U(pxp \otimes p + p^\perp xp \otimes p)R^* = R(pxp \otimes p)R^* + (p \otimes p^\perp xp)FR^*,$$

$$x(p \otimes p) = pxp \otimes p + p^\perp xp \otimes p.$$

As R commutes with $p \otimes p$, this implies $p^\perp xp = 0$, and analogously $pxp^\perp = 0$.
Similarly,

$$UxU^*(p \otimes p^\perp) = U(xp^\perp \otimes p)F = U(p^\perp xp^\perp \otimes p)F = p \otimes p^\perp xp^\perp,$$

$$x(p \otimes p^\perp) = pxp \otimes p^\perp.$$

Taking partial traces, we find $pxp = c \cdot p$, $p^\perp xp^\perp = c \cdot p^\perp$ with $c \in \mathbb{C}$. Thus $x = c \in \mathbb{C}$, and (7.8) is satisfied. □

This result gives us many R-matrices that are not ergodic but do not have any non-trivial algebraic fixed points either. Consider an involutive R-matrix N of normal form, that is,

$$N = \boxplus_{i=1}^{n} \varepsilon_i 1_{d_i} \tag{7.9}$$

for some $n \in \mathbb{N}$, with signs $\varepsilon_i \in \{\pm 1\}$ and dimensions $d_i \in \mathbb{N}$, $\sum_{i=1}^n d_i = d$ (see Theorem 5.7(ii)). Then Lemma 7.10(ii) shows that N has non-trivial fixed points if and only if it is trivial, namely $n = 1$ and $N = \pm 1$. We also know if $d_1 = \cdots = d_n = 1$, then N is diagonal and hence ergodic (Corollary 7.7). But all other normal forms N, and, in fact, all R-matrices R equivalent to them, are *not* ergodic, as we show next.

Proposition 7.11. *Let R be ergodic. Then*

$$\|\phi_R(R)\|_2^2 = \tau(R^*\varphi(R)) = \frac{1}{d^2}. \tag{7.10}$$

If R is ergodic and involutive, it is of diagonal type, that is, $R \sim N$ for a normal form (7.9) with $d_1 = \cdots = d_n = 1$.

Proof. We consider the ergodicity condition (7.5) with $i = k$ and $j = l$. Summing over i, j gives

$$d^{-2} = d^{-3} \sum_{i,j=1}^{d} \delta^i_j = d^{-3} \sum_{i,j,n,m=1}^{d} R^{im}_{in} (R^*)^{jn}_{jm} = \tau(\phi_F(R)\phi_F(R^*)).$$

Recalling that $\phi_F(R) = \phi_R(R)$, this gives $\|\phi_R(R)\|_2^2 = d^{-2}$ as claimed. Furthermore,

$$\tau(\phi_R(R)\phi_F(R^*)). = \tau(R\lambda_R(\phi_F(R^*))) = \tau(R\phi_F(R^*)) = \tau(\varphi(R)R^*).$$

We now specialize to the case that $R = R^*$ is involutive. Then we may express $\tau_R(b_1 b_2) = \tau(\varphi(R)R)$, the value of a three-cycle in the character τ_R, in terms of the Thoma parameters α_k, β_l of R. Recall that $d\alpha_k, d\beta_l \in \mathbb{N}$ are the dimensions d_i of the normal form of R, summing to d. Thus, by (5.17),

$$d = d^3\tau(\varphi(R)R) = \sum_k (d\alpha_k)^3 + \sum_l (d\beta_l)^3 = \sum_{i=1}^{n} d_i^3 \geqslant \sum_{i=1}^{n} d_i = d.$$

It follows that $d_i = 1$ for all i. □

We now want to demonstrate the fact hinted at earlier — there exist R-matrices R such that λ_R is ergodic on the C^*-algebra $\mathcal{O}_d$, but not on the von Neumann algebra $\mathcal{M}$ (or, analogously, ergodic on $\mathcal{F}_d$ but not on $\mathcal{N}$). For this, we need a result that improves the absence of non-trivial algebraic fixed points (Proposition 7.9) to absence of non-trivial fixed points in $\mathcal{O}_d$.

The arguments in the following proof are generalizations of arguments given in [**51**]. Note that the YBE is not used here.

