

# Spectral convergence for high contrast elliptic periodic problems with a defect via homogenisation

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## Abstract

We consider an eigenvalue problem for a divergence form elliptic operator  $A_\varepsilon$  with high contrast periodic coefficients with period  $\varepsilon$  in each coordinate, where  $\varepsilon$  is a small parameter. The coefficients are perturbed on a bounded domain of ‘order one’ size. The local perturbation of coefficients for such operator could result in emergence of localised waves - eigenfunctions with corresponding eigenvalues lying in the gaps of the Floquet-Bloch spectrum. We prove that, for the so-called double porosity type scaling, the eigenfunctions decay exponentially at infinity, uniformly in  $\varepsilon$ . Then, using the tools of two-scale convergence for high contrast homogenisation, we prove the strong two-scale compactness of the eigenfunctions of  $A_\varepsilon$ . This implies that the eigenfunctions converge in the sense of the strong two-scale convergence to the eigenfunctions of a two-scale limit homogenised operator  $A_0$ , consequently establishing ‘asymptotic one-to-one correspondence’ between the eigenvalues and the eigenfunctions of these two operators. We also prove by direct means the stability of the essential spectrum of the homogenised operator with respect to the local perturbation of its coefficients. That allows us to establish not only the strong two-scale resolvent convergence of  $A_\varepsilon$  to  $A_0$  but also the Hausdorff convergence of the spectra of  $A_\varepsilon$  to the spectrum of  $A_0$ , preserving the multiplicity of the isolated eigenvalues.

*Keywords:* localised modes, elliptic operators, perturbed periodic operators, multiscale methods, two-scale convergence, high-contrast homogenisation

*AMS Subject Classifications:* 35B27, 35P99

## 1 Introduction

In this paper we consider a high contrast two-phase periodic medium with a small period and with a ‘finite size’ defect filled by a third phase, see Fig. 1. This physically represents, for instance, a simplified model of cross-section of a photonic crystal fiber. Mathematically, the problem relates to a compact perturbation of  $\varepsilon$ -periodic coefficients in a divergence form elliptic operator  $A_\varepsilon$ . The behaviour of  $A_\varepsilon$  and its spectral characteristics as  $\varepsilon \rightarrow 0$  is of the main interest. A similar problem is considered in [16] using the method of asymptotic expansions, but the present study pursues different aims and approaches the problem from another direction, namely developing an appropriate version of the two-scale convergence technique [20, 2, 23]. As a result we obtain a complete description of the asymptotic (with respect to  $\varepsilon$ ) behaviour of the localised modes and other spectral characteristics for

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the operator  $A_\varepsilon$  in terms of an explicitly described (two-scale) limit operator  $A_0$ . For other recent applications of the high contrast homogenisation techniques see also [22, 10, 8, 6, 11, 12, 4, 17].

In the absence of a defect, Zhikov considers in [24] a divergence form elliptic operator  $\widehat{A}_\varepsilon$  (denoted in [24] by  $A_\varepsilon$ ) with periodic coefficients corresponding to a double-porosity model [3, 9] ( $A_\varepsilon$  in our notation is obtained from  $\widehat{A}_\varepsilon$  by a compact perturbation of its coefficients). Operators of such type have the Floquet-Bloch essential spectrum, displaying a band-gap structure. Zhikov proves that the spectra of  $\widehat{A}_\varepsilon$  converge in the sense of Hausdorff to the spectrum of a certain two-scale homogenised operator  $\widehat{A}_0$  with constant coefficients, see also [14, 23], and that  $\widehat{A}_0$  is the limit of  $\widehat{A}_\varepsilon$  in the sense of strong two-scale resolvent convergence. The spectrum of  $\widehat{A}_0$  is purely essential and displays an explicit band-gap structure. It is well known, see e.g. [21, 13], that in the case of a compact perturbation of periodic coefficients in the elliptic operator  $A_\varepsilon$  its essential spectrum remains unperturbed. The only extra spectrum that can emerge in the gaps due to the perturbation is a discrete one (isolated eigenvalues with finite multiplicity).<sup>1</sup> Such an extra spectrum does emerge at least under some assumptions, e.g. [13, 16]. This corresponds physically to localised modes emerging near the defect.

One of the main goals of the paper is to establish the strong two-scale convergence of the eigenfunctions of  $A_\varepsilon$  corresponding to the eigenvalues in the gaps. In order to obtain this we need the strong two-scale compactness of eigenfunctions. This requires in turn an exponential decay of the eigenfunctions *uniform in  $\varepsilon$* .

The problem of wave localization (i.e. of the existence of eigenvalues with corresponding eigenfunctions decaying exponentially) in the gaps of the essential spectrum has been intensively investigated for a wide range of differential operators over the last decades. The results obtained up to date ensure the exponential decay of eigenfunctions of  $A_\varepsilon$  for a *fixed*  $\varepsilon$ , see e.g. [13]. However this is insufficient for establishing the required compactness. Moreover, the developed methods, e.g. [5] and [13] (the latter using the method of Agmon[1]), seem to be insufficient for the present purpose. The reason is that in order to obtain the *uniform* exponential decay one has to perform some kind of two-scale asymptotic analysis, investigating the behaviour of the eigenfunctions on small and large scales simultaneously. To achieve this we supplement the method of [1] by the related two-scale techniques, which play a crucial role. As a result, we obtain a uniform estimate with the decay exponent  $\alpha$  (see (3.1) and (2.14) below) which ensures the compactness, but may also be of an independent interest. On one hand, it is sharp in a sense. On the other hand, it behaves qualitatively entirely differently compared to e.g. the one in [5]: while the one in [5] is proportional to the square root of the distance to the gap end, the decay exponent we derive becomes large on approaching the left end of the gap and small near the right end.

The structure of the paper is the following. We first define the problem in Section 2, describe the two-scale limit operator  $A_0$  and state the main result. We then consider a subsequence of eigenvalues of  $A_\varepsilon$  converging to some point  $\lambda_0$  lying in a gap of the spectrum of  $\widehat{A}_0$ . In Section 3 we prove (Theorem 3.1) the uniform exponential decay for the eigenfunctions of  $A_\varepsilon$ . Section 4 is devoted to the proof of a main auxiliary lemma that is employed in the previous section, which may also be of an independent interest. In Section 5 we list some properties of the two-scale convergence and several related statements which we use in the next section. Employing the uniform exponential decay, we establish in Section 6 (see Theorem 6.1) the strong two-scale compactness of (normalised) eigenfunctions of  $A_\varepsilon$ , see e.g. [23, 24]. This implies that, up to a subsequence, the eigenfunctions two-scale converge to a function, which is eventually proved to be an eigenfunction of the two-scale limit operator  $A_0$  with a defect, which could be considered as a perturbation of  $\widehat{A}_0$ . Accordingly  $\lambda_0$  is an eigenvalue of  $A_0$ . The two-scale convergence of the eigenfunctions together with the results of [16] on the existence of the eigenvalues in the gaps and related error bounds allow us to make a conclusion about the ‘asymptotic one-to-one correspondence’ between eigenfunctions and eigenvalues

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<sup>1</sup>We do not concern in this paper the issue of whether *embedded* eigenvalues can emerge on the bands as a result of the perturbation.

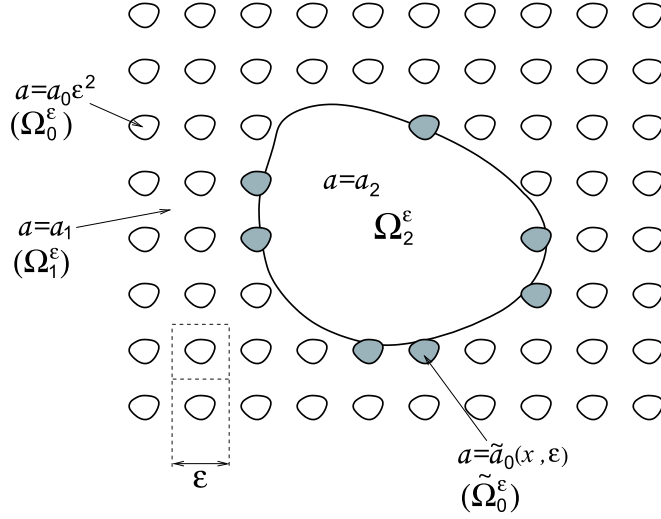


Figure 1: A defect in a rapidly oscillating high contrast periodic medium, cf. [16, Fig. 1].

of the operators  $A_\varepsilon$  and  $A_0$  as  $\varepsilon \rightarrow 0$ . In the last section we prove by direct means (via the Weyl's sequences) the stability of the essential spectrum of  $\hat{A}_0$  with respect to the local perturbation of its coefficients (see Theorem 7.1). Thereby this establishes the convergence of the spectra of  $A_\varepsilon$  to the spectrum of  $A_0$  in the sense of Hausdorff (Theorem 2.1).

## 2 Notation, problem formulation, limit operator and the main result

We will use the following notation for the geometric configuration visualised on Figure 1, cf. [16]. Consider a periodic set of unit cubes

$$\{Q : Q = [0, 1]^n + \xi, \xi \in \mathbb{Z}^n\}. \quad (2.1)$$

Let  $F_0$  be an open periodic set with period one in each coordinate such that  $F_0 \cap Q \Subset Q$  is a connected domain with infinitely smooth boundary. We denote  $F_0 \cap Q$  by  $Q_0$  and  $Q \setminus \overline{Q_0}$  by  $Q_1$ . Notice that the position of the particular set  $Q_0$ ,  $Q_1$  or  $Q$  depends on  $\xi \in \mathbb{Z}^n$ , however we will not reflect this in the notation to simplify what follows. Regularity assumptions on the boundary could be relaxed.<sup>2</sup> Let  $\Omega_2$  be a bounded domain with a sufficiently smooth boundary, containing the origin; its complement is denoted by  $\Omega_1$ ,  $\Omega_1 = \mathbb{R}^n \setminus \overline{\Omega_2}$ .

We define the ‘inclusion phase’ or the ‘soft phase’  $\Omega_0^\varepsilon$  as

$$\Omega_0^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_1} \varepsilon Q_0,$$

where  $\varepsilon > 0$  is a small parameter. The set of inclusions  $\varepsilon Q_0$  which intersect the boundary of  $\Omega_2$  is denoted by  $\tilde{\Omega}_0^\varepsilon$ . The ‘matrix phase’, denoted by  $\Omega_1^\varepsilon$ , is the complement to the inclusions in  $\Omega_1$ , i.e.  $\Omega_1^\varepsilon = \Omega_1 \setminus (\overline{\Omega_0^\varepsilon} \cup \tilde{\Omega}_0^\varepsilon)$ . ‘Defect domain’  $\Omega_2^\varepsilon$  is defined by  $\Omega_2 \setminus \tilde{\Omega}_0^\varepsilon$ . We also use the notation  $\theta_\Omega$  for the characteristic function of a set  $\Omega$  and  $B_R$  for the open ball of radius  $R$  centered at the origin.

We consider an eigenvalue problem

$$A_\varepsilon u^\varepsilon = \lambda_\varepsilon u^\varepsilon \quad (2.2)$$

<sup>2</sup>In particular, the results on the two-scale convergence stated in the paper remain valid at least under the assumption of Lipschitz regular boundaries. The  $\varepsilon^{1/2}$ -order bounds, as they were obtained in [16], require higher regularity.

for the point spectrum of an elliptic operator  $A_\varepsilon$ , self-adjoint in  $L^2(\mathbb{R}^n)$ ,

$$A_\varepsilon u^\varepsilon := -\nabla \cdot \left( a(x, \varepsilon) \nabla u^\varepsilon(x) \right), \quad x \in \mathbb{R}^n, \quad (2.3)$$

with coefficient  $a(x, \varepsilon)$  given by the formula

$$a(x, \varepsilon) = \begin{cases} a_0 \varepsilon^2, & x \in \Omega_0^\varepsilon, \\ a_1, & x \in \Omega_1^\varepsilon, \\ a_2, & x \in \Omega_2^\varepsilon, \\ \tilde{a}_0(x, \varepsilon), & x \in \tilde{\Omega}_0^\varepsilon, \end{cases} \quad (2.4)$$

where measurable  $\tilde{a}_0(x, \varepsilon)$  is such that

$$\text{either } \tilde{A}_0 \varepsilon^{2-\theta} \leq \tilde{a}_0(x, \varepsilon) \leq \sigma_0 \varepsilon^{2-\theta} \text{ for all } \varepsilon, \text{ or } \tilde{a}_0(x, \varepsilon) = a_0 \varepsilon^2 \text{ for all } \varepsilon. \quad (2.5)$$

Here  $a_0, a_1, a_2, \tilde{A}_0, \tilde{B}_0$  and  $\theta$  are some positive constants independent of  $\varepsilon$ ,  $\theta \in (0, 2]$ . Notice that this includes as particular cases e.g. the case of ‘removed’ boundary inclusions, i.e.  $a(x, \varepsilon) = a_1$  if  $x \in \tilde{\Omega}_0^\varepsilon \cap \Omega_1$ ,  $a(x, \varepsilon) = a_2$  if  $x \in \tilde{\Omega}_0^\varepsilon \cap \Omega_2$ , and the case of the ‘full’ inclusions,  $\tilde{a}_0(x, \varepsilon) = a_0 \varepsilon^2$ . The domain of  $A_\varepsilon$  is defined in a standard way via Friedrichs extension procedure with a bilinear form, see (2.6) below, defined on  $H^1(\mathbb{R}^n)$ .

