Long-wavelength gravitational waves and cosmic acceleration

a thesis submitted to Cardiff University for the degree of

Doctor of Philosophy

by

Edmund Rudolph Schluessel

September 2011

Long-wavelength gravitational waves and cosmic acceleration

Summary

Multiple observations of distant type Ia supernovae show the deceleration parameter of the universe is negative. The standard cosmological model states expansion should be slowing down.

A new theory is presented which explains cosmic acceleration only through the action of well-supported phenomena in the context of Einstein's general theory of relativity through the use of the Bianchi type IX homogeneous, closed cosmology.

The evidence for acceleration is assessed and previously-unreported biases and insufficiencies in the evidence are revealed and discussed.

The Einstein equations for the Bianchi type IX cosmology are solved to quadratic order in a matter-dominated universe. The first terms of a power-series solution are given for arbitrarily strong growing mode of gravitational waves in a matter-dominated Bianchi IX universe. The effect of these waves on the energy density of the universe is shown to be compatible with available data.

The equations for redshift anisotropy in the Bianchi IX universe are solved to quadratic order. Reported anomalous structure in the cosmic microwave background is considered in the light of these solutions. The Bianchi IX universe is shown to provide an explanation for these anomalies compatible with the CMB.

In order to help typify a new class of standard sources for determining cosmological parameters, a formula relating the time-dependent deflection of light by a massive,

compact binary such as a super-massive black hole binary is derived. This formula is applied to the system 3C66B and finds that in ideal circumstances, the best available observational techniques could detect a time-dependent component to the bending of light by the core of 3C66B.

A solution for the Einstein equations in the Bianchi IX universe is found which explains cosmic acceleration while remaining compatible with the CMB and other cosmological parameters as reported by WMAP.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

Signed .... (candidate) Date ....

STATEMENT 1

This thesis is being submitted in partial fulfillment of the requirements for the degree of PhD.

Signed .... (candidate) Date ....

STATEMENT 2

This thesis is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by explicit references.

Signed .... (candidate) Date ....

STATEMENT 3

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed .... (candidate) Date ....

STATEMENT 4: PREVIOUSLY APPROVED BAR ON ACCESS

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loans after expiry of a bar on access previously approved by the Graduate Development Committee.

Signed .... (candidate) Date ....

Acknowledgments

I thank Prof Leonid Grishchuk and Prof Peter Coles of Cardiff University for continuing advice, support and supervision throughout this research. Prof Sergei Kopeikin of University of Missouri-Columbia, Dr Simon Dye of Cardiff University, Dr Naomi Ridge formerly of Harvard University and Dr Rockhee Sung of University of Cape Town provided helpful conversations. I thank my parents for their ongoing encouragement and aid. This work is dedicated to the memory of Dr Didier Biron.

Material from CHAPTER 3, prepared in the course of this research, has been published as "Constraining the parameters of binary systems through time-dependent light deflection" in *General Relativity and Gravitation*, Volume 40, Issue 5, pp.1013-1027.

