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Damped Trend forecasting and the Order-Up-To replenishment policy

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Abstract

We develop a *z*-transform transfer function model of the Damped Trend forecasting mechanism from which we determine its stability boundary. We show that the Damped Trend forecasting mechanism is stable for a much larger proportion of the parametrical space than is currently acknowledged in the literature. We incorporate the Damped Trend forecasting mechanism into an Order-Up-To (OUT) replenishment policy and investigate the frequency response of this system. We prove that Naïve, Exponential Smoothing and Holts forecasts, when used within the OUT policy, will always generate bullwhip, for every possible demand process, for any lead-time. However, the Damped Trend forecasting mechanism, when used within the OUT policy, behaves differently. Sometimes it will generate bullwhip and sometimes it will not. Bullwhip avoidance occurs when demand is dominated by low frequencies in some instances. In other instances bullwhip avoidance happens at high frequencies. We are also able to demonstrate a complex odd-even lead-time effect exists. Bullwhip may be avoided when the lead-time is odd for a particular demand pattern, but re-appears when the lead-time changes to an even number.

Keywords: Damped Trend, Forecasting, Order-Up-To Replenishment Policy, Bullwhip, Stability.

1. Introduction

The Damped Trend (DT) forecasting method developed by [1] has often been promoted as the most accurate forecasting technique in the so-called M-competitions [2]. [3] find the DT method is the best method for 84% of the 3003 time series in the M3 forecasting competition when using local initial values. It was the best method 70% of the time when using global initial values. [4] also conclude that DT forecasting can "reasonably claim to be a benchmark forecasting method for all others to beat". The great virtual of DT is that future forecasts are not simply flat line extensions of the current, next period forecast. It is able to detect and forecast trends and future forecasts change with these trends. The DT forecasting methodology also contains at least eleven different forecasting methods when all parameters are selected from the real $[0,1]$ interval, [3]. This makes it a powerful and very general forecasting approach as tuning the DT parameters effectively automates model selection.

The frequency response approach that we take is particularly powerful as we are able to generate results that are applicable for any demand processes. This is because all demand processes can be decomposed into a set of harmonic frequencies via the Fourier Transform. By understanding how the system reacts to the complete set of harmonic frequencies (via the Amplitude Ratio within the frequency response graph) we are than able to gain insights and draw conclusions that are valid for all possible demand pattern. Many of the results that we obtain are also valid for any lead-time. Our findings strengthen, sharpen and refine the arguments of [6].

In this paper we derive a discrete-time transfer function of the DT mechanism in section 2. In section 3 we identify the stability boundaries of DT forecasting mechanism via Jury's Inners Approach [5]. Section 4 describes the replenishment policy used. We incorporate the DT forecasting methodology into the Order-Up-To (OUT) replenishment policy, develop a discrete-time *z*-transform transfer function representation of the combined forecasting and replenishment system and analyze its frequency response plot. Section 5 provides summary numerical results, confirming our theoretical findings. Section 6 concludes.

2. The Damped Trend forecasting method

DT forecasts [1] are generated by

$$\left.\begin{aligned} \hat{a}_t &= (1-\alpha)\left(\hat{a}_{t-1} + \phi \hat{b}_{t-1}\right) + \alpha d_t \\ \hat{b}_t &= (1-\beta)\phi \hat{b}_{t-1} + \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) \\ \hat{d}_{t,t+k} &= \hat{a}_t + \hat{b}_t \sum_{i=1}^{k} \phi^i \end{aligned}\right\}. \quad (1)$$

Here d_t is the time series being forecasted. a_t is the current estimate of the level, exponentially smoothed by the constant α. b_t is the current estimate of the trend, exponentially smoothed by the constant β. b_0 is the initial value of the trend, assumed to be zero, $b_0 = 0$. ϕ is the damping parameter that can be interpreted as a measure of the persistence of the trend. k is the number of periods ahead that the forecast is required to predict. $\hat{d}_{t,t+k}$, is the forecast, made at time *t* of demand in the period $t+k$.

Several well-known forecasting approaches are encapsulated within the DT model. These include Holts method where there is no damping of the trend component when $\phi = 1$, Simple Exponential Smoothing (SES) when $\beta = 0$ and Naïve forecasting when $\alpha = 1$ and $\beta = 0$, see Table 1.

Forecasting method	**Parameter settings**	**Notes**
Holts method	$\phi = 1$	By setting $\phi = 1$, $\hat{d}_{t,t+k} = \hat{a}_t + k\hat{b}_t$ results. The future forecasts then becomes a linear extrapolation of the current estimate of the trend.
Simple exponential smoothing, SES	$\beta = 0$	$\beta = 0$ implies that $b_t = 0 \ \forall t \geq 0$. It then follows that $\hat{a}_t = (1-\alpha)\hat{a}_{t-1} + \alpha d_t$. This in turn means $\hat{d}_{t,t+k} = \hat{a}_t$. Here we have made explicit the fact that the SES forecast of all future forecasts (k periods ahead) is simply the forecast of the next periods demand. Ignoring the subscript that gives information on which period we are forecasting yields the common SES formula, $\hat{d}_t = (1-\alpha)\hat{d}_{t-1} + \alpha d_t$.
Naïve forecasting	$\beta = 0$, $\alpha = 1$	This is easy to see from the exponential smoothing formula as these parameters yields $\hat{d}_{t,t+k} = d_t$.

Table 1. Three popular forecasting methods encapsulated with the Damped Trend method

Transfer functions are useful tools for studying linear systems, as they allow convolution in the time domain to be replaced by simply algebra in the complex frequency domain. In the frequency domain there is also a wide range of tools developed by control engineering theorists for understanding the dynamic behaviour of such systems. It is a relatively simple task to develop a block diagram of (1) and manipulate it to obtain the transfer function of the DT forecasting mechanism (Figure 1). We refer interested readers to [7] for information on how to achieve this.