For any  $\varepsilon > 0$  the operator  $A_\varepsilon$  is an operator with  $\varepsilon$ -periodic coefficients, which are compactly perturbed (within bounded domain  $\Omega_2^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$ ). This implies (e.g. [21, 13]) that its essential spectrum coincides with the Floquet-Bloch spectrum of the associated ‘unperturbed’ operator  $\hat{A}_\varepsilon$ , with only extra spectrum being hence the discrete spectrum in the gaps of  $\hat{A}_\varepsilon$ .<sup>3</sup> Note that the spectrum of  $\hat{A}_\varepsilon$  contains gaps for small enough  $\varepsilon$ , cf. [14, 23, 24], and there is often an extra discrete spectrum in the gaps of  $\sigma_{\text{ess}}(A_\varepsilon)$ , e.g. [13, 16]. By definition,  $u^\varepsilon \in H^1(\mathbb{R}^n)$ ,  $u^\varepsilon \not\equiv 0$ , is an eigenfunction of the eigenvalue problem (2.2) with an eigenvalue  $\lambda_\varepsilon$  if

$$\int_{\mathbb{R}^n} a(x, \varepsilon) \nabla u^\varepsilon \cdot \nabla w \, dx = \lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon w \, dx \quad (2.6)$$

for all  $w \in H^1(\mathbb{R}^n)$ .

The aim of this work is to establish that as  $\varepsilon \rightarrow 0$  the operator  $A_\varepsilon$  converges in the appropriate sense (namely, in the sense of two-scale convergence, see Section 5) to a ‘two-scale’ limit operator  $A_0$ , which we describe next. For the rest of the present section we assume that  $Q = [0, 1]^n$ , considering all functions of two variables  $(x, y)$  to be 1-periodic in each coordinate with respect to  $y$ . The ‘two-scale’ limit operator  $A_0$  is analogous to the one introduced in the defect free setting by Zhikov [23, 24] and acts in a Hilbert space

$$\mathcal{H}_0 := \left\{ u(x, y) \in L^2(\mathbb{R}^n \times Q) \mid u(x, y) = u_0(x) + v(x, y), u_0 \in L^2(\mathbb{R}^n), v \in L^2(\Omega_1; L^2(Q_0)) \right\}, \quad (2.7)$$

with the natural inner product inherited from  $L^2(\mathbb{R}^n \times Q)$  and  $\mathcal{H}_0$  being its closed subspace, cf. [24]. It is implied that  $v$  is extended by zero for  $y \in Q_1$  or  $x \in \Omega_2$ . The operator  $A_0$  is defined as generated by a (closed) symmetric and bounded from below bilinear form  $B_0(u, w)$  acting in a dense subspace

$$\mathcal{V} = H^1(\mathbb{R}^n) + L^2(\Omega_1, H_0^1(Q_0)) \quad (2.8)$$

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<sup>3</sup>This does not rule out possible emergence of embedded eigenvalues on the bands, not considered in this paper.

of  $\mathcal{H}_0 = L^2(\mathbb{R}^n) + L^2(\Omega_1, L^2(Q_0))$ , which is defined as follows: for  $u = u_0 + v, w = w_0 + z \in \mathcal{V}$ ,

$$B_0(u, w) = a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v \cdot \nabla_y z \, dy \, dx. \quad (2.9)$$

Here  $A^{\text{hom}} = (A_{ij}^{\text{hom}})$  is the standard ‘‘porous’’ homogenised (symmetric, positive-definite) matrix for the periodic medium as described above but when no defect is present and with  $a_0 = 0$ , see e.g. [15, §3.1]:

$$A_{ij}^{\text{hom}} \xi_i \xi_j = \inf_{w \in C_{\text{per}}^\infty(Q)} \int_{Q_1} a_1 |\xi + \nabla w|^2 \, dy \quad (\xi \in \mathbb{R}^n). \quad (2.10)$$

Here  $C_{\text{per}}^\infty(Q)$  stands for the set of infinitely smooth functions with periodic boundary conditions. Then one can see (cf. [24]) that the form is indeed bounded from below, densely defined and closed. Hence, according to the standard Friedrichs extension procedure, e.g. [21],  $A_0$  can be defined as a self-adjoint operator with a domain  $\mathcal{D}(A_0) \subset \mathcal{V}$ . A function  $u^0(x, y) = u_0(x) + v(x, y) \in \mathcal{V}$ ,  $u^0(x, y) \not\equiv 0$ , is an eigenfunction of the limit operator  $A_0$  corresponding to an eigenvalue  $\lambda_0$  if and only if

$$B_0(u^0, w) = \lambda_0 \int_{\mathbb{R}^n} \int_Q (u_0 + v)(w_0 + z) \, dy \, dx. \quad (2.11)$$

for any  $w = w_0 + z \in \mathcal{V}$  (we assume where it is possible that a function defined on a smaller domain is extended by zero on a larger domain).<sup>4</sup>

The ‘unperturbed’ operators  $\widehat{A}_\varepsilon$  and  $\widehat{A}_0$  could be defined analogously to  $A_\varepsilon$  and  $A_0$  formally setting above  $\Omega_2 = \emptyset$  and  $\Omega_1 = \mathbb{R}^n$ . (See also [23, 24], where these operators are denoted by  $A_\varepsilon$  and  $A$  respectively.)

We next describe a function  $\beta(\lambda)$  which was introduced by Zhikov [23, 24] (cf. also [8]) and plays an important role in our considerations. Let  $\lambda_j$  and  $\varphi_j$ ,  $j = 1, 2, \dots$ , be eigenvalues and corresponding orthonormalised eigenfunctions of operator  $T$  defined as

$$Tf := -a_0 \Delta f, \quad f \in H_0^1(Q_0) \cap H^2(Q_0). \quad (2.12)$$

Note that the eigenvalues of  $T$  belong to the spectrum of  $\widehat{A}_0$ , see [23]. For  $\lambda \neq \lambda_j$ ,  $j \geq 1$ , denote by  $b$  the solution to

$$Tb - \lambda b = -a_0 \Delta b - \lambda b = 1, \quad b \in H_0^1(Q_0). \quad (2.13)$$

The function  $\beta(\lambda)$  is defined by

$$\beta(\lambda) := \lambda(1 + \lambda \langle b \rangle_y) = \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle_y^2}{\lambda_j - \lambda} \quad (2.14)$$

where  $\langle f \rangle_y := \int_Q f(y) \, dy$ . It is well-defined for any  $\lambda$  except  $\lambda = \lambda_j$  with  $\langle \varphi_j \rangle_y \neq 0$ , monotonically

increasing between such points, see Figure 2. This function describes the structure of  $\sigma(\widehat{A}_0)$ , see [23]. Namely, the intervals where  $\beta(\lambda) \geq 0$  correspond to the bands of the spectrum of  $\widehat{A}_0$ . Isolated points of the spectrum of  $\widehat{A}_0$ , i.e.  $\lambda_j$  such that  $\langle \varphi_j \rangle_y = 0$  and  $\beta(\lambda_j) < 0$ , can also be regarded as degenerate bands. The intervals on which  $\beta(\lambda) < 0$  (excluding  $\lambda_j$ ) are gaps.

<sup>4</sup>Explicit examples in [16, §5] ensure the existence of isolated eigenvalues of  $A_0$  of finite multiplicity in the gaps of  $\sigma(\widehat{A}_0)$  in particular situations.

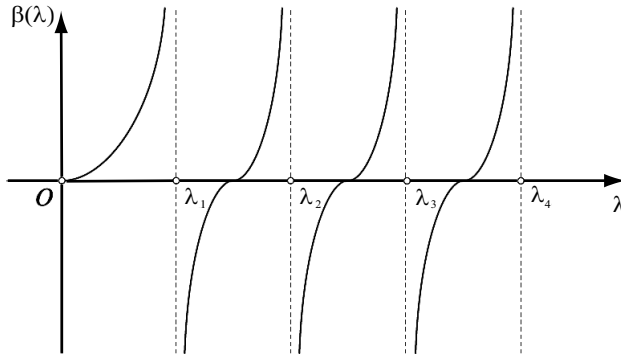


Figure 2:  $\beta(\lambda)$ , cf. [24].

It was shown in [24] (see also [14, 23]) that  $\sigma(\widehat{A}_\varepsilon)$  converges in the sense of Hausdorff to  $\sigma(\widehat{A}_0)$ , while  $\widehat{A}_\varepsilon$  converges to  $\widehat{A}_0$  in the sense of the strong two-scale resolvent convergence (cf. Sections 5 and 6 below) implying the convergence of spectral projectors, etc.

We aim at showing that similar as well as some further results hold for the perturbed operators. Namely, our main result is the following

**Theorem 2.1.** *The operator  $A_\varepsilon$  converges to  $A_0$  in the sense of the strong two-scale resolvent convergence. Hence the spectral projectors also strongly two-scale converge away from the point spectrum of  $A_0$ . The spectrum of  $A_\varepsilon$  converges in the sense of Hausdorff to the spectrum of  $A_0$ . Let  $\lambda_0$  be an isolated eigenvalue of multiplicity  $m$  of the operator  $A_0$  in the gap of its essential spectrum. Then, for small enough  $\varepsilon$ , there exist exactly  $m$  eigenvalues  $\lambda_{\varepsilon,i}$  of  $A_\varepsilon$  (counted with their multiplicities) such that*

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2}, \quad i = 1, \dots, m, \quad (2.15)$$

with a constant  $C$  independent of  $\varepsilon$ .<sup>5</sup> If for some sequence  $\varepsilon_k \rightarrow 0$  a sequence of eigenvalues  $\lambda_{\varepsilon_k}$  of  $A_{\varepsilon_k}$  converges to  $\lambda_0$  which is in the gap of the essential spectrum of  $A_0$ , then  $\lambda_0$  is an isolated eigenvalue of  $A_0$  of a finite multiplicity  $m$  and for large enough  $k$ ,  $\lambda_{\varepsilon_k} \in \{\lambda_{\varepsilon_k,i}, i = 1, \dots, m\}$ .

A key part in establishing the latter is in controlling the behaviour at infinity of the eigenfunctions corresponding to the extra point spectrum which may appear in the spectral gaps of the unperturbed operator. A central property providing this is a uniform exponential decay of the eigenfunctions which we prove next.

### 3 Uniform exponential decay of the eigenfunctions of $A_\varepsilon$

Let  $\lambda_0$  be a point in a gap of  $\sigma(\widehat{A}_0)$ , i.e. such that  $\beta(\lambda_0) < 0$  and  $\lambda_0 \neq \lambda_j$  for all  $j$ . Assume  $\lambda_0$  is an accumulation point of the point spectra of  $A_\varepsilon$ , i.e. for some subsequence  $\varepsilon_k \rightarrow 0$  there exist eigenvalues  $\lambda_{\varepsilon_k}$  of  $A_{\varepsilon_k}$  such that  $\lambda_{\varepsilon_k} \rightarrow \lambda_0$  as  $k \rightarrow \infty$ . (Notice that the results of [13, 16] ensure in particular that such sequences do exist.) We formulate the main result of this section (and also one of the principal results of the paper) in the following.