Contents

1 Introduction **2**

1.1 Background 2

1.1.1 Standard cosmology predicts an expanding universe 3

1.1.2 Simple cosmology predicts a decelerating universe 6

1.1.3 Observations say the universe is accelerating 7

1.2 Dark energy 8

1.2.1 Cosmological constant 9

1.2.2 Quintessence 10

1.2.3 Local inhomogeneity 11

1.2.4 Exotic models 12

1.3 Tensorial theories for acceleration in a flat universe 13

2 Evidence for acceleration 15

2.1 Introduction 15

2.2 Surveys of acceleration 17

2.3 Analysis 22

2.4 Conclusions 26

2.4.1 Recommendations 27

3 Constraining the parameters of binary systems through time-dependent light deflection 30

3.1 Introduction 30

3.2 Theory 32

3.2.1 Notations, definitions & assumptions 32

3.2.2 General theory 34

3.2.3 Application to 3C66B 46

3.3 Observational techniques 50

3.3.1 Interferometry 50

3.3.2 Pulsar timing 51

3.4 Conclusions 52

4 The Bianchi IX cosmology **54**

4.1 The Bianchi classification scheme 55

4.2 The Kasner universe 58

4.2.1 Scale factor 60

4.2.2 Dynamics in the Kasner universe 64

4.3 Gravitational wave nature of Bianchi IX 69

4.3.1 Einstein equations in the tetrad formalism 73

4.3.2 The curvature tensor for Bianchi IX 76

4.4 Einstein equations for Bianchi IX 78

4.4.1 Exact equations 78

4.4.2 Solutions to the Einstein equations at zero order 83

4.4.3 Solutions at linear order 83

4.4.4 Solutions at quadratic order 90

4.4.5 Strong growing waves in the quasi-isotropic regime 101

4.4.6 Dynamics 103

4.4.7 Back-reaction 110

4.4.8 Amplification of gravitational waves 113

4.5 Conclusions 113

5 The Cosmic Microwave Background of a Bianchi IX universe 116

5.1 Geodesic equations . . . 118

5.2 Redshift and CMB variations . . . 122

5.3 Comparison with the observed CMB . . . 124

5.3.1 CMB anomalies . . . 125

5.4 Conclusions . . . 128

6 An accelerating Bianchi IX universe preserving an almost-isotropic CMB 130

6.1 Cosmological parameters . . . 131

6.2 Compatibility with the redshift . . . 133

6.3 Acceleration in the Bianchi IX universe . . . 136

6.3.1 Order of magnitude estimates for gravitational wave amplitudes . . . 136

6.3.2 Quasi-isotropic, strong growing mode acceleration . . . 139

6.4 Conclusions 145

7 Conclusions 148

7.1 Directions for future research 149

7.2 Implications of the Bianchi IX cosmological model 151

List of Tables

2.1 Summary of results from surveys indicating acceleration 23

4.1 The Bianchi classification scheme 58

7.1 Supernova observations used in analysis of acceleration 157

List of Figures

2.1 Sky positions of supernovae used as evidence for acceleration 22

3.1 Configuration of a binary deflector 39

3.2 Light deflection by a binary system 45

6.1 Deceleration parameter versus time 143

Abstract

Strong long-scale gravitational waves can explain cosmic acceleration within the context of general relativity without resorting to the assumption of exotic forms of matter such as quintessence. The existence of these gravitational waves in sufficient strength to cause observed acceleration can be compatible with the cosmic microwave background under reasonable physical circumstances. An instance of the Bianchi IX cosmology is demonstrated which also explains the alignment of low-order multipoles observed in the CMB. The model requires a closed cosmology but is otherwise not strongly constrained. Recommendations are made for further observations to verify and better constrain the model.

CHAPTER 3 has been previously published as ([152]). Equations describing cosmological gravitational waves at quadratic order and in quasi-isotropic approximation in a matter-dominated Bianchi IX universe; equations describing second-order corrections to the cosmic microwave background resulting from quadratic-order-strong gravitational waves; and analytic calculations of the dynamics of a Bianchi IX universe, including the explicit illustration that such a universe can undergo cosmic acceleration are original to this work, as are conclusions following from that mathematical anaysis.

Chapter 1

Introduction

1.1 Background

The observational confirmation that the universe has been expanding from a condition of extreme density and minute size since some point in the finite past represents a major triumph of Einstein's theory of gravitation in providing an elegant explanation for cosmology, without the addition of exotic, heretofore-unobserved substances or fundamental forces. This notion has however faced a serious challenge since Riess's 1998 discovery[1] of cosmic acceleration. The purpose of this research is to evaluate the following question: can the back-reaction of long-wavelength gravitational waves in a closed universe contribute to cosmic acceleration while remaining compatible with observational constraints?

1.1.1 Standard cosmology predicts an expanding universe

The full Einstein equations read[1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the once-contracted Riemann tensor, R is the Ricci curvature scalar, $T_{\mu\nu}$ is the energy-momentum tensor, Λ is the "cosmological constant" and the constant $k \equiv 8\pi G/c^4 \approx 2.08\times10^{-43}\mathrm{kg}^{-1}\mathrm{m}^{-1}\mathrm{s}^2$. We approximate that all matter in the universe is, on large scales, an isotropic fluid ($T_{11} = T_{22} = T_{33}$) so, in a Gaussian ($g_{00} = 1$) and synchronous ($g_{0i} = 0$) coordinate system, we have [4]:

$$R^0_0 - \frac{1}{2}R = kT^0_0 + \Lambda \tag{1.2}$$

$$-R = kT^\mu_\mu - 2\Lambda. \tag{1.3}$$

Cosmological parameters

In discussions of cosmology it is conventional to track the expansion of an isotropic metric by introducing a "scale factor", which is a positive function of time only. In general the scale factor has no specific geometric meaning other than to compare distances in the metric at different points in time. Furthermore, the scale factor loses unique meaning when the universe becomes non-isotropic. Let the coordinate

[1] Throughout this document, indices written with Greek letters μ, ν etc. run over 0,1,2,3 and indices written in Roman letters i, j etc. run over 1,2,3. The sign of the metric tensor reads $+,-,-,-$.

x^0, which is privileged in the $g_{11} = g_{22} = g_{33} =$ constant Minkowski special case of the metric, be designated as "time". We will denote this scale factor function as $a(t)$ in analogy with its definition in the Robertson-Walker metric, where it appears as [5]

$$ds^2 = dt^2 - a^2(t) \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{1 + \frac{1}{4}K\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right]^2} \tag{1.4}$$

and the symbol K has the value 0 in a flat universe, 1 in a closed universe, and -1 in an open universe. In this case the Einstein equations read [5][2]

$$\frac{3}{a^2}\left(\dot{a}^2 + K\right) = k\epsilon + \Lambda \tag{1.5}$$

$$-6\frac{\ddot{a}}{a} = k(\epsilon + 3p) - 2\Lambda \tag{1.6}$$

where ϵ denotes the energy density of matter described by the energy-momentum tensor and p denotes the pressure of matter described in that tensor.