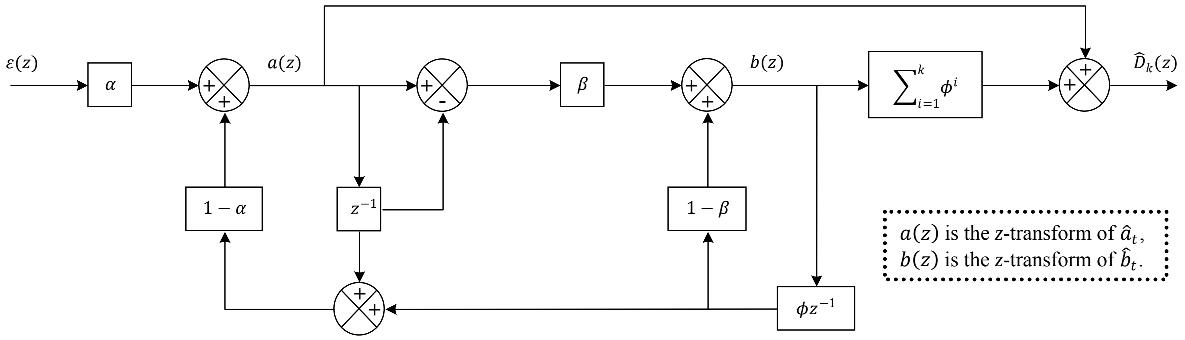

Figure 1. Block diagram of Damped Trend forecasting

The discrete time transfer function of (1) is given by

$$\frac{\hat{D}_k(z)}{\varepsilon(z)}=\frac{z\alpha\left(\phi\left(1+\phi\left(\beta-1\right)\right)-\beta\phi^{k+1}\right)+z^2\alpha\left(\beta\phi^{k+1}-1-\phi\left(\beta-1\right)\right)}{\left(\phi-1\right)\left(1-\alpha\right)\phi+z\left(\phi-1\right)\left(\alpha-\phi-1+\alpha\beta\phi\right)+z^2\left(\phi-1\right)}, \tag{2}$$

which, is in standard form as coefficients of the *z*-transform operator, $z=\sum_{n=0}^{\infty}x\left(n\right)z^{-n}$. We note from (2) that the co-efficient of the highest power of *z* in the denominator is positive when $\phi>1$ and negative when $\phi<1$. This is important as Jury's Inners approach, which will be exploited in the next section, requires this to be positive [5].

3. Stability of Damped Trend forecasts via Jury's Inners Approach

The question of stability is a fundamental aspect of dynamic systems. A stable system will react to a finite input and return to steady state conditions in a finite time. An unstable system will either diverge exponentially to positive or negative infinity or oscillate with ever increasing amplitude. A critically stable system will fall into a limit cycle of constant amplitude to any finite input. Oscillations in the forecasts and order rates in supply chains are costly. So, as a first step to dynamically designing a supply chain replenishment rule, we must ensure that a replenishment rule and all the components (such as the forecasting system) are stable. [5] shows that the necessary and sufficient conditions for stability of a linear discrete system are given by: $A(1)>0$, $(-1)^n A(-1)>0$, and the matrices $\Delta_{n-1}^{\pm}=\mathrm{X}_{n-1}\pm\mathrm{Y}_{n-1}$ are positive innerwise.

For the system we are studying here, $A(z)$ is the denominator of (2), and the co-efficient of the highest power of *z* – in this case a_2, see (3) – must be positive. This can be easily achieved by multiplying both numerator and denominator by -1. However, it is interesting to note that no matter whether we need to do the multiplication or not, the values of the coefficients in the denominator $A(z)$ always remain the same. Therefore, before using Jury's approach, we rewrite the denominator of (2) by substituting $\phi-1$ with its absolute value $\left|\phi-1\right|$ to simplify future analysis. Then, $A(z)$ can be expressed as

$$A(z)=\underbrace{\left|\phi-1\right|(1-\alpha)\phi}_{a_0}+z\underbrace{\left|\phi-1\right|(\alpha-\phi-1+\alpha\beta\phi)}_{a_1}+z^2\underbrace{\left|\phi-1\right|}_{a_2}, \tag{3}$$

and $\Delta_{n-1}^{\pm}$ are simply scalars as the original transfer function (2) is only of second order ($n=2$). Specifically $\Delta_{n-1}^{\pm}$ are

$$\Delta^{+}_{n-1} = a_2 + a_0 = |\phi - 1| + |\phi - 1|(1-\alpha)\phi\,,$$
$$\Delta^{-}_{n-1} = a_2 - a_0 = |\phi - 1| - |\phi - 1|(1-\alpha)\phi\,. \tag{4}$$

Taking each criteria in turn:

- $A(1)$ must be greater than zero: $A(1) = A(z)\big|_{z\to 1}$, that is $A(1)$ is given by (3) with the z is replaced with 1,

$$A(1) = |\phi - 1|\alpha\left(1 + \phi(\beta - 1)\right) > 0\,. \tag{5}$$

  (5) splits the $\{\alpha,\beta\}$ parametrical plane into quarters along the lines given by $\alpha = 0$ and $\beta = (\phi - 1)/\phi$.

- $(-1)^n A(-1) > 0$ must be greater than zero. In the same manner as above, $(-1)^n A(-1)$ is given by (3) with the z is replaced by -1 and $n = 2$,

$$(-1)^n A(-1) = |\phi - 1|\left(\alpha - 2 + \phi\left(\alpha - 2 + \alpha\beta\right)\right) > 0 \tag{6}$$

  (6) divides the $\{\alpha,\beta\}$ parametrical plane along the curve $\beta = (2 - \alpha + 2\phi - \alpha\phi)/\alpha\phi$, which has an asymptote at $\alpha = 0$.