**Theorem 3.1.** *Let  $\lambda_{\varepsilon_k}$  and  $u^{\varepsilon_k}$  be sequences of eigenvalues of the operator  $A_{\varepsilon_k}$  and corresponding eigenfunctions normalised in  $L^2(\mathbb{R}^n)$ , where  $\varepsilon_k$  is some positive sequence converging to zero as  $k \rightarrow \infty$ . Let  $\lambda_0$  be such that  $\beta(\lambda_0)$  is negative and  $\lambda_0$  is not an eigenvalue of the operator  $T$  given by (2.12).*

<sup>5</sup>The error bound (2.15) employs the results of [16] requiring, as stated, higher regularity of  $\partial Q_0$ . The rest of the statement of the theorem applies potentially to less regular boundaries.

Suppose that  $\lambda_{\varepsilon_k}$  converges to  $\lambda_0$ . Then for small enough  $\varepsilon_k$  eigenfunctions  $u^{\varepsilon_k}$  decay uniformly exponentially at infinity, namely, for

$$0 < \alpha < \sqrt{-\beta(\lambda_0)/a_1} \quad (3.1)$$

the following holds:

$$\|e^{\alpha|x|}u^{\varepsilon_k}\|_{L^2(\mathbb{R}^n)} \leq C,$$

uniformly in  $\varepsilon_k$ , i.e. for any  $0 < \varepsilon_k < \varepsilon(\alpha)$ , with  $C = C(\alpha)$  independent of  $\varepsilon$ .

*Proof.* We drop the index  $k$  in  $\varepsilon_k$  for the sake of simplification of notation. So, when we say, for instance, ‘sequence  $\lambda_\varepsilon$ ’ we actually mean ‘subsequence  $\lambda_{\varepsilon_k}$ ’.

The plan of the proof is the following. We first derive ‘elementary’ a priori estimates for the eigenfunction  $u^\varepsilon$  outside the set of inclusions  $\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$ . Next we study the structure of the eigenfunction at the small scale and deduce some vital inequalities for  $\varepsilon \nabla u^\varepsilon$  inside the inclusions. As a central technical step, we then employ in the integral identity (2.6) a test function with exponentially growing weight  $g^2(|x|)$ , see (3.12)–(3.13) below, and perform some delicate two-scale uniform estimates to achieve the result. The main auxiliary technical results are proven in Lemma 3.3 and Proposition 4.1.

**Step 1.** Setting  $w = u^\varepsilon$  in (2.6) we have

$$\varepsilon^2 a_0 \|\nabla u^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}^2 + a_1 \|\nabla u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}^2 + a_2 \|\nabla u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}^2 + \|\tilde{a}_0^{1/2}(x, \varepsilon) \nabla u^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)}^2 = \lambda_\varepsilon \|u^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 = \lambda_\varepsilon.$$

Therefore

$$\|u^\varepsilon\|_{H^1(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon))} \leq C \quad (3.2)$$

uniformly in  $\varepsilon$ . From now on  $C$  denotes a generic constant whose precise value is insignificant and can change from line to line.

**Step 2.** Let us consider the function  $u^\varepsilon$  in a cell  $\varepsilon Q$  corresponding to such  $\xi = \xi(\varepsilon) \in \mathbb{Z}^n$ , see 2.1, that the corresponding ‘inclusion’  $\varepsilon Q_0$  has a nonempty intersection with  $\Omega_1$ . There exists an extension  $\tilde{u}^\varepsilon$  of  $u^\varepsilon|_{\varepsilon Q_1}$  to the whole cell  $\varepsilon Q$  such that

$$\|\tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq C \|u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \quad \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq C \|\nabla u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \quad (3.3)$$

where  $C$  does not depend on  $\varepsilon$  or  $\xi$ , see e. g. [19, Ch. 3, §4, Th. 1], which is a version of the so-called ‘extension lemma’, see also e.g. [15, §3.1, L. 3.2]. In particular, we can choose the following extension:

$$\begin{aligned} \tilde{u}^\varepsilon &\equiv u^\varepsilon, & x &\in \Omega_1^\varepsilon \cup \Omega_2^\varepsilon, \\ -\nabla \cdot (a(x, \varepsilon) \nabla \tilde{u}^\varepsilon(x)) &= 0, & x &\in \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon, \end{aligned}$$

which minimises  $\|a^{1/2}(x, \varepsilon) \nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)}$  subject to the prescribed boundary conditions, with (2.4) and (2.5) ensuring that (3.3) still holds. From (3.2) and (3.3) we conclude that

$$\|\tilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)} \leq C. \quad (3.4)$$

We represent  $u^\varepsilon$  in the form

$$u^\varepsilon(x) = \tilde{u}^\varepsilon(x) + v^\varepsilon(x) \quad (3.5)$$

and consider the function  $v^\varepsilon \in H_0^1(\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon)$ .<sup>6</sup> In each inclusion  $\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$  we have the following boundary value problem for  $v^\varepsilon(x)$ :

$$-\nabla \cdot (a(x, \varepsilon) \nabla v^\varepsilon) - \lambda_\varepsilon v^\varepsilon = \lambda_\varepsilon \tilde{u}^\varepsilon, \quad x \in \varepsilon Q_0; \quad v^\varepsilon(x) = 0, \quad x \in \partial(\varepsilon Q_0). \quad (3.6)$$

<sup>6</sup>In a sense, (3.5) decomposes  $u^\varepsilon$  into a slowly varying part  $\tilde{u}^\varepsilon$  and rapidly varying  $v^\varepsilon$ . The two are coupled and subsequently analyzed simultaneously, which is the essence of two-scale asymptotic analysis.

When  $a(x, \varepsilon) = a_0\varepsilon^2$ , i.e. everywhere in  $\Omega_0^\varepsilon$  and also in  $\tilde{\Omega}_0^\varepsilon$  in the case  $\tilde{a}_0(x, \varepsilon) = a_0\varepsilon^2$ , after changing the variables  $x \rightarrow y = x/\varepsilon$  we obtain

$$-a_0\Delta_y v^\varepsilon(\varepsilon y) - \lambda_\varepsilon v^\varepsilon(\varepsilon y) = \lambda_\varepsilon \tilde{u}^\varepsilon(\varepsilon y), \quad y \in Q_0, \quad v^\varepsilon(\varepsilon y) = 0, \quad y \in \partial Q_0. \quad (3.7)$$

Since  $\lambda_0 \neq \lambda_j$  by the assumptions of the theorem,  $\lambda_\varepsilon$  is separated uniformly from the spectrum of operator (2.12) for small enough  $\varepsilon$ . Hence the resolvent of  $T$  at  $\lambda_\varepsilon$  is bounded uniformly in  $\varepsilon$  and (3.7) implies

$$\|v^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \leq C\|\tilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}. \quad (3.8)$$

In the case when  $\tilde{A}_0\varepsilon^{2-\theta} \leq \tilde{a}_0(x, \varepsilon) \leq \tilde{B}_0\varepsilon^{2-\theta}$ ,  $\theta \in (0, 2]$ , we multiply equation (3.6) by  $v^\varepsilon$  and integrate by parts to obtain after rescaling

$$\varepsilon^{-2} \int_{\varepsilon Q_0} \tilde{a}_0(\varepsilon y, \varepsilon) |\nabla_y v^\varepsilon(\varepsilon y)|^2 dx - \lambda_\varepsilon \int_{\varepsilon Q_0} (v^\varepsilon(\varepsilon y))^2 dx = \lambda_\varepsilon \int_{\varepsilon Q_0} \tilde{u}^\varepsilon(\varepsilon y) v^\varepsilon(\varepsilon y) dx. \quad (3.9)$$

Notice that  $\varepsilon^{-2}\tilde{a}_0(\varepsilon y, \varepsilon) \geq \tilde{A}_0\varepsilon^{-\theta} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . Then using Poincaré inequality one easily derives

$$\varepsilon^{-2} \|\tilde{a}_0^{1/2} \nabla_y v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2 + \|v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2 \leq C\|\tilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2, \quad (3.10)$$

for small enough  $\varepsilon$ . Returning in (3.8) and (3.10) to the variable  $x$  we arrive at the following inequality that describes the behaviour of  $v^\varepsilon$  and its gradient in  $\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$ ,

$$\|a^{1/2} \nabla v^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2 + \|v^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2 \leq C\|\tilde{u}^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2, \quad (3.11)$$

with an  $\varepsilon$ -independent constant  $C$ .

**Step 3.** In order to get the uniform exponential decay of the eigenfunctions we next substitute in (2.6) a test function of a special form:

$$w = g^2(|x|)\tilde{u}^\varepsilon(x). \quad (3.12)$$

Here we define function  $g$  as follows

$$g(t) = \begin{cases} e^{\alpha t}, & t \in [0, R], \\ e^{\alpha R}, & t \in (R, +\infty), \end{cases} \quad (3.13)$$

where  $R$  is some arbitrary positive number. The exponent  $\alpha$  will be chosen later. This method was employed e.g. by Agmon, see [1], but in the present case its realization is not straightforward. Namely, to obtain the desired estimates we have to implement the approach of [1] in the context of the two-scale analysis. We will show that  $g(|x|)\tilde{u}^\varepsilon(x)$ , and consequently  $g(|x|)u^\varepsilon(x)$ , are bounded in  $L^2(\mathbb{R}^n)$  uniformly with respect to  $R$  and  $\varepsilon$ . Then we will show via passing to the limit as  $R \rightarrow \infty$  that we can replace  $g(|x|)$  by  $e^{\alpha|x|}$ .

**Remark 3.2.** We cannot use  $e^{2\alpha|x|}\tilde{u}^\varepsilon(x)$  as a test function directly, since it is not known at this stage that this function is square integrable.

The following identity holds by direct inspection

$$\nabla \tilde{u}^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) = |\nabla (g \tilde{u}^\varepsilon)|^2 - |\nabla g|^2 (\tilde{u}^\varepsilon)^2. \quad (3.14)$$

Notice that the absolute value of  $\nabla g$  is bounded by  $g$  with  $\alpha$  (uniformly in  $R$ ):

$$|\nabla g(|x|)| \leq \alpha g(|x|). \quad (3.15)$$



After the substitution of (3.12) into (2.6) we have, via (3.5) and (3.14),

$$\begin{aligned}
& \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) dx + \int_{\mathbb{R}^n \setminus \Omega_0^\varepsilon} a(x, \varepsilon) |\nabla (g \tilde{u}^\varepsilon)|^2 dx - \\
& - a_1 \int_{\Omega_1^\varepsilon} |\nabla g|^2 (\tilde{u}^\varepsilon)^2 dx - \lambda_\varepsilon \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\tilde{u}^\varepsilon)^2 dx - \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 v^\varepsilon \tilde{u}^\varepsilon dx = \\
& = \lambda_\varepsilon \int_{\tilde{\Omega}_0^\varepsilon} g^2 u^\varepsilon \tilde{u}^\varepsilon dx + \lambda_\varepsilon \int_{\Omega_2^\varepsilon} g^2 (\tilde{u}^\varepsilon)^2 dx + \int_{\Omega_2^\varepsilon \cup \tilde{\Omega}_0^\varepsilon} a(x, \varepsilon) |\nabla g|^2 (\tilde{u}^\varepsilon)^2 dx.
\end{aligned} \tag{3.16}$$

Notice that the right hand side is bounded by some constant  $C$  independent of  $\varepsilon$  and  $R$  due to (3.2), (3.4), (3.11) and the boundedness of the domains of integration.

We employ (3.4), (3.11) and the boundedness of  $\tilde{a}_0$  to conclude that the second term on the left hand side of (3.16) tends to zero (uniformly in  $R$ ):

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) dx \right| \leq C \|\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \rightarrow 0, \tag{3.17}$$

as follows. Let us take an arbitrary subsequence  $\tilde{u}^\varepsilon$ . Since  $\|\tilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)}$  is bounded uniformly in  $\varepsilon$ , see (3.4), the set of functions  $\tilde{u}^\varepsilon$  is weakly compact in  $H^1(B_R)$ , hence strongly compact in  $L^2(B_R)$  for any  $R$ ; we take  $R$  large enough so that  $\Omega_2 \subset\subset B_R$ . Then there exists further subsequence  $\tilde{u}^\varepsilon$  that converges to some function  $u_0$  strongly in  $L^2(B_R)$ . Then

$$\|\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \leq \|u_0\|_{L^2(\tilde{\Omega}_0^\varepsilon)} + \|\tilde{u}^\varepsilon - u_0\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \rightarrow 0$$

as Lebesgue measure of the set  $\tilde{\Omega}_0^\varepsilon$  tends to zero. Since we have chosen in the beginning an arbitrary subsequence  $\tilde{u}^\varepsilon$ , (3.17) follows. From (3.11) and (3.17) we also obtain

$$\|v^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \rightarrow 0. \tag{3.18}$$

**Step 4.** The following Lemma approximates and bounds the last and the first terms (both, in a sense, of a ‘two-scale’ nature) on the left hand side of (3.16).