Proposition 7.12. *Let $U \in \mathcal{U}(\mathcal{F}_d)$ and $v \in \mathcal{U}(\mathcal{F}_d^1)$ such that there exists $i \in \{1, \ldots, d\}$ with $vS_i = z \cdot S_i$ for some $z \in \mathbb{T}$. If $S_i \in (\lambda_v, \lambda_U)$, then $\mathcal{O}_d^{\lambda_U} = \mathbb{C}$.*

Proof. In view of Proposition 7.3, it is enough to show that $\mathcal{F}_d^{\lambda_U} = \mathbb{C}$. Let $x \in \mathcal{F}_d^{\lambda_U}$ be a fixed point. Writing $T := S_i$ for the intertwiner, the assumption $T \in (\lambda_v, \lambda_U)$ implies

$$T\lambda_v(x) = \lambda_U(x)T = xT \;\Rightarrow\; x = \lambda_v^{-1}(T^*xT). \tag{7.11}$$

Since $\lambda_v^{-1}(T) = v^{-1}S_i = \frac{1}{z}T$, we see that λ_v^{-1} commutes with $\operatorname{ad} T^*$. We therefore have $x = T^*\lambda_v^{-1}(x)T$, which we may iterate to

$$x = (T^*)^n \lambda_v^{-n}(x) T^n, \quad n \in \mathbb{N}. \tag{7.12}$$

We now show that this implies $x \in \mathbb{C}$. Indeed, if x lies in $\mathcal{F}_d^m$ for some $m \in \mathbb{N}$, then so does $\lambda_v^{-n}(x)$, and thus $T^{*n}\lambda_v^{-n}(x)T^n \in \mathbb{C}$ for all $n \geqslant m$. This already shows that λ_U admits no non-trivial algebraic fixed points.

If $x \in \mathcal{F}_d$ is a non-algebraic fixed point of λ_U, we consider a sequence $(x_k)_{k\in\mathbb{N}} \subset {}^0\mathcal{F}_d$ converging in norm to x. For any k, there exists $n(k) \in \mathbb{N}$ such that for all $n \geqslant n(k)$, we have $T^{*n}\lambda_v^{-n}(x_k)T^n = \mu_k \cdot 1$ for an n-independent complex number μ_k. Given $k, l \in \mathbb{N}$, we then have for $n \geqslant \max\{n(k), n(l)\}$

$$|\mu_k - \mu_l| = \|T^{*n}\lambda_v^{-n}(x_k - x_l)T^n\| \leqslant \|x_k - x_l\|,$$

and it follows that μ_k converges to a limit μ as $k \to \infty$.

To show that $x = \mu \cdot 1$, let $n, k \in \mathbb{N}$ be arbitrary. We have

$$\begin{aligned}\|x - \mu\| &= \|T^{*n}\lambda_v^{-n}(x)T^n - \mu\| \\ &\leqslant \|T^{*n}\lambda_v^{-n}(x - x_k)T^n\| + \|T^{*n}\lambda_v^{-n}(x_k)T^n - \mu_k\| + |\mu_k - \mu| \\ &\leqslant \|x - x_k\| + \|T^{*n}\lambda_v^{-n}(x_k)T^n - \mu_k\| + |\mu_k - \mu|.\end{aligned}$$

Given $\varepsilon > 0$, we can choose k large enough such that $\|x - x_k\| < \varepsilon$ and $|\mu - \mu_k| < \varepsilon$. Choosing $n > n(k)$, we also have $T^{*n}\lambda_v^{-n}(x_k)T^n - \mu_k = 0$ and conclude $\|x - \mu\| < 2\varepsilon$. □

As an aside, we mention that this proposition still holds when U is an arbitrary unitary in $\mathcal{O}_d$. Since we will not need this stronger version, we refrain from giving the proof. Let us now look at an explicit example.

EXAMPLE 7.13. Consider the normal form R-matrix $N := 1_2 \boxplus 1_1 \in \mathcal{R}(3)$. We claim that

$$\mathcal{O}_3^{\lambda_N} = \mathbb{C}, \quad \mathcal{N}^{\lambda_N} \neq \mathbb{C}. \tag{7.13}$$

The non-ergodicity of λ_N on $\mathcal{N}$, that is, $\mathcal{N}^{\lambda_N} \neq \mathbb{C}$, follows from Proposition 7.11 because N is an involutive normal form with dimensions $d_1 = 2, d_2 = 1$.