- $\Delta^{\pm}_{n-1} = \mathrm{X}_{n-1} \pm \mathrm{Y}_{n-1}$ must be positive innerwise. A matrix is positive innerwise if its determinant is positive and all the determinants of its Inners are also positive. Because the order of the transfer function $n = 2$, then the $\Delta^{\pm}_{n-1}$ matrices only contain one element [8]. To ensure that the elements are positive innerwise, it is enough that

$$\left.\begin{aligned}\Delta^{+}_{n-1} = a_2 + a_0 = |\phi - 1|\left(1 + (1-\alpha)\phi\right) > 0\\ \Delta^{-}_{n-1} = a_2 - a_0 = |\phi - 1|\left(1 - (1-\alpha)\phi\right) > 0\end{aligned}\right\}. \tag{7}$$

  The criteria Δ^{+}_{n-1} divides the parametric plane along $\alpha = (\phi + 1)/\phi$, Δ^{-}_{n-1} divides the parametric plane along $\alpha = (\phi - 1)/\phi$.

Figure 2 provides a conceptual map of the stability boundary and how it changes for different ϕ. It is common practice in exponential smoothing models to restrict the smoothing parameters to the $[0,1]$ interval [9],[10]. A series of papers [1],[11] have also proposed the damping parameter is restricted to $0 \le \phi \le 1$. However, it is interesting to note that there are stable DT forecasts for a much broader range of parameter values than those usually recommended in the literature. Similar findings were observed for Holts method and SES. When $\phi = 1$, when we have the Holts method, the stability conditions are $0 < \alpha < 2$, $0 < \beta < (4 - 2\alpha)/\alpha$. When $\beta = 0$, the SES stability boundary can be observed, $0 < \alpha < 2$.

4. Using Damped Trend Forecasting within the Order-Up-To Policy

A single retailer first receives goods in each period t. He observes and satisfies customer demand within the replenishment period, d_t. Any unfilled demand is backlogged. The retailer observes his inventory level and places a replenishment order, o_t, at the end of each period. There is a fixed time period of T_p between placing an order and receiving that order in stock. We assume that the retailer follows a simple Order-Up-To inventory policy. In an OUT policy, orders are placed to raise the inventory position ip_t up to an OUT level or base stock level s_t,

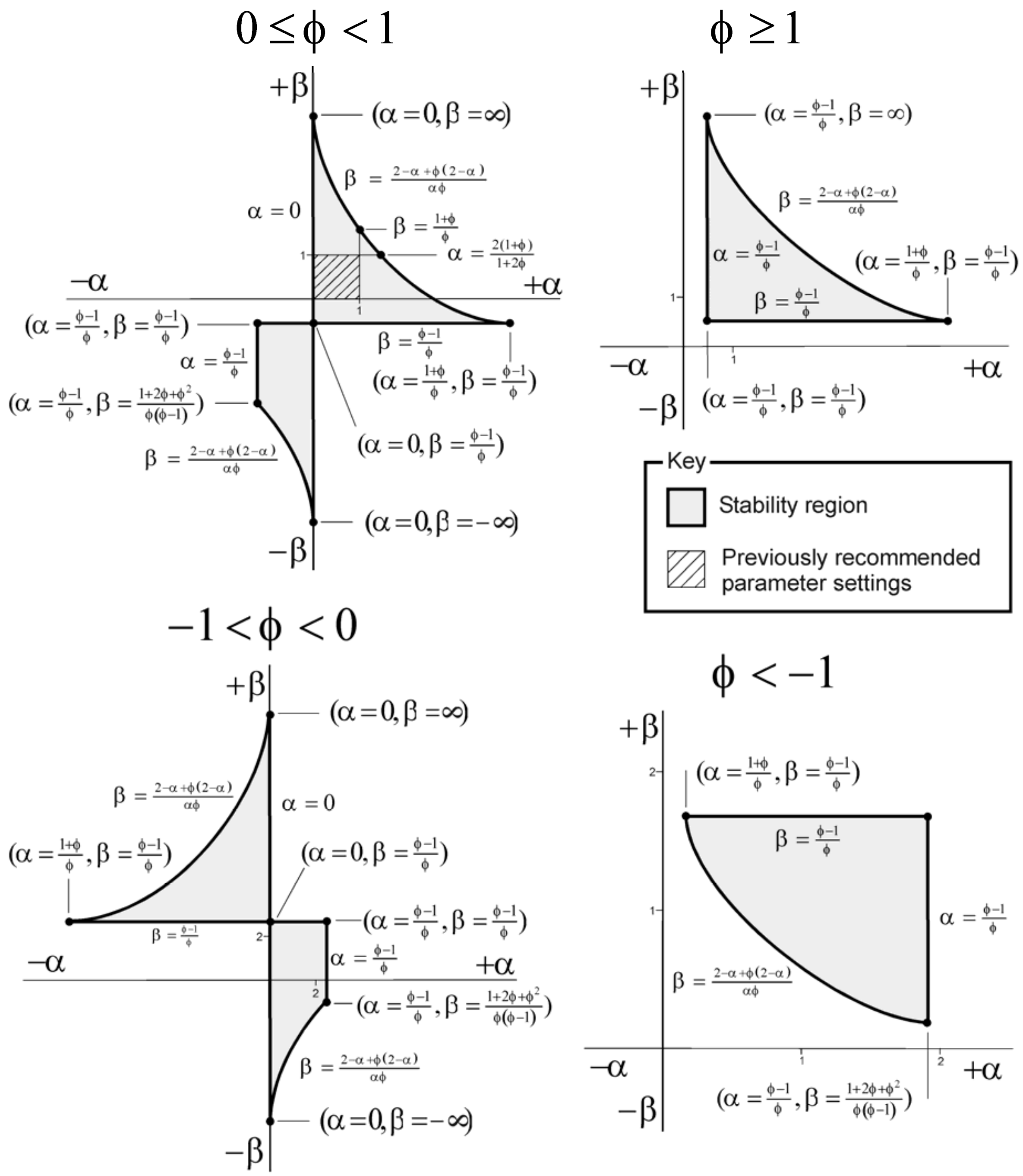

Figure 2. The Damped Trend Stability region

$$o_t = s_t - ip_t\,. \tag{8}$$

The inventory position is the amount of inventory on-hand + inventory on-order − backlog. The amount of inventory on-hand minus the backlog is the net stock ns_t level. The inventory on-order is also known as the Work-In-Progress (WIP), wip_t. The inventory position at time period t, ip_t is given by