**Lemma 3.3.** *There exists  $\varepsilon_0 > 0$  such that for all positive  $\varepsilon < \varepsilon_0$  the following estimates are valid*

$$\begin{aligned}
& \left| \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 v^\varepsilon \tilde{u}^\varepsilon dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\tilde{u}^\varepsilon)^2 dx \right| \leq \\
& \leq C \varepsilon \left( \|\nabla (g \tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 \right) + C,
\end{aligned} \tag{3.19}$$

and

$$\left| \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) dx \right| \leq C \varepsilon \left( \|\nabla (g \tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + C \right), \tag{3.20}$$

where  $C$  does not depend on  $\varepsilon$  and  $R$ .

The proof of this lemma is quite technical and we give it in the next section. We make use of Lemma 3.3 and convergence (3.17) to transform identity (3.16) into the following inequality, valid for small enough  $\varepsilon$ :

$$a_1 \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 - a_1 \|(\nabla g)\tilde{u}^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}^2 - \beta(\lambda_\varepsilon) \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 - 2C\varepsilon \left( \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 \right) \leq C,$$

where  $C$  is independent of  $\varepsilon$  and  $R$ . Notice that  $\beta(\lambda_\varepsilon)$  is negative and uniformly bounded away from zero as  $\lambda_\varepsilon \rightarrow \lambda_0$ . Applying (3.15) to the second term on the left hand side we arrive at

$$(a_1 - 2C\varepsilon) \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + (-\beta(\lambda_\varepsilon) - \alpha^2 a_1 - 2C\varepsilon) \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 \leq C. \quad (3.21)$$

Hence we should choose  $\alpha$  such that  $-\beta(\lambda_0) - \alpha^2 a_1$  is positive, i.e.

$$\alpha < \sqrt{-\beta(\lambda_0)/a_1}.$$

Since  $g(|x|)$  coincides with  $e^{\alpha|x|}$  on the ball  $B_R$ , restricting the  $L^2$ -norms to  $B_R$  we obtain that

$$\left\| e^{\alpha|x|} \tilde{u}^\varepsilon \right\|_{L^2(B_R)} \leq C$$

for small enough  $\varepsilon$ , where  $C$  does not depend on  $\varepsilon$  and  $R$ . Then passing to the limit as  $R \rightarrow \infty$  we obtain

$$\left\| e^{\alpha|x|} \tilde{u}^\varepsilon \right\|_{L^2(\mathbb{R}^n)} \leq C. \quad (3.22)$$

**Step 5.** Now we easily get the same estimate for the function  $u^\varepsilon$ :

$$\left\| e^{\alpha|x|} u^\varepsilon \right\|_{L^2(\mathbb{R}^n)} \leq \left\| e^{\alpha|x|} \tilde{u}^\varepsilon \right\|_{L^2(\mathbb{R}^n)} + \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon} \left\| e^{\alpha|x|} v^\varepsilon \right\|_{L^2(\varepsilon Q_0)}.$$

In each cell we use inequality (3.11) and

$$\sup_{x' \in \varepsilon Q} e^{\alpha|x'|} \leq e^{\alpha\sqrt{n}\varepsilon} e^{\alpha|x|}, \quad \forall x \in \varepsilon Q,$$

to obtain

$$\left\| e^{\alpha|x|} v^\varepsilon \right\|_{L^2(\varepsilon Q_0)} \leq C e^{\alpha\sqrt{n}\varepsilon} \left\| e^{\alpha|x|} \tilde{u}^\varepsilon \right\|_{L^2(\varepsilon Q_0)} \leq C \left\| e^{\alpha|x|} \tilde{u}^\varepsilon \right\|_{L^2(\varepsilon Q_0)},$$

and hence, finally,

$$\left\| e^{\alpha|x|} u^\varepsilon \right\|_{L^2(\mathbb{R}^n)} \leq C$$

uniformly in  $\varepsilon$ . □

**Remark 3.4.** From (3.3), (3.15) and (3.21) it also follows that the gradient of  $\tilde{u}^\varepsilon$  decays exponentially at infinity,

$$\left\| e^{\alpha|x|} \nabla \tilde{u}^\varepsilon \right\|_{L^2(\mathbb{R}^n)} \leq C \quad (3.23)$$

uniformly in  $\varepsilon$ .

**Remark 3.5.** Estimate (3.1) is sharp in a sense. As we will show later,  $u_\varepsilon$  strongly two-scale converges to  $u_0$ , for which  $\sqrt{-\beta(\lambda_0)/a_1}$  is the optimal estimate for its decay exponent, cf. (7.15).

## 4 Proof of Lemma 3.3.

*Proof. Step 1.* First we decompose the function  $v^\varepsilon$  in  $\Omega_0^\varepsilon$  into the sum of two functions:

$$v^\varepsilon = \tilde{v}^\varepsilon + \hat{v}^\varepsilon, \quad (4.1)$$

solving the following equations (cf. (3.7)):

$$-a_0 \Delta_y \tilde{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon \tilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon \langle \tilde{u}^\varepsilon(\varepsilon y) \rangle_y, \quad y \in Q_0, \quad \tilde{v}^\varepsilon(\varepsilon y) = 0, \quad y \in \partial Q_0, \quad (4.2)$$

$$-a_0 \Delta_y \hat{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon \hat{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon (\tilde{u}^\varepsilon(\varepsilon y) - \langle \tilde{u}^\varepsilon(\varepsilon y) \rangle_y), \quad y \in Q_0, \quad \hat{v}^\varepsilon(\varepsilon y) = 0, \quad y \in \partial Q_0. \quad (4.3)$$

The solution of (4.2) could be presented in the form

$$\tilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon \langle \tilde{u}^\varepsilon \rangle_y b_\varepsilon(y), \quad (4.4)$$

where  $b_\varepsilon$  is a solution of (2.13) with  $\lambda = \lambda_\varepsilon$ . Due to the uniform (with respect to  $\varepsilon$ ) boundedness of the resolvent of the operator  $T$  in the neighborhood of  $\lambda_0$ , the solution of (4.3) is bounded as follows,

$$\|\tilde{v}^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \leq C \|\tilde{u}^\varepsilon(\varepsilon y) - \langle \tilde{u}^\varepsilon \rangle_y\|_{L^2(Q_0)} \leq C \|\nabla_y \tilde{u}^\varepsilon\|_{L^2(Q_0)},$$

here we also employed the Poincaré inequality. In particular

$$\|\tilde{v}^\varepsilon(x)\|_{L^2(\varepsilon Q_0)} \leq \varepsilon C \|\nabla \tilde{u}^\varepsilon(x)\|_{L^2(\varepsilon Q)}, \quad (4.5)$$

where  $C$  in the inequality does not depend on  $\varepsilon$  or  $\xi \in \mathbb{Z}^n$ .

**Step 2.** At this stage we will need several inequalities which follow from the properties of  $g$  and  $\tilde{u}^\varepsilon$ .

**Proposition 4.1.** *The following estimates are valid for small enough  $\varepsilon$  with constants independent of  $\varepsilon$  and the choice of particular  $\varepsilon Q$ :*

$$\|g^2 \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \leq C \left( \|\nabla(g \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right), \quad (4.6)$$

$$\|\tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla(g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q)} \leq C \left( \|\nabla(g \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right), \quad (4.7)$$

$$\|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla(g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q)} \leq C \left( \|\nabla(g \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right). \quad (4.8)$$

*Proof.* Notice that

$$\sup_{x' \in \varepsilon Q} g(x') \leq e^{\alpha \sqrt{n} \varepsilon} g(x), \quad x \in \varepsilon Q. \quad (4.9)$$

We apply (3.3), (3.15) and (4.9) to get (4.6):

$$\begin{aligned} & \|g^2 \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \leq C \|g^2 \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \leq C \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|g \nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} = \\ & = C \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla(g \tilde{u}^\varepsilon) - (\nabla g) \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \leq C \left( \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \|\nabla(g \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)} + \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right) \leq \\ & \leq C \left( \|\nabla(g \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right). \end{aligned}$$

The proof of (4.7) and (4.8) is analogous.  $\square$

Let us show that the entity  $\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon dx$  is relatively small (compared to the first term on the right hand side of (3.19)). Indeed, applying inequalities (4.5) and (4.6) in each cell we obtain

$$\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon dx \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \|g^2 \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \|\widehat{v}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon C \left( \|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right). \quad (4.10)$$

Considering sets

$$\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \quad \text{and} \quad \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1,$$

one can notice that they are “nearly” equal to

$$\Omega_0^\varepsilon \cup \Omega_1^\varepsilon \quad \text{and} \quad \Omega_1^\varepsilon,$$

respectively. Namely,

$$\begin{aligned} \Omega_0^\varepsilon \cup \Omega_1^\varepsilon &= \left( \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon, \\ \Omega_1^\varepsilon &= \left( \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1 \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon, \end{aligned}$$

where

$$\begin{aligned} \Omega_{1,-}^\varepsilon &= \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \cap \Omega_2, \\ \Omega_{1,+}^\varepsilon &= \bigcup_{\varepsilon Q_0 \cap \Omega_2 \neq \emptyset} \varepsilon Q \cap \Omega_1^\varepsilon. \end{aligned}$$

We introduce two ‘correctors’

$$r^\varepsilon = \|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\Omega_{1,-}^\varepsilon)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_{1,-}^\varepsilon)}^2,$$

and

$$r_1^\varepsilon = \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_{1,+}^\varepsilon \cup \Omega_{1,-}^\varepsilon)}^2.$$

Then inequality (4.10) transforms into

$$\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon dx \leq \varepsilon C \left( \|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + r^\varepsilon \right). \quad (4.11)$$

**Step 3.** Now we consider the term  $\int_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon dx$  (cf. (3.19)) using also (4.4) and (2.14):

$$\begin{aligned} & \left| \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2 dx \right| \leq \\ & \leq C \varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \int_Q g^2 \widetilde{u}^\varepsilon(\varepsilon y) b_\varepsilon(y) \langle \widetilde{u}^\varepsilon \rangle_y dy - \langle b_\varepsilon \rangle_y \int_Q g^2(\varepsilon y) (\widetilde{u}^\varepsilon(\varepsilon y))^2 dy \right| + C r_1^\varepsilon \leq \\ & \leq C \varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \langle \widetilde{u}^\varepsilon \rangle_y \int_Q (g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y) b_\varepsilon dy \right| + \left| \langle b_\varepsilon \rangle_y \int_Q (g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y) \widetilde{u}^\varepsilon dy \right| + C r_1^\varepsilon. \end{aligned} \quad (4.12)$$

Notice that the mean value of  $\tilde{u}^\varepsilon$  is bounded by its norm in  $L^2$

$$|\langle \tilde{u}^\varepsilon(\varepsilon y) \rangle_y| = \left| \int_Q \tilde{u}^\varepsilon dy \right| \leq \|\tilde{u}^\varepsilon(x)\|_{L^2(Q)}. \quad (4.13)$$

Similarly,

$$\langle b_\varepsilon \rangle_y \leq \|b_\varepsilon\|_{L^2(Q_0)} \leq C, \quad (4.14)$$

where  $C$  does not depend on  $\varepsilon$  due to the uniform boundedness of  $(T - \lambda)^{-1}$  in the neighborhood of  $\lambda_0$ . Via the Poincaré inequality we derive

$$\left| \int_Q (g^2 \tilde{u}^\varepsilon - \langle g^2 \tilde{u}^\varepsilon \rangle_y) \tilde{u}^\varepsilon dy \right| \leq \|g^2 \tilde{u}^\varepsilon - \langle g^2 \tilde{u}^\varepsilon \rangle_y\|_{L^2(Q)} \|\tilde{u}^\varepsilon\|_{L^2(Q)} \leq C \|\nabla_y (g^2 \tilde{u}^\varepsilon)\|_{L^2(Q)} \|\tilde{u}^\varepsilon\|_{L^2(Q)}, \quad (4.15)$$

and

$$\left| \int_Q (g^2 \tilde{u}^\varepsilon - \langle g^2 \tilde{u}^\varepsilon \rangle_y) b_\varepsilon dy \right| \leq C \|\nabla_y (g^2 \tilde{u}^\varepsilon)\|_{L^2(Q)}, \quad (4.16)$$

with constants independent of  $\varepsilon$  and  $\xi$ . Applying inequalities (4.13)–(4.16) and then (4.7) to (4.12) we arrive at

$$\begin{aligned} & \left| \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 \tilde{v}^\varepsilon \tilde{u}^\varepsilon dx + (\lambda_\varepsilon - \beta(\lambda_\varepsilon)) \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\tilde{u}^\varepsilon)^2 dx \right| \leq \\ & \leq \varepsilon C \left( \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + r^\varepsilon \right) + C r_1^\varepsilon, \end{aligned} \quad (4.17)$$

where  $C$  is  $\varepsilon$ -independent. Since the correctors  $r^\varepsilon$ ,  $r_1^\varepsilon$  are uniformly bounded, inequalities (4.11) and (4.17) together imply the validity of (3.19).