When discussing cosmology it is common [4] to define cosmological "dynamic", that is time-dependent, quantities in relation to the scale factor through the means of a Taylor expansion. Let the subscript *0* denote a function evaluated at a particular moment in time t_0 (which is how we will generally use the subscript *0*):

$$\frac{a_0}{a} \approx a_0 \left[\frac{1}{a_0} + (t - t_0)\left(\frac{d}{dt}\frac{1}{a}\right)_{t=t_0} + \frac{1}{2}(t - t_0)^2\left(\frac{d^2}{dt^2}\frac{1}{a}\right)_{t=t_0}\right] = \tag{1.7}$$

$$= 1 - (t - t_0)\frac{\dot{a}_0}{a_0} + \frac{1}{2}(t - t_0)^2\left(\frac{2\dot{a}_0^2 - a_0\ddot{a}_0}{a_0^2}\right). \tag{1.8}$$

[2] A single dot denotes a derivative with respect to t; two dots denote a second derivative with respect to t.

In isotropic cosmology we define the Hubble constant $H_0 \equiv \dot{a}_0/a_0$ and the deceleration parameter[3] $Q_0 \equiv -\ddot{a}_0 a_0/\dot{a}_0^2$ so

$$\frac{a_0}{a} \approx 1 - H_0\,(t-t_0) + H_0^2\left(1+\frac{1}{2}Q_0\right)(t-t_0)^2\,. \tag{1.9}$$

Because the universe seems flat and dominated by ordinary matter over small scales, it is common to move terms arising from K to the right-hand side of the equation, where they act as elements of an "effective energy-momentum tensor". It is also customary to state contributors to cosmological expansion as dimensionless parameters Ω_i in comparison to the "critical density" $\epsilon_{\text{critical}}$, that is, the energy-density of ordinary matter required for the universe to be flat: $k\epsilon_{\text{critical}} = 3H_0^2$ so

$$\begin{gathered} 3H_0^2 = k\epsilon_0 + \Lambda - 3\tfrac{K}{a_0^2} \\ \Omega_K + \Omega_M + \Omega_R + \Omega_\Lambda = 1 \end{gathered} \tag{1.10}$$

where

$$\Omega_M + \Omega_R \equiv k\epsilon_0/3H_0^2 \tag{1.11}$$

$$\Omega_\Lambda \equiv \Lambda/3H_0^2 \tag{1.12}$$

$$\Omega_K \equiv -K/\dot{a}_0^2. \tag{1.13}$$

Multiple observations, most recently by WMAP, have confirmed that $\Omega_R \ll \Omega_M$ [16] and so to the limit of the precision with which these quantities can be evaluated

[3] Q has been defined with a minus sign for historical reasons. $Q > 0$ denotes a decelerating universe; $Q < 0$ denotes an accelerating universe. We have avoided the more common notation q in favor of Q to avoid confusion when interpreting the source material.

$\Omega_K + \Omega_M + \Omega_\Lambda = 1$.[4] When we discuss some field of unknown character contributing to the energy density, we will designate it with the subscript X (for example, such a field would be related to a density parameter Ω_X).

1.1.2 Simple cosmology predicts a decelerating universe

If the scale factor a measures a distance, it is reasonable to say by analogy that $\dot{a}$ can be compared to a velocity and $\ddot{a}$ an acceleration. Let the time-dependent Hubble parameter be defined by $H \equiv \dot{a}/a$. We define the time-dependent deceleration parameter Q by:

$$Q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{d}{dt}\frac{1}{H} - 1. \tag{1.14}$$

Dividing (1.6) by (1.10) we easily obtain

$$Q = \frac{1}{2}\frac{k\,(\epsilon + 3p) - 2\Lambda}{k\epsilon + \Lambda - K/a^2} = \frac{1}{2}\Omega_M - \Omega_\Lambda \tag{1.15}$$

in a matter-dominated universe: a flat universe with no cosmological constant must always decelerate. While the properties of so-called "dark matter" remain undetermined, the localisibility of dark matter's distribution and its slow motion implies it can be treated as $w = 0$ dust.