$$ip_t = ns_t + wip_t\,. \tag{9}$$

The OUT level is often estimated from the observed demand. It can be written as

$$s_t = tns + \hat{d}_{t,t+T_p+1} + \underbrace{\sum_{i=1}^{T_p} \hat{d}_{t,t+i}}_{dwip_t}, \tag{10}$$

where $\hat{d}_{t,t+T_p+1}$ is the forecasted demand in period $t+T_p+1$ made in period *t*. The Target Net Stock, *tns*, is a safety stock used to ensure a strategic level of inventory availability. *tns* is a time invariant constant. Under the assumptions of normally distributed forecast errors and piece-wise linear convex inventory holding (*h*) and backlog costs (*b*) then it is common to assume $tns = z\sigma_{ns}$; $z = \Phi^{-1}\left[\frac{b}{b+h}\right]$. Here σ_{ns} is the standard deviation of the net stock levels and $\Phi^{-1}\left[x\right]$ is the inverse of the cumulative normal distribution function evaluated at *x*. The time varying Desired Work In Progress, $dwip_t = \sum_{i=1}^{T_p} \hat{d}_{t,t+i}$ is the sum of the forecasts, made at time *t* in the periods from $t+1$ to $t+T_p$. [12] show the order decision can be rewritten as

$$\left.\begin{aligned} o_t &= \underbrace{tns + \hat{d}_{t,t+T_p+1} + dwip_t}_{s_t} - \underbrace{\left(wip_t + ns_t\right)}_{ip_t} \\ &= \underbrace{tns + \hat{d}_{t,t+T_p+1} + \sum_{i=1}^{T_p} \hat{d}_{t,t+i}}_{s_t} - \underbrace{\sum_{i=1}^{T_p} o_{t-i}}_{wip_t} - ns_t = s_t - s_{t-1} + d_t \end{aligned}\right\}. \tag{11}$$

The *z*-transform transfer function for the order rate, expressed in a manner in which the forecasting system has been left unspecified, is given by

$$\frac{O(z)}{\varepsilon(z)} = (1 - z^{-1}) \underbrace{\left(\frac{\hat{D}_{T_p+1}(z)}{\varepsilon(z)} + \frac{DWIP(z)}{\varepsilon(z)} \right)}_{s_t} + 1. \tag{12}$$

(12) is a useful departure point for further analysis as the forecasting components can be simply "slotted" into $\hat{D}_{T_p+1}\left(z\right)$ and $DWIP\left(z\right)$ to yield the system transfer function. We notice in (11) and that the OUT policy requires two forecasts. One of these forecasts is a prediction, made at time *t* of the demand in the period $t+T_p+1$. Adapting the DT forecast to achieve this is done with

$$\hat{d}_{t,t+T_p+1} = \hat{a}_t + \hat{b}_t \sum_{i=1}^{T_p+1} \phi^i = \hat{a}_t + \hat{b}_t \frac{\phi(\phi^{T_p+1} - 1)}{\phi - 1}. \tag{13}$$

The other forecast required by the OUT policy is a prediction, made a time *t*, of demand over the lead-time. That is, the demand in periods $\left[t+1, t+2, ..., t+T_p\right]$. In the time domain this is

$$dwip_t = \hat{d}_{t,\left[t+1,t+T_p\right]} = \sum_{i=1}^{T_p} \hat{d}_{t,t+i} = \hat{a}_t T_p + \hat{b}_t \sum_{i=1}^{T_p} \frac{\phi(\phi^i - 1)}{\phi - 1} = \hat{a}_t T_p + \hat{b}_t \frac{\phi\left(T_p - T_p\phi + \phi\left(\phi^{T_p+1} - 1\right)\right)}{\left(\phi - 1\right)^2}. \tag{14}$$

Figure 3 shows that the transfer functions of the DT forecast and WIP target can be built up from 2 auxiliary variables, $a\left(z\right)$ and $b\left(z\right)$. These are