**Step 4.** Finally, it is not difficult to obtain similarly (3.20) via (3.11), (4.7) and (4.8):

$$\begin{aligned} & \left| \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) dx \right| \leq \varepsilon^2 C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \|\nabla u^\varepsilon\|_{L^2(\varepsilon Q_0)} \|\nabla (g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_0)} \leq \\ & \leq \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left( \|\varepsilon \nabla v^\varepsilon\|_{L^2(\varepsilon Q_0)} + \varepsilon \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \right) \|\nabla (g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_0)} \leq \\ & \leq \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left( \|\tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \|\nabla (g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_0)} + \varepsilon \|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \|\nabla (g^2 \tilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_0)} \right) \leq \\ & \leq \varepsilon C \left( \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + r^\varepsilon \right) \leq \varepsilon C \left( \|\nabla(g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + C \right) \end{aligned}$$

for small enough  $\varepsilon$ .

Notice that all the estimates obtained in this section are independent of  $R$ .  $\square$

## 5 Some properties of two-scale convergence

In this section we list the definitions and some properties of the two-scale convergence, see [2, 20, 23, 24]. We also formulate several statements (analogous to those in [23]) which are necessary for obtaining the two-scale convergence of the eigenfunctions of  $A_\varepsilon$  and derivation of the limit equation.

Let  $\Omega$  be an arbitrary region in  $\mathbb{R}^n$ , in particular  $\Omega = \mathbb{R}^n$ . Denote by  $\square$  the unit cube  $[0, 1]^n$ . We consider all functions of the form  $u(x, y)$  to be 1-periodic in  $y$  in each coordinate.

**Definition 5.1.** We say that a bounded in  $L^2(\Omega)$  sequence  $v_\varepsilon$  is weakly two-scale convergent to a function  $v \in L^2(\Omega \times \square)$ ,  $v_\varepsilon(x) \xrightarrow{2} v(x, y)$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) \varphi(x) b\left(\frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int v(x, y) \varphi(x) b(y) dy dx$$

for all  $\varphi \in C_0^\infty(\Omega)$  and all  $b \in C_{\text{per}}^\infty(\square)$  (where  $C_{\text{per}}^\infty(\square)$  is the set of 1-periodic functions from  $C^\infty(\mathbb{R}^n)$ ).

**Definition 5.2.** We say that a bounded in  $L^2(\Omega)$  sequence  $u_\varepsilon$  is strongly two-scale convergent to a function  $u \in L^2(\Omega \times \square)$ ,  $u_\varepsilon(x) \xrightarrow{2} u(x, y)$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) v_\varepsilon(x) dx = \int_{\Omega} \int u(x, y) v(x, y) dy dx$$

for all  $v_\varepsilon(x) \xrightarrow{2} v(x, y)$ .

**Proposition 5.3.** (Properties of the two-scale convergence.)

(i) If  $u_\varepsilon(x) \xrightarrow{2} u(x, y)$  and  $a \in L_{\text{per}}^\infty(\square)$  then

$$a(x/\varepsilon) u_\varepsilon(x) \xrightarrow{2} a(y) u(x, y).$$

(ii)  $v_\varepsilon(x) \xrightarrow{2} v(x, y)$  if and only if  $v_\varepsilon(x) \xrightarrow{2} v(x, y)$  and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon^2 dx = \int_{\Omega} \int v^2 dy dx.$$

(iii) If  $f_\varepsilon(x) \rightarrow f(x)$  in  $L^2(\Omega)$ , then  $f_\varepsilon(x) \xrightarrow{2} f(x)$ .

**Proposition 5.4.** (The mean value property of periodic functions.) Let  $\Phi(y) \in L_{\text{per}}^1(\square)$ . Then for each  $\phi(x) \in C_0^\infty(\mathbb{R}^n)$  we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \phi(x) \Phi(x/\varepsilon) dx = \langle \Phi \rangle_y \int_{\mathbb{R}^n} \phi(x) dx.$$

Potential vector space  $V_{\text{pot}}$  is defined as a closure of the set  $\{\nabla\varphi : \varphi \in C_{\text{per}}^\infty(\square)\}$  in  $L^2(\square)^n$ . We say that a vector  $b \in L^2(\square)^n$  is solenoidal ( $b \in V_{\text{sol}}$ ) if it is orthogonal to all potential vectors. Thus,

$$L^2(\square)^n = V_{\text{pot}} \oplus V_{\text{sol}},$$

and

$$L^2(\Omega \times \square)^n = L^2(\Omega, V_{\text{pot}}) \oplus L^2(\Omega, V_{\text{sol}}).$$

**Lemma 5.5.** Let  $u_\varepsilon$  and  $\varepsilon \nabla u_\varepsilon$  be bounded in  $L^2(\mathbb{R}^n)$ . Then (up to a subsequence)

$$\begin{aligned} u_\varepsilon(x) &\xrightarrow{2} u(x, y) \in L^2(\mathbb{R}^n, H_{\text{per}}^1), \\ \varepsilon \nabla u_\varepsilon(x) &\xrightarrow{2} \nabla_y u(x, y), \end{aligned}$$

where  $H_{\text{per}}^1 = H_{\text{per}}^1(\square)$  is the Sobolev space of periodic functions.

**Lemma 5.6.** Let  $u_\varepsilon \in H^1(\mathbb{R}^n)$ ,

$$u_\varepsilon(x) \xrightarrow{2} u(x) \in H^1(\mathbb{R}^n), \quad (5.1)$$

and  $\nabla u_\varepsilon$  is bounded in  $L^2(\mathbb{R}^n)$ . Then, up to a subsequence,

$$\nabla u_\varepsilon(x) \xrightarrow{2} \nabla u(x) + v(x, y), \text{ where } v \in L^2(\mathbb{R}^n, V_{\text{pot}}). \quad (5.2)$$

**Lemma 5.7.** Let (5.1) and (5.2) be valid. Let also

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^{\ddagger}} a_1 \nabla u_\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x) \varphi(x) dx = 0 \quad (5.3)$$

for any  $\varphi \in C_0^\infty(\Omega_1)$  and  $w \in C_{\text{per}}^\infty(\square)$ . Then the following weak convergence of the flows takes place:

$$a_1 \theta_{Q_1}(\varepsilon^{-1}x) \nabla u_\varepsilon(x) \rightharpoonup A^{\text{hom}} \nabla u(x) \text{ in } \Omega_1,$$

where homogenised matrix  $A^{\text{hom}}$  is defined by (2.10).

The proofs of the listed statements repeat the proofs of the corresponding assertions in [23] with no or only small alterations, and are not given here.

**Definition 5.8.** Let  $A_\varepsilon$ ,  $\varepsilon > 0$ , and  $A_0$  be non-negative self-adjoint operators in  $L^2(\mathbb{R}^n)$  and  $\mathcal{H}_0 \subset L^2(\mathbb{R}^n \times Q)$ , see (2.7), respectively. We say that  $A_\varepsilon \xrightarrow{2} A_0$  in the sense of the strong two-scale resolvent convergence if  $(A_\varepsilon + I)^{-1} f_\varepsilon \xrightarrow{2} (A_0 + I)^{-1} f_0$  as long as  $f_\varepsilon \xrightarrow{2} f_0$ .

## 6 Strong two-scale convergence of the eigenfunctions and multiplicity of the eigenvalues of $A_\varepsilon$

In this section we will show that the normalised eigenfunctions  $u_\varepsilon$  are compact in the sense of strong two-scale convergence. Namely, provided  $\lambda_\varepsilon \rightarrow \lambda_0$ , a sequence of normalised eigenfunctions  $u_\varepsilon$  of the operator  $A_\varepsilon$  strongly two-scale converges, up to a subsequence, to a function  $u^0(x, y)$ . This implies that  $u^0(x, y)$  is an eigenfunction corresponding to the eigenvalue  $\lambda_0$  of the limit operator  $A_0$ . This, together with results of [16], establishes an ‘asymptotic one-to-one correspondence’ between isolated eigenvalues and corresponding eigenfunctions of the operators  $A_\varepsilon$  and  $A_0$ .

**Theorem 6.1.** Under the assumptions of Theorem 3.1  $\lambda_0$  is an eigenvalue of the operator  $A_0$ . Moreover, there exists a subsequence  $\varepsilon$  such that eigenfunctions  $u^\varepsilon$  of the operator  $A_\varepsilon$  strongly two-scale converge to an eigenfunction  $u^0(x, y)$  of  $A_0$  corresponding to the eigenvalue  $\lambda_0$ .

*Proof. Step 1.* In order to establish strong two-scale convergence of the eigenfunctions  $u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon$  we prove it for each of its components separately. From (3.22) and (3.23) it follows that

$$\|\tilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n \setminus B_R)} \leq C e^{-\alpha R} \quad (6.1)$$

with  $C$  independent of  $\varepsilon$  and  $R$ . From this one can easily conclude that  $\tilde{u}^\varepsilon$  is weakly compact in  $H^1(\mathbb{R}^n)$  and strongly compact in  $L^2(\mathbb{R}^n)$ . Indeed, since  $\tilde{u}^\varepsilon$  are bounded in  $H^1(\mathbb{R}^n)$  uniformly in  $\varepsilon$ ,

$$\tilde{u}^\varepsilon \rightharpoonup u_0 \text{ in } H^1(\mathbb{R}^n), \quad (6.2)$$

up to a subsequence. For any fixed  $R$  function  $\tilde{u}^\varepsilon$  converges to  $u_0$  weakly in  $H^1(B_R)$  and, hence, strongly in  $L^2(B_R)$  up to a subsequence. Considering a sequence of balls  $B_R$ ,  $R \in \mathbb{N}$ , one can use the method of extracting a diagonal subsequence such that

$$\tilde{u}^\varepsilon \rightarrow u_0 \text{ in } L^2(B_R) \quad (6.3)$$

for any  $R > 0$ .

For any  $\delta > 0$  we can choose  $R$  such that  $\|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$  and  $\|\tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$  for sufficiently small  $\varepsilon$  (the latter follows from (6.1)). From (6.3) it follows that  $\|u_0 - \tilde{u}^\varepsilon\|_{L^2(B_R)} < \delta/3$  for sufficiently small  $\varepsilon$ . Then, up to a subsequence,

$$\|u_0 - \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|u_0 - \tilde{u}^\varepsilon\|_{L^2(B_R)} + \|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)} + \|\tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta$$

for small enough  $\varepsilon$ . Hence, up to a subsequence, we have

$$\tilde{u}^\varepsilon \rightarrow u_0 \text{ in } L^2(\mathbb{R}^n).$$

Then from properties of the two-scale convergence we conclude that

$$\tilde{u}^\varepsilon \xrightarrow{2} u_0. \quad (6.4)$$

**Step 2.** Now let us consider  $v^\varepsilon$ . We denote by  $v_1^\varepsilon$  and  $v_2^\varepsilon$  its restrictions  $v_\varepsilon|_{\Omega_0^\varepsilon}$  and  $v_\varepsilon|_{\tilde{\Omega}_0^\varepsilon}$  respectively, extended by zero to the rest of  $\mathbb{R}^n$ .