To demonstrate ergodicity of λ_N on $\mathcal{O}_3$, we will verify the conditions of Proposition 7.12 with $v = 1$ and $i = 3$, that is, show that S_3 is an intertwiner from id to λ_N. We have to show $S_3S_i = NS_iS_3$ and $S_3S_i^* = S_i^*NS_3$ for $i = 1, 2, 3$ (note that $N = N^*$).

The R-matrix is here $N = \sum_{j,k,l,m=1}^3 N_{lm}^{jk} S_jS_kS_m^*S_l^*$ and its matrix elements satisfy $N_{3l}^{kj} = \delta_3^j\delta_l^k = N_{l3}^{jk}$ by definition of N (note that $N = FNF$). Thus, for $i = 1, 2, 3$, the conclusion follows by a routine calculation.

In Section 8, we discuss another example in which the algebraic part of the fixed point algebra is infinite-dimensional and can be described explicitly (Proposition 8.2).

8. *Two-dimensional R-matrices*

As a concrete family of examples, we consider in this section R-matrices in dimension $d = 2$. In [**35**], all solutions to the YBE have been computed, including non-unitary and non-involutive ones. In [**22**], the unitary solutions have been singled out: $\mathcal{R}(2)$ consists precisely of all those matrices R which are of the form $R = (Q \otimes Q)R_i(Q \otimes Q)^{-1}$, where R_i, $i = 1, \ldots, 4$, is one of the following R-matrices and $Q \in \operatorname{End}\mathbb{C}^2$ is invertible and satisfies certain restrictions, ensuring that R is unitary[†].

$$R_1 = q \cdot 1, \quad q \in \mathbb{T}, \quad R_2 = \begin{pmatrix} p & & & \\ & & q & \\ & r & & \\ & & & s \end{pmatrix}, \quad p, q, r, s \in \mathbb{T}, \tag{8.1}$$

$$R_3 = \begin{pmatrix} & & & p \\ & q & & \\ & & q & \\ r & & & \end{pmatrix}, \quad q, p \cdot r \in \mathbb{T}, \quad R_4 = \frac{q}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & 1 & 1 \end{pmatrix}, \quad q \in \mathbb{T}. \tag{8.2}$$

Note that R_3 is not always unitary because only $|pr| = 1$ is required, and also Q is not necessarily unitary. For our purposes, it is better to present the elements of $\mathcal{R}(2)$ in the form $\lambda_u(R_i) \cong (u \otimes u)R_i(u \otimes u)^{-1}$, where both $u \in \mathcal{F}_2^1$ and $R_i \in \mathcal{F}_2^2$ are unitary.

THEOREM 8.1. *A matrix $R \in \mathcal{F}_2^2$ lies in $\mathcal{R}(2)$ if and only if there exists $u \in \mathcal{U}(\mathcal{F}_2^1)$ and $i \in \{1, \ldots, 4\}$ such that $R = \lambda_u(R_i)$, where all parameters p, q, r, s appearing in the representatives $R_1, \ldots, R_4$ have modulus* 1.

[†]In this section (only), the notation R_i refers to the specific R-matrices listed here, and not to (2.3).

Proof. The 'if' part of the statement follows by noting that when the parameters p, q, r, s have modulus 1, then $R_1, \ldots, R_4 \in \mathcal{R}(2)$. For the 'only if' statement, we first note that for $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a = \sqrt{|p|}$, the transformed matrix $(Q \otimes Q)R_3(Q \otimes Q)^{-1}$ is of the same form as R_3, but with all parameters having unit modulus. We may therefore without loss of generality take all parameters to have unit modulus, that is, all representatives $R_1, \ldots, R_4$ to be unitary.

Let now $R = (Q \otimes Q)R_i(Q \otimes Q)^{-1}$ for some invertible $Q \in \operatorname{End}\mathbb{C}^2$ and R_i unitary. Then $R^* = R^{-1}$ is equivalent to R_i commuting with $|Q|^2 \otimes |Q|^2$, where $|Q|^2 = Q^*Q$. Thus, R_i also commutes with $|Q| \otimes |Q|$. Proceeding to the polar decomposition $Q = U|Q|$, $U \in \mathcal{U}(\mathcal{F}_2^1)$, we then have

$$
\begin{aligned}
R = (Q \otimes Q)R_i(Q \otimes Q)^{-1} &= (U \otimes U)(|Q| \otimes |Q|)R(|Q|^{-1} \otimes |Q|^{-1})(U^{-1} \otimes U^{-1}) \\
&= (U \otimes U)R(U^{-1} \otimes U^{-1}) = \lambda_U(R_i).
\end{aligned}
$$