We can also immediately say that in a universe with no cosmological constant,

[4] Chernin, in [81], elegantly derives a description of the scale factor in an open Friedmann cosmology which can be used when there is a significant amount of relativistic matter in a cold matter-dominated universe. Chernin's equation is easily generalized to the closed Friedmann universe.

acceleration is possible under the condition

$$\frac{2}{1+3w}\left(1 - K/a^2 k\epsilon\right) < 0 \tag{1.16}$$

so, because ϵ must be positive, acceleration is only possible when $w < -1/3$.

1.1.3 Observations say the universe is accelerating

Acceleration in and of itself is not a newcomer to cosmology. The de Sitter cosmology [82], discovered in 1917, is driven solely by a cosmological constant and consequently has a constant deceleration parameter of $Q = -1$. Bondi, Gold & Hoyle's "steady state" universe [83] similarly accelerates with $Q = -1$, this value being associated with a universe whose expansion is driven solely by a field whose energy density is not dependent on the scale factor (Ω_X = constant). With the proposal of "big bang" nucleosynthesis [84] and the subsequent discovery of the cosmic microwave background (CMB) [85], consensus came to settle on the simplest matter-filled model, the Friedmann universe [86].

Throughout the 1990s, astronomical observations began to indicate that the matter energy density of the universe was far below the critical density, leading some (for example [87]) to propose the resurrection of the cosmological constant in order to preserve the observed near-flatness of space.

In 1998, Riess *et al.* published an analysis [1] of the light from a small number of type Ia supernovae with $0.16 \leq z \leq 0.62$ and concluded from this set that the recent universe is accelerating with $Q_0 = -1.0 \pm 0.4$. Further observations and analysis (see CHAPTER 2) have also provided evidence that the universe has $Q_0 < 0$.

While Riess *et al.* did not exclude the possibility of a universe with $K \neq 0$, the assumption of a flat universe remains predominant throughout the field of cosmology as observations, both from supernova data (see SECTION 2) and WMAP (see SECTION 6.1), have shown that the universe is, on observable scales, very close to flat – although it is impossible to distinguish between a universe that is genuinely flat, with $\Omega_K = 0$ and one with Ω_K very close to but not equal to zero.

1.2 Dark energy

Since the discovery of acceleration, numerous explanations for the phenomenon have been proposed, all depending on an isotropic field creating additional, invisible energy. Turner and Huterer[6] introduce the term "dark energy", analogous to dark matter in the sense that dark energy does not interact electromagnetically with ordinary matter and has the property of negative pressure, to describe this additional fluid, which appears to make up over 70% of the total energy content of the universe.[16]

The assumption of a flat homogeneous cosmology demands that cosmic acceleration comes from a cosmological constant or a scalar field. Most scalar theories for explaining cosmic acceleration fall into two classes: an exotic form of matter with negative energy density, or surrender of the cosmological principle. Other scalar theories sacrifice different assumptions, such as homogeneity, or invoke more exotic explanations unsupported by laboratory physics.

1.2.1 Cosmological constant

The simplest, most familiar variation on the Robertson-Walker cosmological model which allows an accelerating universe is the "ΛCDM" model – a universe dominated by "cold" (non-relativistic, $p = 0$) matter with both baryonic and dark components, and with the existence of a non-zero cosmological constant. In such a universe the Einstein equations read [5]

$$3H^2 = k\epsilon + \Lambda \tag{1.17}$$

$$-6\frac{\ddot{a}}{a} = k\epsilon - 2\Lambda \tag{1.18}$$

so when $k\epsilon/\Lambda$ is small such that $(k\epsilon/\Lambda)^2$ is negligible, that is, the universe is dominated by a cosmological constant,

$$Q = \frac{1}{2}\frac{k\epsilon - 2\Lambda}{k\epsilon + \Lambda} \approx -1 + \frac{3k\epsilon}{2\Lambda} \tag{1.19}$$

which at first glance appears to neatly explain Riesse *et al.*'s result. However, as will be shown (see SECTION 2.3), the case for a cosmological constant is not definite. Furthermore, the theoretical background explaining the strength of the cosmological constant is not well developed, relying on an understanding of quantum gravity which does not yet exist [72]. While the cosmological constant can always be said to have a "right to exist" in the Einstein equations, current physics does not explain why it should have any particular strength and as such the cosmological constant should be treated as the simplest form of a scalar field of exotic matter.

1.2.2 Quintessence

More general than the cosmological constant but similar in structure is the proposal of "quintessence" [6], a novel form of matter with a time-dependent equation of state that can take on negative values. Many forms of these have been proposed; one form of these, for example, is the "Chaplygin gas" [88], which has equation of state $p = -A/\epsilon$ for $A > 0$. Quintessence theories are particularly motivated by the idea that acceleration is a cosmologically recent phenomenon, noting limited data (see Chapter 2) that the equation of state of dark energy may be evolving with time.