**Lemma 6.2.** *The following convergence properties are valid for  $v_1^\varepsilon$  (up to a subsequence):*

$$\begin{aligned} v_1^\varepsilon(x) &\xrightarrow{2} v(x, y) \in L^2(\Omega_1, H_0^1(Q_0)), \\ \varepsilon \nabla v_1^\varepsilon(x) &\xrightarrow{2} \nabla_y v(x, y), \end{aligned}$$

where  $v(x, y)$  is a solution to the following problem:

$$-a_0 \Delta_y v - \lambda_0 v = \lambda_0 u_0, \quad y \in Q_0. \quad (6.5)$$

Here  $u_0$  is a function from (6.4).

*Proof.* Function  $v_1^\varepsilon \in H^1(\Omega_0^\varepsilon)$  satisfies the following differential equation:

$$-\varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_\varepsilon v_1^\varepsilon = \lambda_\varepsilon \tilde{u}^\varepsilon \text{ in } \Omega_0^\varepsilon. \quad (6.6)$$

The right hand side of this equation is of the form  $\lambda_\varepsilon \theta_{\Omega_0^\varepsilon} \tilde{u}^\varepsilon$ . By (6.4) and the properties of the two-scale convergence we have

$$\lambda_\varepsilon \theta_{\Omega_0^\varepsilon}(x) \tilde{u}^\varepsilon(x) \xrightarrow{2} \lambda_0 \theta_{Q_0}(y) \theta_{\Omega_1}(x) u_0(x). \quad (6.7)$$

Following [23] we consider more general problem

$$z_\varepsilon \in H^1(\Omega_0^\varepsilon), \quad -\varepsilon^2 a_0 \Delta z_\varepsilon - \lambda_\varepsilon z_\varepsilon = f_\varepsilon, \quad f_\varepsilon \in L^2(\Omega_0^\varepsilon). \quad (6.8)$$

(It is implicit that  $f_\varepsilon = z_\varepsilon = 0$  in  $\mathbb{R}^n \setminus \Omega_0^\varepsilon$ .)

**Proposition 6.3.** *Let*

$$f^\varepsilon(x) \xrightarrow{2} f(x, y). \quad (6.9)$$

Then

$$\begin{aligned} z^\varepsilon(x) &\xrightarrow{2} z(x, y) \in L^2(\Omega_1, H_0^1(Q_0)), \\ \varepsilon \nabla z^\varepsilon(x) &\xrightarrow{2} \nabla_y z(x, y), \end{aligned}$$

where function  $z(x, y)$  solves the following equation:

$$-a_0 \Delta_y z - \lambda_0 z = f, \quad y \in Q_0. \quad (6.10)$$



*Proof.* One can easily derive an estimate for  $z^\varepsilon$  analogous to (3.11), applying to (6.8) a reasoning similar to those for the solution of equation (3.6). This gives us the weak two-scale convergence of  $z^\varepsilon$  and  $\varepsilon \nabla z^\varepsilon$  via Lemma 5.5. The result follows by a straightforward passing to the limit in the integral identity corresponding to (6.8) with appropriately chosen test function. The full proof could be found in [23] and applies to the present situation with no alteration.  $\square$

The above proposition together with (6.7) establishes a “weak” form of the statement of the lemma, i.e. weak two-scale convergence of  $v_1^\varepsilon$ . We now prove that the convergence is actually strong, following again [23]. Multiply (6.6) and (6.8) by  $z^\varepsilon$  and  $v_1^\varepsilon$  respectively and integrate by parts. The left hand sides of the resulting equalities are identical. So, equating the right hand sides, we obtain the following identity

$$\int_{\Omega_1} f^\varepsilon v^\varepsilon dx = \lambda_\varepsilon \int_{\Omega_1} \tilde{u}^\varepsilon z^\varepsilon dx.$$

By the definition of the strong two-scale convergence we have

$$\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon \int_{\Omega_1} \tilde{u}^\varepsilon z^\varepsilon dx = \lambda_0 \int_{\Omega_1} \int_{Q_0} u_0(x) z(x, y) dy dx.$$

Multiplying (6.5) and (6.10) by  $z$  and  $v$  respectively and integrating by parts it is easy to see that

$$\lambda_0 \int_{\Omega_1} \int_{Q_0} u_0(x) z(x, y) dy dx = \int_{\Omega_1} \int_{Q_0} f(x, y) v(x, y) dy dx.$$

Thus, we have a convergence of the integrals:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_1} f^\varepsilon v_1^\varepsilon dx = \int_{\Omega_1} \int_{Q_0} f(x, y) v(x, y) dy dx$$

for any weakly two-scale convergent sequence  $f^\varepsilon$ . Hence, by the definition,

$$v_1^\varepsilon(x) \xrightarrow{2} v(x, y).$$

$\square$

**Lemma 6.4.** *Sequence of functions  $v_2^\varepsilon$  converges to zero in the sense of strong two-scale convergence:*

$$v_2^\varepsilon \xrightarrow{2} 0 \text{ as } \varepsilon \rightarrow 0.$$

*Proof.* Straightforward from (3.18) and Proposition 5.3 (iii).  $\square$

Combining (6.4) with Lemmas 6.2 and 6.4, we arrive at

$$u^\varepsilon(x) \xrightarrow{2} u^0(x, y) = u_0(x) + v(x, y), \tag{6.11}$$

where  $u_0 \in H^1(\mathbb{R}^n)$ ,  $v \in L^2(\Omega_1, H_0^1(Q_0))$ .

**Step 3.** Now it remains to show that  $u^0(x, y)$  is an eigenfunction and  $\lambda_0$  is the corresponding eigenvalue of the limit operator  $A_0$ , i.e. that  $u^0(x, y)$  satisfies (2.11). In order to do that we need to choose appropriate test-function  $\psi^\varepsilon$  and pass to the limit in the integral identity

$$\begin{aligned} \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx + a_1 \int_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx + \\ + a_2 \int_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx = \lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon \psi^\varepsilon dx \end{aligned} \tag{6.12}$$

corresponding to the original eigenvalue problem (2.2)–(2.3). Let us take

$$\begin{aligned}\psi^\varepsilon(x) &= \psi_0(x) + \varphi(x)b(\varepsilon^{-1}x), \\ \psi_0 &\in C_0^\infty(\mathbb{R}^n), \varphi \in C_0^\infty(\Omega_1), b(y) \in C_0^\infty(Q_0),\end{aligned}\tag{6.13}$$

and consider each term of (6.12) separately. Let us expand the first term:

$$\begin{aligned}\varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon dx &= \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla \tilde{u}^\varepsilon \nabla \psi^\varepsilon dx + \\ &+ \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla v^\varepsilon (\nabla \psi_0 + b(\varepsilon^{-1}x) \nabla \varphi) dx + a_0 \int_{\Omega_0^\varepsilon} \varepsilon \nabla v^\varepsilon \varphi \nabla_y b(\varepsilon^{-1}x) dx.\end{aligned}$$

As  $\nabla \tilde{u}^\varepsilon$  is bounded in  $L^2$ -norm and  $|\nabla \psi^\varepsilon| \leq C\varepsilon^{-1}$  then the first term on the right hand side tends to zero. From (3.11) and the boundedness of  $\nabla \psi_0 + b \nabla \varphi$  we conclude that the second term also converges to zero. Since by Lemma 6.2  $\varepsilon \nabla v^\varepsilon$  converges two-scale weakly, from the definition of the weak two-scale convergence we obtain

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon dx = a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v(x, y) \varphi(x) \nabla_y b(y) dy dx.\tag{6.14}$$

Let us show that convergence property (5.3) holds for  $u^\varepsilon$ . To this end we substitute into (6.12) a test function of the form  $\varepsilon w(\varepsilon^{-1}x)\varphi(x)$ ,  $\varphi \in C_0^\infty(\Omega_1)$ ,  $w \in C_{\text{per}}^\infty(\square)$ , cf. [23]. Then all the terms except, possibly,

$$\int_{\Omega_1^\varepsilon} a_1 \nabla u^\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x) \varphi(x) dx$$

converge to zero. As a result, the above term also converges to zero. We then apply Lemma 5.6 for  $u_\varepsilon$  replaced by  $\tilde{u}^\varepsilon$ . Since  $\tilde{u}^\varepsilon$  coincides with  $u^\varepsilon$  on  $\Omega_1^\varepsilon$ , by Lemma 5.7 applied to the second term on the left hand side of (6.12) with  $\psi^\varepsilon$  as in (6.13) we obtain

$$\lim_{\varepsilon \rightarrow 0} a_1 \int_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx = \lim_{\varepsilon \rightarrow 0} a_1 \int_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_0 dx = \int_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla \psi_0 dx.\tag{6.15}$$

For small enough  $\varepsilon$  the function  $\psi^\varepsilon$  is equal to  $\psi_0$  in  $\tilde{\Omega}_0^\varepsilon$ , so  $\nabla \psi^\varepsilon$  is bounded in  $\tilde{\Omega}_0^\varepsilon$ . Since  $\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 |\nabla u^\varepsilon|^2 dx$  is bounded uniformly in  $\varepsilon$  and  $|\tilde{\Omega}_0^\varepsilon| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we have

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla u^\varepsilon \nabla \psi^\varepsilon dx \right| \leq C \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 |\nabla u^\varepsilon| dx \leq C |\tilde{\Omega}_0^\varepsilon|^{1/2} \tilde{a}_0^{1/2} \left( \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 |\nabla u^\varepsilon|^2 dx \right)^{1/2} \rightarrow 0.\tag{6.16}$$

The function  $u^\varepsilon$  coincides with  $\tilde{u}^\varepsilon$  on  $\Omega_2^\varepsilon$ . Then, via (6.2) we have convergence of the last term on the left hand side of (6.12):

$$\lim_{\varepsilon \rightarrow 0} a_2 \int_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon dx = \lim_{\varepsilon \rightarrow 0} \left[ a_2 \int_{\Omega_2} \nabla \tilde{u}^\varepsilon \cdot \nabla \psi_0 dx - a_2 \int_{\tilde{\Omega}_0^\varepsilon \cap \Omega_2} \nabla \tilde{u}^\varepsilon \cdot \nabla \psi_0 dx \right] = a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 dx.\tag{6.17}$$

Thus, passing to the limit as  $\varepsilon \rightarrow 0$  on the left hand side of (6.12) via (6.14)–(6.17), and on the right hand side via (6.11), we arrive at

$$a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v \cdot \varphi \nabla_y b \, dy \, dx + \int_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla \psi_0 \, dx + a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx = \lambda_0 \int_{\mathbb{R}^n} \int_Q (u_0 + v)(\psi_0 + \varphi b) \, dy \, dx.$$

Since the space of functions from (6.13) is dense in  $\mathcal{V}$  (see (2.8)), the latter is equivalent to (2.11). It follows from (6.11), Proposition 5.3 (ii) and the normalization of  $u^\varepsilon$  that  $u^0(x, y) \not\equiv 0$ . Thus we have proved that  $\lambda_0$  and  $u^0(x, y)$  are respectively an eigenvalue and an eigenfunction of the operator  $A_0$ , completing the proof of the theorem.  $\square$

**Remark 6.5.** Theorem 6.1 combined with [13, Theorem 2] implies the existence of eigenvalues of  $A_0$  in the gaps of its essential spectrum, provided  $\Omega_2$  is large enough and/or  $a_2$  is small enough.