This establishes that R is of the claimed form. □

In Cuntz algebra notation, the representatives $R_1, \ldots, R_4$ take the form

$$R_1 = q \cdot 1, \tag{8.3}$$

$$R_2 = p\, S_1S_1S_1^*S_1^* + q\, S_1S_2S_1^*S_2^* + r\, S_2S_1S_2^*S_1^* + s\, S_2S_2S_2^*S_2^*, \tag{8.4}$$

$$R_3 = p\, S_1S_1S_2^*S_2^* + q\, S_1S_2S_2^*S_1^* + q\, S_2S_1S_1^*S_2^* + r\, S_2S_2S_1^*S_1^*, \tag{8.5}$$

$$R_4 = \frac{q}{\sqrt{2}}(1 + (S_1S_1^* - S_2S_2^*)\varphi(-S_1S_2^* + S_2S_1^*)). \tag{8.6}$$

By explicit calculations, one verifies that if $R = \lambda_u(R_i)$, then also its adjoint R^* and its flipped version FRF are of this form, that is, $R^* = \lambda_{u'}(R_i)$ and $FRF = \lambda_{u''}(R_i)$ for suitable $u', u'' \in \mathcal{U}(\mathcal{F}_2^1)$, and the same[†] i. In particular, equivalences of type 1 and type 3 (see p. 18) leave the families $\{\lambda_u(R_i) : u \in \mathcal{U}(\mathcal{F}_d^1)\}$ invariant.

However, type 2 equivalences can change the representative R_i. Indeed, $\lambda_u(R_3) = R_3$ for $u = \left(\begin{smallmatrix} 0 & a \\ 1 & 0 \end{smallmatrix}\right)$ with $a = \sqrt{p/q}$, but $\varphi(u)R_3\varphi(u)^*$ equals the second representative R_2 after suitable identification of parameters.

Below we give a table summarizing key features of the endomorphisms corresponding to the R-matrices $R = \lambda_u(R_i)$, $i = 1, \ldots, 4$. Note that irreducibility and ergodicity of R do not depend on u as both properties are invariant under type 1 equivalences. The index in the third column is $[\mathcal{N} : \lambda_R(\mathcal{N})] = \operatorname{Ind}_{E_R}(\lambda_R)$.

#	Representative	$\mathcal{M}_{R,1}$	Ind.	Fixed point algebras
1	$q \cdot 1$	$\mathbb{C}$ (automorphism)	1	$\mathcal{O}_2^{\lambda_R} \cong \mathcal{F}_2 \quad \operatorname{ord}(q) = \infty$ $\mathcal{O}_2^{\lambda_R} \cong \mathcal{O}_{2^{\operatorname{ord}(q)}}$
2	$\begin{pmatrix} p & & & \\ & & q & \\ & r & & \\ & & & s \end{pmatrix}$	$M_2 \quad p = r, q = s$ $\mathbb{C} \oplus \mathbb{C} \quad \text{else}$	4	$\mathcal{N}^{\lambda_R} = \mathbb{C}$
3	$\begin{pmatrix} & & & p \\ & q & & \\ & & q & \\ r & & & \end{pmatrix}$	$\mathbb{C} \oplus \mathbb{C} \quad q^2 = pr$ $\mathbb{C} \quad q^2 \neq pr$	4	$\mathcal{N}^{\lambda_R} = \mathbb{C}$
4	$\frac{q}{\sqrt{2}}\begin{pmatrix} 1 & 1 & \\ -1 & 1 & \\ & 1 & -1 \\ & 1 & 1 \end{pmatrix}$	$\mathbb{C}$	2	$\dim \mathcal{F}_2^{\lambda_R} = \infty$ see Proposition 8.2

[†]The only non-trivial thing to do is to find $u \in \mathcal{U}(\mathcal{F}_2^1)$ such that $FR_4F = \lambda_u(R_4)$; here $u = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} -1 & i \\ -i & 1 \end{smallmatrix}\right)$ works.

Proof of the claims in the table. We go through families 1–4. The R-matrices in family 1 define automorphisms (hence $\operatorname{Ind}\lambda_R = 1$), and the form of the fixed point algebra is easy to deduce.