At the most fundamental level, all theories of quintessence propose the existence of a kind of matter which:

- has never been observed experimentally;
- does not interact with ordinary matter via the electromagnetic force;
- has a negative equation of state, that is, a positive energy density produces a negative pressure;
- plays a prominent role at current energy levels, as opposed to effects such as unification of forces thought to have taken place only in the very early universe.

In the absence of any compelling experimental evidence whatsoever for any kind of quintessence, quintessence and quintessence-like models should be regarded as highly speculative explanations for dark energy.

1.2.3 Local inhomogeneity

A more mundane explanation which has been offered for acceleration is the "Hubble bubble" [1, 61], regions of lower density in the intergalactic medium. If the vicinity of the Milky Way had lower matter energy density, expansion in its vicinity would increase [93], causing the illusion of cosmic acceleration.

Not only would the density deficit in such a "bubble" have to be quite large in order to cause acceleration, but the theory, which has the advantage of requiring no new physics, supposes either the existence of a rare or unique void that the Milky Way happens to be in – a violation of the cosmological principle in the sense that it makes observers in the Milky Way privileged – or a preponderance of voids whose presence makes the universe inhomogeneous not just in small patches but on average [91, 94].

1.2.4 Exotic models

Modified relativity

Some proposals to explain dark energy propose modifications to the Einstein equations. The best-known of these is the Cardassian Expansion model[95], which proposes time-dependent variation of the equation of state of matter. The Cardassian model is of particular interest in that it proposes an equation for the density perturbation

$$\kappa''(x) + 2\frac{s}{x}\kappa' - \frac{3}{2}s^2\kappa = 0 \tag{1.20}$$

for unknown constant s, which equation begins to resemble that for weak gravitational waves in a closed universe (cf. EQUATION (4.133)). Like Chaplygin gas and the "DGP" model[96], the Cardassian model justifies itself based on theories about higher-dimensional manifolds which remain untested.

Topological defects

The existence of cosmic strings would change the overall equation of state of the matter in the universe by a constant [67, 97], creating acceleration through simple deviation from the Friedmann model. While theories of cosmic inflation predict the formation of cosmic strings and other topological defects, such defects remain completely undetected.

1.3 Tensorial theories for acceleration in a flat universe

If we wish to preserve the theory of general relativity at the same time as retaining cosmic homogeneity, while at the same time relying only on effects with good experimental basis, scalar fields appear to be excluded as an explanation for acceleration. Ergo within the context of general relativity the next place to search for an answer is in tensor theories, which include the possibility of gravitational waves.

Lifshitz's theory of cosmological perturbations [79] appears to exclude tensorial answers to the problem of acceleration: gravitational waves have the same equation of state as radiation, and local clumps of gravitational waves in the theory (where "local" means bounded within an region smaller than the radius of curvature of the universe) both decay rapidly and collapse spatially. Rodrigues [113] takes a first step in discussing anisotropic dark energy, but limits his analysis to a flat universe and thus creates the problem of an anisotropic "big rip".

A high-frequency gravitational wave background has been proposed [92] as the source of cosmic acceleration. While the authors' analysis appears initially promising, similar to many scalar dark energy candidates the theory relies on the existence of an inflation-induced gravitational wave background that remains only hypothetical. Furthermore, the authors obtain their result by selection of an averaging scheme without mathematical rigor – surely choosing a mathematical model based on the desired results cannot be considered scientific. At any rate, the strength of the background that inflationary theory predicts is not sufficiently great to explain the observed large acceleration.

Chapter 2

Evidence for acceleration

2.1 Introduction

Numerous assumptions have been made in developing the predominant theories for acceleration that must be examined in detail to be understood. If some of these assumptions have been made on a weak basis, our range of compelling models for dark energy must change and new paths for the exploration of possible models will open.

The theory of tests to evaluate the deceleration parameter using supernovae as standard candles began with Wagoner [45] in 1977. Starting from assumptions of an isotropic Friedmannian cosmology which is not necessarily flat, Wagoner notes the approximate relation

$$d_E = H_0^{-1}\left[z - \frac{1}{2}\left(1 + Q_0\right)z^2 + \mathcal{O}\left(z^3\right)\right] \tag{2.1}$$

which, when H_0 and z are known, relates the deceleration parameter to the distance d_E as determined by the dimming of the supernova (where Wagoner was originally discussing Type II supernova events)[1]. This relation is valid when z is small such that z^3 is negligible, limiting its usefulness above $z \sim 1$, and requires the assumption of only small changes in the Hubble constant H_0 (that is, in a Friedmann cosmology $\dot{a}_F/a_F$ evaluated near the observer) on the interval from $z \approx 0$ to $z \approx 1$.