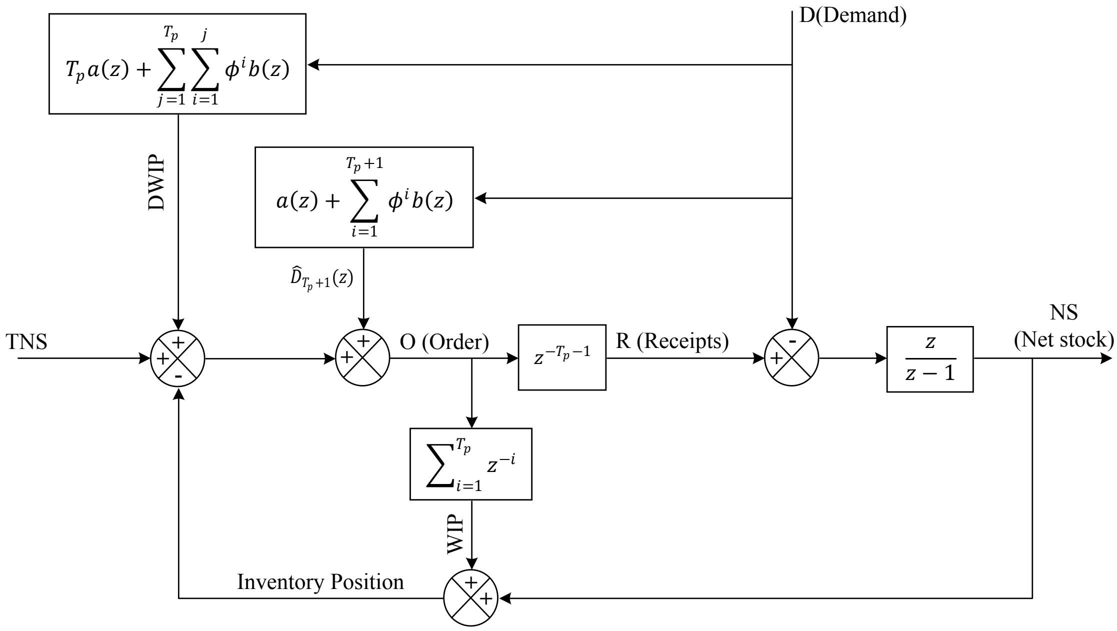

Figure 3. Block diagram of OUT policy with Damped Trend forecasts

$$\left.\begin{aligned} a(z) &= \frac{z\alpha\left(z+\phi(\beta-1)\right)}{\phi(1-\alpha)+z\left(\alpha-1+\phi(\alpha\beta-1)\right)+z^2} \\ b(z) &= \frac{(z-1)z\alpha\beta}{\phi(1-\alpha)+z\left(\alpha-1+\phi(\alpha\beta-1)\right)+z^2} \end{aligned}\right\}. \tag{15}$$

The z-transforms of the two DT forecasts required by the OUT policy are

$$\frac{\hat{D}_{T_p+1}(z)}{\varepsilon(z)} = \frac{z\alpha\left(\beta\phi^{T_p+2}(z-1)-(z-\phi)\left(1+\phi(\beta-1)\right)\right)}{(\phi-1)\left(z^2+z(\alpha-\phi-1+\alpha\beta\phi)+\phi(1-\alpha)\right)}, \tag{16}$$

$$\frac{DWIP(z)}{\varepsilon(z)} = \frac{z\alpha\left(\beta\phi^2(z-1)\left(\phi^{T_p}-1\right)-T_p(z-\phi)(\phi-1)\left(1+\phi(\beta-1)\right)\right)}{(\phi-1)^2\left(z^2+z(\alpha-\phi-1+\alpha\beta\phi)+\phi(1-\alpha)\right)}. \tag{17}$$

The transfer functions of the two required forecasts when Holts Method is used are

$$\left.\begin{aligned} \frac{\hat{D}_{T_p+1}(z)}{\varepsilon(z)} &= \frac{z\alpha\left(z\left(1+\beta T_p\right)+\beta\left(1-T_p\right)-1\right)}{1+z^2-\alpha+z\left(\alpha(1+\beta)-2\right)} \\ \frac{DWIP(z)}{\varepsilon(z)} &= \frac{z^2\alpha T_p\left(2+\beta\left(1+T_p\right)\right)+z\alpha T_p\left(\beta\left(1-T_p\right)-2\right)}{2\left(z^2+z\left(\alpha(1+\beta)-2\right)+1-\alpha\right)} \end{aligned}\right\}. \tag{18}$$

As the trend component of the demand process is not explicitly forecasted in the SES and the Naïve forecasting models, the majority of scholars consider the DWIP term to be simply the product of the lead-time and the most recent forecast [6], [13]. The transfer functions of $\hat{D}_{T_p+1}$

and *DWIP* for SES are then

$$\left.\frac{\hat{D}_{T_p+1}(z)}{\varepsilon(z)}=\frac{z\alpha}{z+\alpha-1};\ \frac{DWIP(z)}{\varepsilon(z)}=\frac{z\alpha T_p}{z+\alpha-1}\right\}. \tag{19}$$

The transfer functions of $\hat{D}_{T_p+1}$ and *DWIP* and Naïve forecasting methods are

$$\left.\frac{\hat{D}_{T_p+1}(z)}{\varepsilon(z)}=1;\ \frac{DWIP(z)}{\varepsilon(z)}=T_p\right\}. \tag{20}$$

Once we have the transfer functions of these two forecasts, we can substitute them into (12) to obtain the order rate transfer function. The order rate transfer function is important as it contains information on the well-known bullwhip effect. The bullwhip effect is present if the variance of the orders is greater than the variance of the demand. If customer demand is independently and identically distributed (i.i.d.) then the following relationship holds,

$$Bullwhip=\frac{\sigma_o^2}{\sigma_d^2}=\sum\nolimits_{t=0}^{\infty}(o_t)^2=\sum\nolimits_{t=0}^{\infty}\left(Z^{-1}\left[O(z)\right]\right)^2. \tag{21}$$

In (21) $Z^{-1}[x]$ is the inverse *z*-transform operator. Via Parseval's theorem, we can make the link between the bullwhip effect and the frequency response.

$$\sum\nolimits_{t=0}^{\infty}(o_t)^2=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\left|O(e^{i\omega})\right|^2 d\omega=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{O(e^{i\omega})O(e^{-i\omega})}_{\text{Amplitude Ratio, AR}}d\omega. \tag{22}$$

Here $O(z)$ is the *z*-transform of o_t and ω represents the angular frequency of o_t. If we investigate the Amplitude Ratio (AR) of different frequencies we are able to gain insight into how the forecasting and replenishment system behaves to any demand pattern. This is because all demand patterns can be decomposed into a set of harmonic frequencies. i.i.d. demands mean that all frequencies are present with equal density in the demand signal. The frequency response of discrete time systems is a function with a periodicity of 2π. However, we only need to study the AR for frequencies in the period $[0,\pi]$, as the frequency response plot on $[-\pi,0]$ is a simple reflection of $[0,\pi]$ about the origin. Furthermore we note that $AR|_{\omega=0}=1$ and $\frac{dAR}{d\omega}\big|_{\omega=0}=0$ for all systems. Let's now take a look at the frequency response for the OUT policy with different forecasting mechanisms.