For the diagonal R-matrices in family 2, Lemma 6.3 shows that λ_R decomposes into two quasi-free automorphisms which are either equivalent (if $p = r$ and $q = s$) or inequivalent (if $p \neq r$ or $q \neq s$). This implies the claimed form of the relative commutant and shows $\operatorname{Ind}\lambda_R = 4$ in both cases. Since R_2 is diagonal, its ergodicity follows from Corollary 7.7.

For the 'anti-diagonal' R-matrices in family 2, one computes

$$\mathcal{M}_{R_3,1} = \{x \in \mathcal{F}_2^1 \,:\, R_3^* x R_3 = \varphi(x)\} = \begin{cases} \mathbb{C} & q^2 \neq pr \\ \mathbb{C} \oplus \mathbb{C} & q^2 = pr \end{cases}.$$

In the second case, λ_R is equivalent to the direct sum of two inequivalent automorphisms, and $\operatorname{Ind}\lambda_R = 4$. In the first case, λ_R is irreducible and R has the three distinct eigenvalues $q, \sqrt{pr}, -\sqrt{pr}$. As the cardinality of the spectrum is a lower bound for $\operatorname{Ind}\lambda_R$ (6.4), and in $d = 2$, the index of λ_R may only take the values 1,2, or 4 [**13**, Proposition 9.9], we see $\operatorname{Ind}\lambda_R = 4$ also in this case.

Each member of family 3 is type 2 equivalent to a member of family 2, that is, $R_3 \approx R_2$, and the equivalence relation $\approx$ preserves ergodicity (Remark 7.4). Hence family 3 is ergodic as well.

Due to the block form of the representative R_4 for the last family, $S_1 S_1^* \in \mathcal{F}_2^1$ is seen to be a fixed point of λ_{R_4}. Its fixed point algebra will be described in more detail below. It is easy to see that λ_{R_4} is irreducible.

The R-matrix R_4 (8.2) is special from various points of view: Up to applying quasi-free automorphisms, R_4 is the unique non-trivial R-matrix in $\mathcal{R}(2)$ for which λ_R is not ergodic, and the unique R-matrix in $\mathcal{R}(2)$ with index 2. We also mention that R_4 generates a representation of the Temperley–Lieb algebra at loop parameter $\delta = \frac{1}{2}$, and satisfies $R_4^4 \in \mathbb{C}$. Furthermore, $\lambda_{R_4}(\mathcal{O}_2)$ is the fixed point algebra of an explicit order two automorphism $\alpha \in \operatorname{Aut}\mathcal{O}_2$ [**11**]. The images of the braid group representations $\rho_R(B_n)$ are described in [**25**] in terms of extraspecial 2-groups, and its relevance for topological quantum computing is discussed in [**43**]. A variation of R_4 also appears in the exchange algebra of light-cone fields in the Ising model [**54**].

In view of this interest in R_4, it might be useful to indicate how it can be obtained systematically from the results of this article. We look for a non-trivial matrix $R \in M_2 \otimes M_2 \cong M_4$ that is a unitary solution of the YBE such that λ_R is irreducible and has non-trivial fixed points in $\mathcal{F}_d^1$. Then we know that (a) R has trivial left and right partial traces $\phi_F(R) = \phi_F(FRF) = \tau(R)$, and (b) there is a one-dimensional projection $p \in \mathcal{F}_d^1$ that commutes with R. Choose a basis of $\mathbb{C}^2$ such that $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (this amounts to applying a quasi-free automorphism to R). Then (a) and (b) imply that R is of the form

$$\begin{pmatrix} a & b & & \\ c & d & & \\ & & d & -b \\ & & -c & a \end{pmatrix}, \tag{8.7}$$

with $a, b, c, d \in \mathbb{C}$. At this stage, it is not difficult to implement the requirements that R is unitary and solves the YBE. One finds that non-triviality requires $b, c \neq 0$, and the YBE then implies $d = a$ and $c = -a^2/b$. Implementing unitarity yields the form (8.2).

To conclude this discussion, we now describe the fixed points of λ_{R_4} in $\mathcal{F}_2$ in more detail. To this end, we use the standard Pauli matrices $\sigma_0, \ldots, \sigma_3$ as a basis for $M_2 \cong \mathcal{F}_2^1$, with $\sigma_0 = 1$.