**Remark 6.6.** It is not hard to show that there holds the strong two-scale resolvent convergence  $A_\varepsilon \xrightarrow{2} A_0$ , see Definition 5.8. Namely, considering the resolvent equation

$$A_\varepsilon w^\varepsilon + w^\varepsilon = f^\varepsilon,$$

where  $f^\varepsilon \xrightarrow{2} f^0$ , and employing essentially the same arguments as above (cf. also [23, Theorem 5.1]), one can pass to the limit as  $\varepsilon \rightarrow 0$  in the weak form of the resolvent equation choosing appropriate test functions, cf. (6.12)–(6.17), to obtain that  $w^\varepsilon \xrightarrow{2} w^0$ , with

$$A_0 w^0 + w^0 = f^0.$$

Further, arguing as in [23, §4.3], cf. also proof of Lemma 6.2 above, one can show that the above weak two-scale convergence implies the strong one, i.e.  $w^\varepsilon \xrightarrow{2} w^0$  as long as  $f^\varepsilon \xrightarrow{2} f^0$ , which means the strong two-scale resolvent convergence by the definition. The latter implies in particular the strong two-scale convergence of spectral projectors ( $P_\varepsilon(\lambda) \xrightarrow{2} P_0(\lambda)$  if  $\lambda$  is not an eigenvalue of  $A_0$ ), see [21, 24], and has other nice properties, however it does not imply in its own the convergence of the spectra. The latter requires an additional (two-scale) compactness property to hold, which Theorem 6.1 provides.

**Remark 6.7.** The function  $v(x, y)$  could be represented as a product of  $u_0(x)|_{\Omega_1}$  and  $\lambda_0 b(y)$ , where  $b(y)$  solves (2.13) with  $\lambda = \lambda_0$ . Then  $v(x, \varepsilon^{-1}x)$  strongly two-scale converges to  $v(x, y)$  by the mean value property and the properties of two-scale convergence. Then

$$u^{\text{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \mathbb{R}^n \setminus \Omega_0^\varepsilon, \end{cases} \quad (6.18)$$

also strongly two-scale converges to  $u^0(x, y)$ . Hence it approximates the eigenfunction  $u^\varepsilon(x)$ :

$$\|u^{\text{appr}} - u^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \rightarrow 0. \quad (6.19)$$

Now, using the result of Theorem 6.1 we will discuss the multiplicity properties of the eigenvalues  $\lambda_\varepsilon$  and  $\lambda_0$ . Let us assume that the multiplicity of the eigenvalue  $\lambda_0$  of  $A_0$  is  $m$ . Suppose that for a subsequence  $\varepsilon_k \rightarrow 0$  there exist  $l$  (accounting for multiplicities) eigenvalues of  $A_\varepsilon$ ,  $\lambda_{\varepsilon_k, 1} \leq \lambda_{\varepsilon_k, 2} \leq \dots \leq \lambda_{\varepsilon_k, l}$ , such that  $\lambda_{\varepsilon_k, i} \rightarrow \lambda_0$ ,  $i = 1, \dots, l$ . Let  $u_i^{\varepsilon_k}$  be the corresponding eigenfunctions orthonormalised in  $L^2(\mathbb{R}^n)$ . It follows from Theorem 6.1 that there exists a subsequence  $k_m$  such that

$$u_i^{\varepsilon_{k_m}} \xrightarrow{2} u_i^0, \quad i = 1, \dots, l,$$

where  $u_i^0$  are eigenfunctions of  $A_0$  corresponding to  $\lambda_0$ . In particular, due to the strong two-scale convergence, we have convergence of the inner products as a consequence of the convergence of norms:

$$(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} \rightarrow (u_i^0, u_j^0)_{\mathcal{H}_0}.$$

However  $(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} = \delta_{ij}$ . Then  $u_i^0$ ,  $i = 1, \dots, l$  are also orthonormal (in  $\mathcal{H}_0$ ), i.e. there exist at least  $l$  linearly independent eigenfunctions of  $A_0$  corresponding to  $\lambda_0$ . Thus,  $l \leq m$ .

The results presented in [16] remain also valid for the setting of the problem in the present paper, i.e. when the coefficients of the divergence form operator  $A_\varepsilon$  are of the form (2.4). By Theorem 4.1 of [16], if  $\lambda_0$  is an eigenvalue of the limit operator  $A_0$  lying in a gap of its essential spectrum, then for small enough  $\varepsilon$ , there exist eigenvalues (or at least one eigenvalue) of  $A_\varepsilon$  such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2}, \quad i = 1, \dots, l(\varepsilon).$$

Moreover, again by [16, Thm 4.1], for any eigenfunction  $u_i^0$  of  $A_0$  corresponding to  $\lambda_0$  the related  $u_i^{\text{appr}}$ , see (6.18), can be approximated by a linear combination of the eigenfunctions of  $A_\varepsilon$  corresponding to  $\lambda_{\varepsilon,i}$ ,  $i = 1, \dots, l(\varepsilon)$ . Since, by the above,  $(u_i^{\text{appr}}, u_j^{\text{appr}})_{L^2(\mathbb{R}^n)} \rightarrow \delta_{ij}$ , as  $\varepsilon \rightarrow 0$ ,  $i, j = 1, \dots, m$ , it is not hard to show that  $l(\varepsilon) \geq m$ . Hence we conclude that there exist exactly  $m$  eigenvalues (counted with their multiplicities) of  $A_\varepsilon$  such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2}, \quad i = 1, \dots, m,$$

where  $m$  is a multiplicity of  $\lambda_0$ . In other words there is an ‘‘asymptotic one-to-one correspondence’’ between isolated eigenvalues and eigenfunctions of the operators  $A_\varepsilon$  and  $A_0$ .

## 7 Identity of the essential spectra of $\widehat{A}_0$ and $A_0$ , convergence of the spectra of $A_\varepsilon$ in the sense of Hausdorff

By definition, the Hausdorff convergence of spectra,  $\sigma(A_\varepsilon) \xrightarrow{H} \sigma(A_0)$  as  $\varepsilon \rightarrow 0$ , means that

- for all  $\lambda \in \sigma(A_0)$  there are  $\lambda_\varepsilon \in \sigma(A_\varepsilon)$  such that  $\lambda_\varepsilon \rightarrow \lambda$ ;
- if  $\lambda_\varepsilon \in \sigma(A_\varepsilon)$  and  $\lambda_\varepsilon \rightarrow \lambda$ , then  $\lambda \in \sigma(A_0)$ .

We remind that  $\widehat{A}_\varepsilon$  and  $\widehat{A}_0$  denote the ‘unperturbed’ operators corresponding to  $A_\varepsilon$  and  $A_0$ , see Section 2. It was shown in [24] that  $\sigma(\widehat{A}_\varepsilon) \xrightarrow{H} \sigma(\widehat{A}_0)$  (the spectra of both  $\widehat{A}_\varepsilon$  and  $\widehat{A}_0$  are purely essential). In [13] it is proved that the essential spectrum of a divergence form operator  $-\nabla \cdot a(x)\nabla$  (where  $a(x) \geq \delta > 0$  is a scalar function) remains unperturbed with respect to the local perturbation of the coefficient  $a(x)$ . Applying this assertion to the operator  $\widehat{A}_\varepsilon$  and its perturbation  $A_\varepsilon$  we conclude that  $\sigma(\widehat{A}_\varepsilon) = \sigma_{\text{ess}}(A_\varepsilon) \xrightarrow{H} \sigma(\widehat{A}_0)$ . Let us assume that  $\sigma(\widehat{A}_0) = \sigma_{\text{ess}}(A_0)$ . Then  $\sigma_{\text{ess}}(A_\varepsilon) \xrightarrow{H} \sigma_{\text{ess}}(A_0)$ . In this case Theorem 6.1 together with the results of [16] imply the convergence of the discrete spectra in the gaps ( $\sigma_{\text{disc}}(A_\varepsilon) \xrightarrow{H} \sigma_{\text{disc}}(A_0)$ ) and, consequently, we would have  $\sigma(A_\varepsilon) \xrightarrow{H} \sigma(A_0)$ . However, we cannot apply the result of [13] as it is stated to the case of the two-scale operators  $\widehat{A}_0$  and  $A_0$ . In this section we prove the stability of the essential spectrum of  $\widehat{A}_0$  with respect to the local perturbation of its coefficients, establishing thereby the missing part of the reasoning. We do this by direct means using the Weyl’s criterium for the essential spectrum of an operator, see e.g. [7].

**Theorem 7.1.** *The essential spectra of the operators  $\widehat{A}_0$  and  $A_0$  coincide.*

*Proof. Step 1.* First we describe the domains of  $\widehat{A}_0$  and  $A_0$ . According to the Friedrichs extension procedure, see e.g. [21], a function  $u$  belongs to  $\mathcal{D}(A_0)$  if and only if  $u = u_0(x) + v(x, y) \in \mathcal{V}$  and there exists  $h \in \mathcal{H}_0$  such that

$$B_0(u, w) = (h, w)_{\mathcal{H}_0}$$

for all  $w \in \mathcal{V}$ , see (2.7)–(2.9). If  $u = u_0 + v \in \mathcal{D}(A_0)$  then  $u_0, v \in \mathcal{D}(A_0)$ . Due to the regularity properties of solutions of elliptic equations,  $u_0 \in H_{\text{loc}}^2$  everywhere away from the boundary of  $\Omega_2$ .

Operator  $\widehat{A}_0$  acting in the Hilbert space  $\widehat{\mathcal{H}}_0$  was described in [24] and is generated by a (closed) symmetric and bounded from below bilinear form  $\widehat{B}_0(u, w)$  on a dense subspace  $\widehat{\mathcal{V}}$  of  $\widehat{\mathcal{H}}_0$ , where  $\widehat{\mathcal{H}}_0$ ,  $\widehat{\mathcal{V}}$  and  $\widehat{B}_0(u, w)$  are defined by (2.7)–(2.9) with  $\Omega_2 = \emptyset$  and  $\Omega_1 = \mathbb{R}^n$ . A function  $u$  belongs to domain  $\mathcal{D}(\widehat{A}_0)$  if and only if  $u = u_0(x) + v(x, y) \in \widehat{\mathcal{V}}$  and there exists  $h \in \widehat{\mathcal{H}}_0$  such that

$$\widehat{B}_0(u, w) = (h, w)_{\widehat{\mathcal{H}}_0}$$

for all  $w \in \widehat{\mathcal{V}}$ . If  $u = u_0 + v \in \mathcal{D}(\widehat{A}_0)$  then  $u_0, v \in \mathcal{D}(\widehat{A}_0)$ ,  $u_0 \in H^2(\mathbb{R}^n)$ .

Let  $A$  be a self-adjoint operator with domain  $\mathcal{D}(A)$  acting in a Hilbert space  $H$ . By the Weyl's criterium, see e.g. [7], condition  $\lambda \in \sigma_{\text{ess}}(A)$  is equivalent to the existence of a singular sequence  $u^{(k)} \in \mathcal{D}(A)$ , i.e. such that

$$0 < C_1 \leq \|u^{(k)}\|_H \leq C_2, \quad (7.1)$$

$$u^{(k)} \rightharpoonup 0 \text{ weakly in } H, \quad (7.2)$$

$$(A - \lambda)u^{(k)} \rightarrow 0 \text{ strongly in } H. \quad (7.3)$$

**Step 2.** Let  $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$  and  $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x, y)$  be the corresponding singular sequence in  $\mathcal{D}(\widehat{A}_0) \subset \widehat{\mathcal{H}}_0$ . We want to construct on its basis a singular sequence for the operator  $A_0$ , i.e. in  $\mathcal{D}(A_0) \subset \mathcal{H}_0$  and satisfying properties (7.1)–(7.3). First notice that the gradient of  $u_0^{(k)}$  is bounded in  $L^2(\mathbb{R}^n)$ . Indeed, from (2.9) and (7.3) we have

$$\|\nabla u_0^{(k)}\|_{L^2(\mathbb{R}^n)}^2 \leq C\widehat{B}_0(u^{(k)}, u^{(k)}) = C\lambda(u^{(k)}, u^{(k)})_{\widehat{\mathcal{H}}_0} + o(1) \leq C. \quad (7.4)$$

Let us define a cut-off function

$$\eta_{k,R}(x) = \eta\left(\frac{1}{k}(|x| - R)\right),$$

where  $\eta \in C^2(\mathbb{R})$  is such that

$$\eta(t) = \begin{cases} 1, & t \leq 0, \\ 0, & t \geq 1. \end{cases}$$

Consider the following sequence,  $u^{(k)}\eta_{k,R_k} \in \mathcal{D}(\widehat{A}_0)$ , where  $R_k$  is chosen large enough so that  $\|u^{(k)}(1 - \eta_{k,R_k})\|_{\widehat{\mathcal{H}}_0} \leq \frac{1}{k}$ . This sequence obviously satisfies (7.1) regarding the operator  $\widehat{A}_0$ .