4.1 Frequency Response of the OUT Policy with Naïve Forecasts. The AR is strictly increasing in ω as $\frac{dAR}{d\omega}=2\left(T_p^{\,2}+3T_p+2\right)\sin\omega$, $AR|_{\omega=0}=1$ and $AR|_{\omega=\pi}=\left(3+2T_p\right)^2$ within the interval $[0,\pi]$, see Figure 4a.

4.2 Frequency Response of the OUT Policy with SES Forecasts. For stable SES forecasts (Figure 4b) the AR is a strictly increasing within the frequency interval $(0,\pi)$ as

$$\frac{dAR}{d\omega}=\frac{2\alpha^3\left(T_p+1\right)\left(\alpha T_p+2\right)\sin(\omega)}{\left(\alpha^2+2\alpha\cos(\omega)+2-2\alpha-\alpha\cos(\omega)\right)^2}>0. \tag{23}$$

Together with $AR|_{\omega=1} = 1$ we can deduce that the value of AR is always greater than 1 for all frequencies. In another words, the OUT policy with SES forecasting will always produce bullwhip for all demand patterns for all lead-times. This finding is consistent with the results in [6], but they failed to completely characterise the frequency response.

4.3 Frequency Response of the OUT Policy with Holts forecasting. For stable Holts forecasts the AR originates at $AR|_{\omega=0} = 1$ and ends at

$$AR|_{\omega=\pi} = \left(\frac{4+\alpha\left(2+\beta+2T_p\left(2+\beta\left(2+T_p\right)\right)\right)}{\alpha\left(2+\beta\right)-4} \right)^2 > 1, \tag{24}$$

see Figure 4c. However, in between these two points there are two different AR responses. Either the AR is strictly increasing in ω or, when $\alpha < 4\beta/(2+2\beta+\beta^2)$, there is a stationary point within the $\omega \in (0,\pi)$ interval. Then the AR is an increasing function in ω until $\omega = \arccos\left(\left(\alpha+\alpha\beta-2\beta+\alpha\beta^2\right)/\left(\alpha+\alpha\beta-2\beta\right)\right)$ at which point it becomes a decreasing function until $\omega = \pi$. The stationary point, if it exists, will be a maximum. Using these facts we are able to prove that the Order-Up-To with Naïve, SES and Holts forecasts will, for any demand patterns and all lead-times, always generate bullwhip.

4.4 Frequency Response of the OUT Policy with Damped Trend Forecasting.
The DT frequency response is more complex than those previously considered. We first investigate two situations: low-frequency responses (ω near 0) and high-frequency responses (ω near π). We then pay attention to frequencies between 0 and π. Although $AR|_{\omega=0} = 1$ and $\frac{dAR}{d\omega}\big|_{\omega=0} = 1$ the second derivate can be positive, zero or negative. The sign of the second derivate has geometrical implications. If the second derivate is positive, the graph of AR will be convex near $\omega = 0$ with a local minimum at $\omega = 0$. If the second derivate is negative, the AR curve will be concave near the origin and the point at $\omega = 0$ is a local maximum. A concave AR will imply that the DT forecast enabled OUT policy will be able to avoid generating bullwhip for low frequency demand. The lowest-order non-zero derivative is always of even order for any of the DT settings. This means that a stationary point at $\omega = 0$ cannot be an inflection point when the second derivate is zero - it has to be either a local maximum or a local minimum. This fact also concurs with common knowledge of the periodicity of the frequency response.

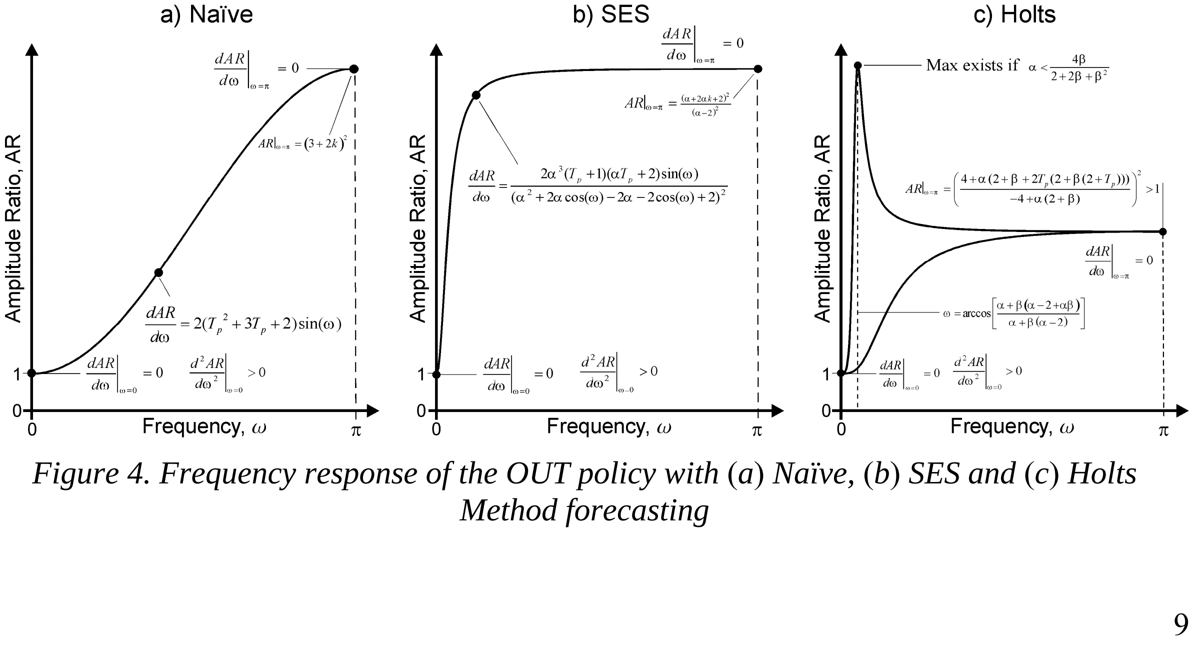

Figure 4. Frequency response of the OUT policy with (*a*) *Naïve,* (*b*) *SES and* (*c*) *Holts Method forecasting*