Type Ia supernovae are thought to be a "standard candle" for the measurement of distance and redshift; that is, supernovae of that type are thought to possess spectral and luminosity curves which are nearly identical. Therefore, observation of extragalactic type Ia supernovae is believed to produce reliable information on both the distance of the event (noting that brightness diminishes as the inverse square of distance) and the redshift of the distance associated with the event (through the change in the peak of the supernovae's spectra), with redshift z related to the scale factor a_F by

$$z + 1 = \frac{a_F\left(t_{\text{observation}}\right)}{a_F\left(t_{\text{emission}}\right)}. \tag{2.2}$$

Analysis of a statistically unbiased dataset of $z\left(t\right)$ therefore gives empirical information on $H\left(t\right)$.

Colgate [44] proposed that Type I supernovae should be used to measure the

[1] EQUATION (2.1) is of course a generalization of the famous distance-redshift approximation $H_0 d_E \approx z$ [4].

deceleration parameter in preference to Type II supernovae. Type I supernovae, specifically the "Type Ia" whose mechanism is thought to be the accretion of matter onto the surface of a white dwarf star, are understood to have a well-defined typical absolute magnitude and spectrum. Assuming this is true, the distance to and redshift of a given Type Ia supernova event (SNe) can easily be determined by fitting its light curve to standard templates. Therefore, with a sufficient sample of extragalactic supernovae of $z \gtrsim 0.2$, the parameters $H(t)$ and thus $Q(t)$ can be measured directly. When an isotropic cosmology with constant deceleration parameter $Q = Q_0$ is assumed, knowledge of H_0 and Q_0 are sufficient to typify the parameters of the universe [4].

With the advent of modern optical astronomy such as adaptive optics [46] and space-based optical telescopy [47], such surveys have become possible, but have produced results contradicting the standard, cold matter-filled Friedmann model of cosmology.

2.2 Surveys of acceleration

Cosmological studies measuring Q have been ongoing since 1997 and consist of analyses of redshifts [1, 3, 9, 11, 12, 13, 15, 43, 48, 49, 51, 50, 52, 53, 55, 56, 54, 57, 58] of type Ia supernovae. Some attempt to measure not only the dark energy equation of state at the present time w_{X0} but the first Taylor coefficient of a time-series expansion w_{Xa}.

The High-z Supernova Search Team's initial study of the deceleration parameter

[1] was the first large study to call attention to the problem of acceleration. Working from a sample of sixteen supernovae (four of which were well-observed "high confidence" sources), the most distant with $z = 0.97$, Riess concluded that the universe has $Q_0 < 0$ to high confidence, although the measurement of Q_0 itself possessed a high degree of uncertainty. Riess also noted the high sensitivity of the result to individual data points. The authors dismiss the closed cosmology despite the data indicating it as preferred [1, Fig. 7]; however the size of their experimental error precludes real evaluation of spatial curvature.

The Supernova Cosmology Project (SCP) made an earlier attempt to evaluate Q_0 with the use of supernovae [55]. This small survey ($n = 7$) on relatively nearby supernovae found a result inconsistent with those that followed it, giving results consistent with a universe with no dark energy and with too high a degree of error to meaningfully evaluate the geometry of the universe.

In contrast, the Supernova Cosmology Project's 1998 evaluation [48, 49] of 42 Type Ia SNe added further evidence that the universe was accelerating and also makes note of the surprising coincidence of the energy density Ω_X's near-equivalence with the total energy density in the current epoch. The SCP also did not consider the closed cosmology despite supernova data favoring it [49, Fig. 7].

The ESSENCE [11] survey was expressly designed to examine cosmic acceleration and detected 102 type-Ia supernovae from $0.10 \leq z \leq 0.78$, of which 60 were used for cosmological analysis. The initial analysis of ESSENCE assumed flatness of the universe. ESSENCE's observational fields were deliberately chosen to overlap the areas of previous surveys and to lie within ten degrees of the celestial equator;

all were also between 23:25 and 02:33 Right Ascension. Combining data from ESSENCE, SNLS and other sources [52] led to a conclusion consistent with other analyses. Exploration of more exotic models [53] found that no model of those tested was a good fit for ESSENCE's data.