PROPOSITION 8.2. *An element $x \in \mathcal{F}_2^n$, $n \in \mathbb{N}$, is a fixed point of λ_{R_4} if and only if it is a linear combination of elements of the form $\sigma_{i_1}\varphi(\sigma_{i_2}) \cdots \varphi^{n-1}(\sigma_{i_n})$, where the following three conditions are satisfied.*

(i) $i_n \in \{0, 3\}$.
(ii) *If* $i_k \in \{0, 2\}$ *for some* $k \in \{2, \ldots, n\}$, *then* $i_{k-1} \in \{0, 3\}$.
(iii) *If* $i_k \in \{1, 3\}$ *for some* $k \in \{2, \ldots, n\}$, *then* $i_{k-1} \in \{1, 2\}$.

We have $\dim(\mathcal{F}_2^n)^{\lambda_R} = 2^n$ *and* $\mathcal{N}^{\lambda_R} = ({}^0\mathcal{F}_2^{\lambda_R})''$.

Proof. The first step is to realize that the R-matrix R_4 has the form $R_4 = \frac{q}{\sqrt{2}}(1 + i\sigma_3\varphi(\sigma_2))$. Thus, $x \in \mathcal{F}_2$ is a fixed point of λ_{R_4} if and only if it commutes with $\varphi^m(S)$, $m \in \mathbb{N}_0$, where $S := \sigma_3\varphi(\sigma_2)$ (cf. Proposition 7.1(ii)). Recall that the Pauli matrices satisfy $\sigma_i = \sigma_i^* = \sigma_i^{-1}$ and

$$\sigma_i\sigma_j\sigma_i = \begin{cases} +\sigma_j & j \in \{0, i\} \\ -\sigma_j & \text{else} \end{cases}. \tag{8.8}$$

Let x be a linear combination of elements of the form $\sigma_{i_1}\varphi(\sigma_{i_2}) \cdots \varphi^{n-1}(\sigma_{i_n})$. In view of the action (8.8), it follows that x is a fixed point if and only if each term in its expansion into this basis is a fixed point, that is, we may take $x = \sigma_{i_1}\varphi(\sigma_{i_2}) \cdots \varphi^{n-1}(\sigma_{i_n})$ without loss of generality.

Since $\sigma_2^2 = 1$, we have $\operatorname{ad}\varphi^{n-1}(S)(x) = \sigma_{i_1}\varphi(\sigma_{i_2}) \cdots \varphi^{n-1}(\sigma_3\sigma_{i_n}\sigma_3)$, which coincides with x if and only if $\sigma_3\sigma_{i_n}\sigma_3 = \sigma_{i_n}$, that is, if and only if $i_n \in \{0, 3\}$ as claimed in (i). Similarly,

$$\operatorname{ad}\varphi^{k-1}(S)(x) = \sigma_{i_1}\varphi(\sigma_{i_2}) \cdots \varphi^{k-1}(\sigma_3\sigma_{i_k}\sigma_3)\varphi^k(\sigma_2\sigma_{i_{k+1}}\sigma_2) \cdots \varphi^{n-1}(\sigma_{i_n}),$$

which coincides with x if and only if either $\sigma_2\sigma_{i_{k+1}}\sigma_2 = \sigma_{i_{k+1}}$ and $\sigma_3\sigma_{i_k}\sigma_3 = \sigma_{i_k}$ or $\sigma_2\sigma_{i_{k+1}}\sigma_2 = -\sigma_{i_{k+1}}$ and $\sigma_3\sigma_{i_k}\sigma_3 = -\sigma_{i_k}$. By (8.8) this gives the listed conditions (ii) and (iii).

A dimension count gives $\dim(\mathcal{F}_2^n)^{\lambda_R} = 2^n$. In view of the product form of $\sigma_3\varphi(\sigma_2)$, it is easy to see that $\mathcal{N}^{\lambda_R}$ is invariant under the τ-preserving conditional expectations $E_n : \mathcal{N} \to \mathcal{F}_2^n$. This invariance implies that any $x \in \mathcal{N}^{\lambda_R}$ can be approximated weakly by the sequence of fixed points $\{E_n(x)\}_{n\in\mathbb{N}}$, and hence, $\mathcal{N}^{\lambda_R} = ({}^0\mathcal{F}_2^{\lambda_R})''$. □

This result implies in particular that $[\mathcal{N} : \mathcal{L}_{R_4}] = \infty$.

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