Consider low frequency behaviour when $T_p = 1$. With $-1 \le \phi \le 1$, $\left.\frac{\mathrm{d}^2 AR}{\mathrm{d}\omega^2}\right|_{\omega=0} > 0$. This means that these settings will always generate bullwhip when ω is near 0. When $\phi > 1$, the AR near $\omega = 0$ is always concave, implying bullwhip is avoided at low frequencies. However when $\phi < -1$, the second derivate can be positive, negative or zero. Figure 5 maps out the areas of the parametric plane where the bullwhip effect can be avoided when the demand process contains low frequency harmonics. The curves which separate out the different classes of bullwhip behaviour for when $-3 < \phi < -1$ are $\alpha = \frac{\phi-1}{\phi}$ and $\beta = \left\{\frac{-2-\alpha}{\alpha\phi}, \frac{\phi-1}{\phi}\right\}$. When $-5 < \phi \le -3$ the $\alpha = \left\{\frac{4+2\phi}{\phi}, \frac{\phi-1}{\phi}\right\}$ and $\beta = \left\{\frac{-2-\alpha}{\alpha\phi}, \frac{\phi-1}{\phi}\right\}$ determine the different classes of bullwhip behaviour. These were all obtained by setting $\frac{\mathrm{d}^2 AR}{\mathrm{d}\omega^2} = 0$ and solving for the relevant variables.

Consider high frequency bullwhip behaviour near $\omega = \pi$ when $T_p = 1$. DT forecasts with $\phi \ge 1$ or $\phi \le -3$ always generate bullwhip for high-frequency demands as $\left.AR\right|_{\omega=\pi} > 1$. If $\phi = -1$, then $\left.AR\right|_{\omega=\pi} = 1$. There are also some circumstances that the $\left.AR\right|_{\omega=\pi} < 1$, see Figure 6. When $\left.AR\right|_{\omega=\pi} < 1$ the DT enabled OUT policy avoids inducing the bullwhip when the demand processes contain only high frequency harmonics. This bullwhip avoidance occurs for: $0 < \phi < 1$ when $\frac{\phi-1}{\phi} < \alpha < 0$ and $\frac{\phi-1}{\phi} < \beta < \frac{-1}{\phi}$; $-1 < \phi < 0$ when $0 < \alpha < \frac{\phi-1}{\phi}$ and $\frac{-1}{\phi} < \beta < \frac{\phi-1}{\phi}$;

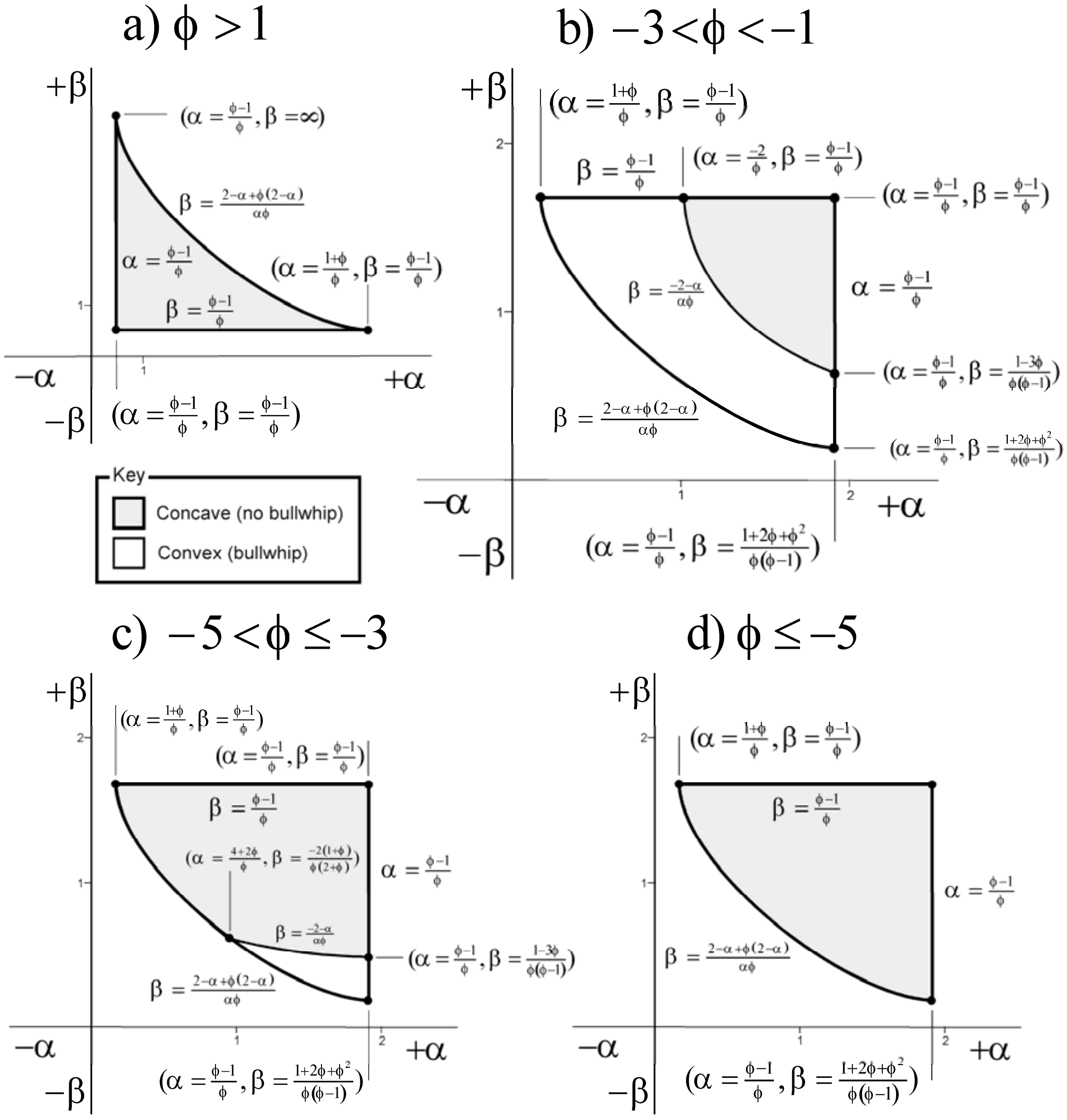

Figure 5. Concavity of AR near $\omega = 0$ for $T_p = 1$

$-3 < \phi < -1$ when $\alpha < \phi - 1$ and $\frac{-(2+\alpha)(1+\phi)}{\alpha\phi(1+2\phi)} < \beta < \frac{-1}{\phi}$.