The Supernova Legacy Survey (SNLS) [12] recorded 472 type-Ia supernovae. While analysis of the SNLS data set [13] provides results consistent with a universe driven by cosmological constant, the uncertainty on analysis of a time-dependent component to the equation of state of dark energy is very large; their analysis also does not consider a closed universe as a possible model [50]. Furthermore, the SNLS team also note the presence of two outliers and only 125 of 472 events were used to evaluate cosmology. SNLS observed SNe in four fields, one of which (field 3) is far above the plane of the celestial equator at 52 degrees declination; this and [54]'s northern field are the only fields with multiple observations in a small area more than 20 degrees from the celestial equator surveyed to date. SNLS also notes [50, section 5.4] that the values of Ω_M evaluated in the four fields are compatible only at a 37% confidence level – a surprising result given that each SNLS field contains at least 60 SNe in quite small (one square degree) areas.

The Hubble Space Telescope or HST survey of supernovae, published in 2004 and reviewed by the Supernova Cosmology Project [14] observed twenty type-Ia supernovae with redshifts $0.63 < z < 1.42$. While the number of SNe observed is small, the HST survey has the advantage of covering a wider area of sky than other SNe surveys. Analysis of the HST dataset suggests a rapidly-evolving dark energy field, although with very high error on measurements greater than $z = 1$ due to the small ($n = 10$) sample size it is impossible to take these results as anything more than suggestive. HST slightly favored a closed model of the universe, when considering interpretations of data that allowed $\Omega_K \neq 0$.

The Supernova Cosmology Project's 2008 analysis of supernova data [51] made a analysis of combined SNLS, ESSENCE and HST data, and attempted to analyze the data in the context of a theory of a time-dependent equation of state for dark energy but concluded "present SN data sets do not have the sensitivity to answer the questions of whether dark energy persists to $z > 1$, or whether it had negative pressure then." The analysis rejected 10% of all SNe from the combined data sets as outliers, many based on their failure to fit with a nearby H_0; Kowalski *et al.*'s rejection of outliers also shifts their analysis from one favoring a closed universe to one favoring a flat one [51, Fig. 11].

Further work by Riess *et al.* [54, 56] produced the so-called "gold" dataset of SNe, a group of supernova events with particularly clear light curves with 33 at $z > 1$. These supernovae were observed in two small (one square degree) fields. [54] claims a great reduction in the uncertainty of the Hubble parameter at $z > 1$ but the Hubble parameter measured in the extended "gold" set gives a value for the Hubble parameter not reconcilable with that in the [56] dataset. Riess *et al.* conclude that w is negative (with large experimental error) in the region $1 < z < 2$, then attempt to extrapolate the behavior of dark energy back to $z = 1089$.

Sollerman *et al*'s analysis of the Sloan Digital Sky Survey-II supernova data [3, 9] is the most recent analysis indicating cosmic acceleration. SDSS-II observed 103 type-Ia supernovae in a long, narrow strip along the celestial equator, including many from lower redshifts than had been previously examined in detail. Sollerman *et al.* also made use of data from the HST, ESSENCE and SNLS surveys, bringing the total number of SNe examined to 288. The primary conclusion to be drawn from SDSS is the sensitivity of cosmological measurements to the specific analysis technique used [60]; analysis of the data with two different curve-fitting algorithms produce two different, albeit somewhat compatible, results.

Further obscuring the neatness of measuring Q, Jha *et al.* noted [59] that the uneven local distribution of galaxies, specifically the existence of voids, can lead to a mis-estimation of H_0 on the order of 6.5% for a given galaxy.

Finally, of note is the WiggleZ dark energy survey [15, 43]. WiggleZ is the most extensive redshift survey thus far conducted, with some 280,000 galaxies with $0.2 < z < 1.0$ used as sources. WiggleZ also covers a wider area of sky than previous surveys, examining some 1000 square degrees in multiple windows around the sky. Two of WiggleZ's windows overlap with SDSS-II's survey area, so while WiggleZ is ongoing, preliminary results [57, 58] can be used to improve the evaluation of Q by improving precision on measurements of z of SNe host galaxies. The authors of [58] note that "the redshift-space clustering pattern is not isotropic in the true cosmological model", attributing the variation to "the coherent, bulk flows of galaxies toward clusters and superclusters". Analysis by the WiggleZ team of pre-existing SNe datasets, using the new, more precise data on galaxy redshifts they obtained, reconfirms the fact of acceleration, and generates results consistent with other surveys, but the data lack sufficient precision to determine the history of Q.

The table in the Appendix details the sky locations of SNe and galaxies used in the determination of acceleration; Figure 2.2 presents these locations graphically.

TABLE 2.1 summarizes the results of these surveys.