Let us check property (7.3). The operator  $\widehat{A}_0$  acts on a function  $u \in H^2(\mathbb{R}^n) \subset \widehat{\mathcal{H}}_0$  as follows<sup>7</sup>, cf. [24]. Let

$$-\nabla \cdot A^{\text{hom}}\nabla u(x) = f(x) \in L^2(\mathbb{R}^n).$$

Then, by the definition of  $\widehat{A}_0$ , we have

$$\widehat{A}_0 u(x) = |Q_1|^{-1}\theta_{Q_1}(y)f(x) \in \widehat{\mathcal{H}}_0.$$

Note that

$$\|\widehat{A}_0 u\|_{\widehat{\mathcal{H}}_0} = |Q_1|^{-1/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

<sup>7</sup>If  $u = u_0(x) + v(x, y)$  then  $\widehat{A}_0 u = h \in \widehat{\mathcal{H}}_0$  implies  $-\nabla \cdot A^{\text{hom}}\nabla u_0 = \langle h \rangle_y$  and  $-a_0\Delta_y v = h(x, y)$ ,  $y \in Q_0$ .

For  $u^{(k)}\eta_{k,R_k}$  we derive

$$\widehat{A}_0 \left( u^{(k)}\eta_{k,R_k} \right) = \eta_{k,R_k} \widehat{A}_0 u^{(k)} - |Q_1|^{-1} \theta_{Q_1}(y) \left( 2\nabla\eta_{k,R_k} \cdot A^{\text{hom}}\nabla u_0^{(k)} + u_0^{(k)}\nabla \cdot A^{\text{hom}}\nabla\eta_{k,R_k} \right).$$

Thus we arrive at

$$\begin{aligned} \left\| (\widehat{A}_0 - \lambda)(u^{(k)}\eta_{k,R_k}) \right\|_{\widehat{\mathcal{H}}_0} &\leq \left\| \eta_{k,R_k}(\widehat{A}_0 - \lambda)u^{(k)} \right\|_{\widehat{\mathcal{H}}_0} + |Q_1|^{-1/2} \left( 2 \left\| \nabla\eta_{k,R_k} \cdot A^{\text{hom}}\nabla u_0^{(k)} \right\|_{L^2(\mathbb{R}^n)} + \right. \\ &\quad \left. + \left\| u_0^{(k)}\nabla \cdot A^{\text{hom}}\nabla\eta_{k,R_k} \right\|_{L^2(\mathbb{R}^n)} \right) = o(1) + \frac{1}{k}O \left( \left\| \nabla u_0^{(k)} \right\|_{L^2(\mathbb{R}^n)} \right) + \frac{1}{k^2}O \left( \left\| u_0^{(k)} \right\|_{L^2(\mathbb{R}^n)} \right). \end{aligned} \quad (7.5)$$

Due to (7.1) and (7.4) the latter converges to 0 as  $k \rightarrow \infty$ . Hence (7.3) holds regarding  $\widehat{A}_0$ .

Now notice that if  $\text{supp } u \cap \overline{\Omega}_2 = \emptyset$ , then  $u \in \mathcal{D}(\widehat{A}_0)$  if and only if  $u \in \mathcal{D}(A_0)$ ; besides  $\widehat{A}_0 u = A_0 u$ . We hence next shift the supports of the elements of the sequence away from  $\Omega_2$  ensuring also that the new sequence is weakly convergent to maintain (7.2). Since  $\text{supp } \eta_{k,R_k}$  is a closed ball of radius  $R_k + k$  centered at the origin, the shift of  $x$  by  $\xi_k := (R_k + 2k + \text{diam}(\Omega_2))\xi$  for every  $k$ , where  $\xi$  is an arbitrary unit vector from  $\mathbb{R}^n$ , will do the job. Hence, for the given  $\lambda$  we have constructed a singular sequence

$$w^{(k)}(x, y) = u^{(k)}(x + \xi_k, y) \eta_{k,R_k}(x + \xi_k),$$

satisfying all the properties (7.1)–(7.3) for the operator  $A_0$ . Namely, the translational invariance of  $\widehat{A}_0$  in  $x$  ensures that (7.1) and (7.3) are satisfied. Finally, (7.2) follows from the pointwise convergence of  $w^{(k)}$  to zero as  $k \rightarrow \infty$  (since for any fixed  $x$ ,  $w^{(k)}(x, y) = 0$  for large enough  $k$ ). Thus  $\lambda \in \sigma_{\text{ess}}(A_0)$ .

**Step 3.** Suppose now that  $\lambda \in \sigma_{\text{ess}}(A_0)$  and  $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x, y)$  is the corresponding singular sequence. Let  $R$  be such that  $\overline{\Omega}_2 \subset B_R$ . There are only two alternative possibilities<sup>8</sup>:

- There exists a sequence  $\delta_i \rightarrow 0$  such that for any  $i \in \mathbb{N}$

$$\|u^{(k)}(1 - \theta_{B_{R+i}})\|_{\mathcal{H}_0} \leq \delta_i \quad (7.6)$$

for all  $k$ .

- There exist a constant  $M > 0$  and subsequences  $k(j) \rightarrow \infty$ ,  $i(j) \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$\|u^{(k(j))}(1 - \theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \geq M \quad (7.7)$$

for all  $j$ .

Let (7.6) take place. The sequence  $\nabla u_0^{(k)}$  is bounded in  $L^2(\mathbb{R}^n)$ , cf. (7.4). From (7.6) and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|f\|_{\mathcal{H}_0}, \text{ for all } f \in L^2(\mathbb{R}^n) \subset \mathcal{H}_0, \quad (7.8)$$

it follows that

$$u_0^{(k)} \rightarrow u(x) \text{ in } L^2(\mathbb{R}^n),$$

up to a subsequence. The reasoning leading to this assertion is essentially identical to the one in (6.1)–(6.4) and is not reproduced. From (7.2) and the latter we conclude that

$$v^{(k)}(x, y) \rightharpoonup -u(x) \text{ weakly in } \mathcal{H}_0.$$

<sup>8</sup>Let  $A_{ki} := \|u^{(k)}(1 - \theta_{B_{R+i}})\|_{\mathcal{H}_0}$  and let  $\delta_i := \sup_k A_{ki}$ . Then either  $\delta_i \rightarrow 0$  giving (7.6) or  $\delta_i \not\rightarrow 0$  yielding (7.7).

Hence, on one hand, we have

$$\left(u, v^{(k)}\right)_{\mathcal{H}_0} \rightarrow -(u, u)_{\mathcal{H}_0} = - \int_{\mathbb{R}^n} u^2 dx.$$

On the other hand,

$$\left(u, v^{(k)}\right)_{\mathcal{H}_0} = \int_{\mathbb{R}^n} \int_{Q_0} u v^{(k)} dy dx = \left(u \theta_{Q_0}(y), v^{(k)}\right)_{\mathcal{H}_0} \rightarrow -(u \theta_{Q_0}(y), u)_{\mathcal{H}_0} = -|Q_0| \int_{\mathbb{R}^n} u^2 dx.$$

Comparing the last two formulas, conclude that at  $u \equiv 0$ , i.e.

$$u_0^{(k)} \rightarrow 0 \text{ in } L^2(\mathbb{R}^n). \quad (7.9)$$

Denote  $A_0 u^{(k)}$  by  $g^{(k)}(x, y) = g_0^{(k)}(x) + h^{(k)}(x, y) \in \mathcal{H}_0$ . From (7.3) and (7.8) we get the following convergence:

$$\begin{aligned} \|g_0^{(k)} - \lambda u_0^{(k)}\|_{L^2(\mathbb{R}^n)} &\rightarrow 0, \\ \|h^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0} &\rightarrow 0. \end{aligned} \quad (7.10)$$

Then from (7.9) we have

$$g_0^{(k)} \rightarrow 0 \text{ in } L^2(\mathbb{R}^n). \quad (7.11)$$

Analogously to [23] we define a self-adjoint operator  $A_y$  acting in  $L^2(\Omega_1 \times Q_0)$  by

$$A_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\Omega_1 \times Q_0).$$

The domain of the operator,  $\mathcal{D}(A_y) \subset L^2(\Omega_1, H_0^1(Q_0))$ , is the set of all the solution of this equation. Similarly we define a self-adjoint operator  $\widehat{A}_y$  acting in  $L^2(\mathbb{R}^n \times Q_0)$ , which corresponds to the defect-free setting,

$$\widehat{A}_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\mathbb{R}^n \times Q_0).$$

One can easily check the following properties:  $\mathcal{D}(A_y) \subset \mathcal{D}(A_0)$ ,  $\mathcal{D}(\widehat{A}_y) \subset \mathcal{D}(\widehat{A}_0)$ ,  $\sigma(A_y) \subset \sigma_{\text{ess}}(A_0)$  and, in particular,

$$\sigma(\widehat{A}_y) \subset \sigma_{\text{ess}}(\widehat{A}_0), \quad \sigma(A_y) = \sigma(\widehat{A}_y). \quad (7.12)$$

It is not difficult to see (by analyzing (2.9), see also [23]) that

$$A_y v^{(k)} = g_0^{(k)} \theta_{\Omega_1}(x) \theta_{Q_0}(y) + h^{(k)}. \quad (7.13)$$

(Note that  $A_y v^{(k)} \neq A_0 v^{(k)}$ .)

Combining (7.10), (7.11) and (7.13) we arrive at

$$\|(A_y - \lambda)v^{(k)}\|_{L^2(\Omega_1 \times Q_0)} = \|g_0^{(k)} \theta_{Q_0}(y) + h^{(k)} - \lambda v^{(k)}\|_{L^2(\Omega_1 \times Q_0)} \rightarrow 0.$$

This implies that  $\lambda$  belongs to the spectrum of  $A_y$  (notice that (7.1) holds for  $v^{(k)}$  via (7.9)). Hence  $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$ , see (7.12).

Now let (7.7) hold. Consider a sequence  $w^{(j)} = u^{(k(j))}(1 - \eta_{i(j), R}) \in \mathcal{D}(\widehat{A}_0)$  (we remind that  $R$  is large enough to ensure  $\Omega_2 \subset\subset B_R$ ). Then

$$\|w^{(j)}\|_{\widehat{\mathcal{H}}_0} \geq \|u^{(k(j))}(1 - \theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \geq M,$$

i.e. (7.1) is satisfied for  $w^{(j)}$ . Since the sequence  $1 - \eta_{i(j), R}$  tends to 0 pointwise, (7.2) is valid. Analogously to (7.5) we derive

$$\|(\widehat{A}_0 - \lambda)w^{(j)}\|_{\widehat{\mathcal{H}}_0} = \|(A_0 - \lambda)w^{(j)}\|_{\mathcal{H}_0} \rightarrow 0, \quad (7.14)$$

yielding (7.3). Thus, we conclude that  $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$ , completing the proof of the theorem.  $\square$

**Remark 7.2.** Theorem 7.1 combined with [24] implies that  $\sigma_{\text{ess}}(A_0) = \{\lambda : \beta(\lambda) \geq 0\} \cup \sigma(A_y)$ . Using the methods of [24] it is not hard to show further that  $\sigma_{\text{ess}}(A_0)$  contains no point spectrum (in particular, no embedded eigenvalues) except if  $\lambda$  is an eigenvalue of  $A_y$  corresponding to an eigenfunction with zero mean. It is natural to conjecture (cf. [24]) that, outside these eigenvalues, the spectrum is absolutely continuous and the “eigenfunctions of the continuous spectrum” are  $u(x, y, \lambda) = u_0(x, \lambda)(1 + \lambda b(y, \lambda))$ , where  $u_0(x, \lambda)$  are solutions of the appropriate scattering problems:

$$\begin{aligned} \nabla \cdot A^{\text{hom}} \nabla u_0 + \beta(\lambda) u_0 &= 0, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}_2, \\ a_2 \Delta u_0 + \lambda u_0 &= 0, \quad x \in \Omega_2 \end{aligned} \tag{7.15}$$

with the appropriate matching condition at  $\partial\Omega_2$  and radiation condition at infinity. A detailed study of this as well as of the convergence of the related generalised eigenfunctions (cf. [24] for the defect-free case) is beyond the scope of the present paper.

Summarizing the main results of the present paper we conclude that Theorems 6.1 and 7.1 together with the results of [13, 16] (see the discussions at the end of Section 6 and in the beginning of the present section) establish the validity of Theorem 2.1.

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