When the lead-time increases, for low frequencies near $\omega = 0$ and $\phi > 1$ the influence of the parameters' settings (Figure 5a) remains exactly the same. That is *AR* < 1 near $\omega = 0$. For high frequency demand near $\omega = \pi$ when $0 < \phi < 1$ to the area of the parametrical plane that is able to avoid the bullwhip effect will become smaller. When $-1 < \phi < 0$, the region within the parametrical plane where bullwhip is attenuated changes in a complex manner. It has different shapes when the lead-time changes from an odd number to an even number. When $\phi < -1$, bullwhip avoiding areas of the parametrical plane for both low-frequency and high frequency demand will disappear and reappear in sophisticated manners when the lead-time switches between an odd number and an even number.

So far we have not been able to determine the characteristics of the two stationary points within the interval $(0, \pi)$. However the results that we have obtained at $\omega = \{0, \pi\}$ indicates that for some demand patterns the OUT policy with DT forecasting mechanism is able to avoid the bullwhip effect. This is a type of dynamic behaviour that is not present when the Naïve, SES and Holts Method is used as a forecasting method within the OUT policy.

5. Numerical verification

We constructed an Excel based simulation of the OUT policy with DT forecasting and unit lead-times. Demand was assumed to be made up of a single sine wave with a mean of 10, unit amplitude and a frequency of $\omega \in \{0.02, 3.1\}$ radians per period. We determined the bullwhip $\frac{\sigma_o^2}{\sigma_d^2}$ and net stock variance amplification ratio, *NSAmp* = $\frac{\sigma_{ns}^2}{\sigma_d^2}$ from 4000 periods after an initialisation period of 1000 periods. Sample numerical results are given below in Table 2. They verify that the OUT policy with DT forecasting can indeed eliminate the bullwhip effect. It is interesting to note that whilst we have not studied the *NSAmp* measure from a theoretical standpoint in this paper, the numerical results in Table 2 hint at the possibility that not only is bullwhip reduced with DT forecasting, but there appears to be good control over inventory levels as well.

6. Concluding remarks

We have studied the stability and bullwhip behaviour of an OUT policy that incorporates DT forecasting. We have demonstrated that the DT forecasts are stable over a much broader range of parameter values than is usually recommended in the literature. We have shown that the OUT with three different types of forecasts, Naïve, SES and the Holts method forecasts, will

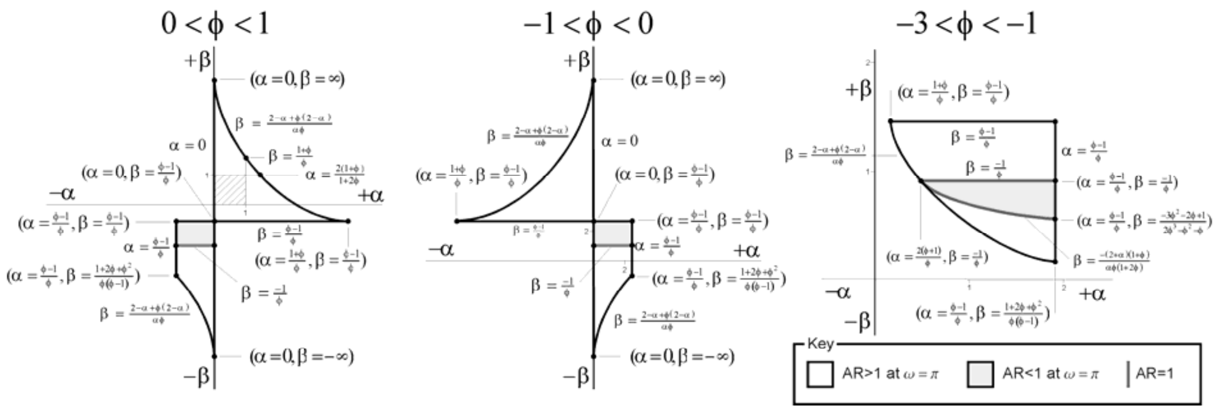

Figure 6. Possible settings that result in $AR \le 1$ near π when $T_p = 1$.

Demand Frequency ω (radians per period)	0.02	0.02	0.02	0.02	3.1	3.1	3.1
α	0.14	1.6	1.1	1.1	-0.5	2	1.4
β	0.14	1.6	1.1	1.1	-1	2	0.45
ϕ	1.1	-1.5	-4.5	-5.5	0.6	-0.6	-2
Bullwhip σ_o^2/σ_d^2	0.9768	0.9964	0.9824	0.9624	0.4278	0.5389	0.1697
NSAmp σ_{ns}^2/σ_d^2	0.3626	0.0056	0.1781	0.8542	0.0309	0.0180	0.1997

Table 2. Numerical results from a 4000 period simulation verifying our theoretical results

always generate bullwhip, for any demand patterns and for all lead-times. For the DT forecasting method, we have shown that for some demand patterns the OUT replenishment policy with DT forecasting mechanism is able to avoid generating bullwhip. This is a qualitatively different bullwhip behaviour that is not present with other, more traditional forecasting policies. This suggests that the DT forecasting methodology deserves much more attention in the OR/OM literature than it currently receives.

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