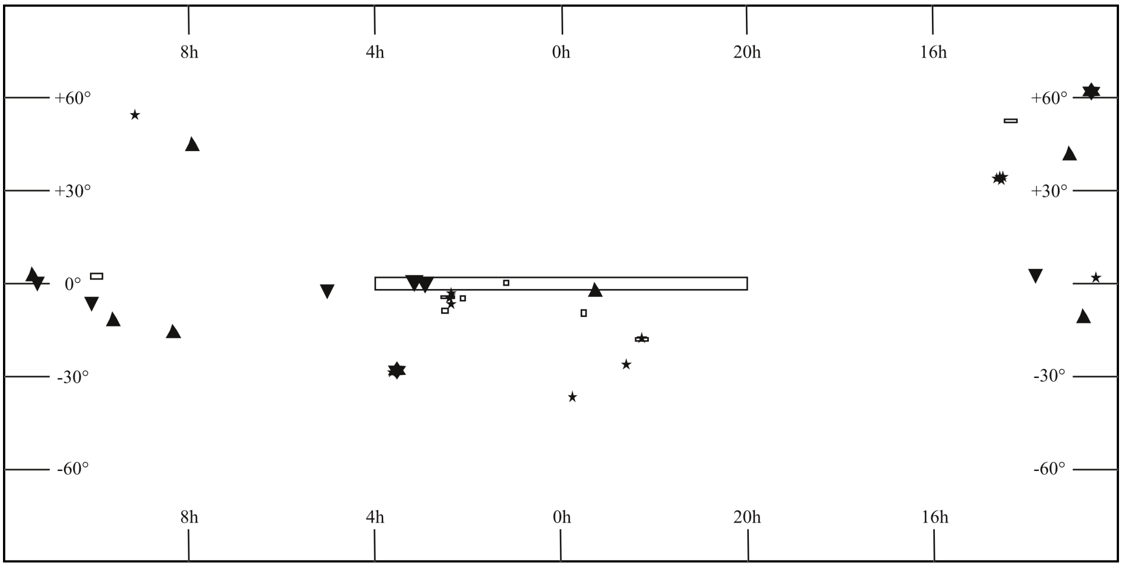

Figure 2.1: Sky positions of supernovae used as evidence for acceleration
Surveys of cosmic acceleration cover a limited portion of the sky, and data are divided into two contiguous, antipodal regions. Most data has been collected in a small area of the sky near the equator. *Triangles: Riess 1998 supernovae. Five-pointed stars: HST SNe. Six-pointed stars: Riess "gold" dataset. The long, thin strip centered on 0,0 is the SDSS-II survey area. Other boxes are the SNLS and ESSENCE survey areas.*

2.3 Analysis

Analysis of supernova data is, in one sense, quite consistent: all surveys apart from [55] agree that for $z < 1$ the universe has a deceleration parameter $Q_0 = -0.6$.

Survey	No of SNe	z
Supernova Cosmology Project 1997[55]	7	$0.35 < z < 0.46$
High-z Supernova Search Team[1]	16	$0.16 < z < 0.97$
Supernova Cosmology Project 1998[49]	42	$0.18 < z < 0.86$
HST[14]	20	$0.63 < z < 1.42$
ESSENCE[11]	102	$0.10 \leq z \leq 0.78$
Supernova Legacy Survey[12, 50]	125	$0.015 < z < 1$
ESSENCE + SNLS[52]	162	$0.015 < z < 1$
Supernova Cosmology Project combined[51]	307	$0.015 < z < 1$
Riess "gold" sample[56, 54]	16	$1.25 < z < 2$
WiggleZ[58]	557	$0.1 < z < 0.9$

Supernova
High-z S
Supernova
Super
ES
Supernova C
Ri
Sloane

Supernova
High-z S
Supernova
Super
ES
Supernova C
Ri
Sloane

Table 2.1: Summary of results from surveys indicating acceleration

*"dne" = "Does not evaluate". *: [56] attempts to analyze w_a with several different constraints but provides no numerical figure for its estimate of w_a's value. †: Where not explicitly stated in the source, Ω_X is evaluated from $\Omega_M + \Omega_K + \Omega_X = 1$. ‡: $Q_0^{flat} = \frac{1}{2}\Omega_M - \Omega_X$. (1): MLCS2K2 evaluation. (2): SALT-II evaluation. (3): ΛCDM model evaluation.*

Deeper analysis suffers from a lack of data at high redshifts and large numbers of free parameters in cosmological models, especially when more exotic models are considered. Meanwhile, while most surveys indicate that the acceleration in recent times acts as though driven by a cosmological constant, with an equation of state compatible with $w_X = -1$, the results from [60] show that this can be the result of the prior assumptions made about the model of dark energy.

No definitive statement can be made about the evolution of H over time from the information thus far available, particularly not statements connecting the state of cosmic acceleration now with the state of acceleration at the epoch of last scattering.

Nor can any definitive statement be made about cosmological models, other than to say that the most conservative, ΛCDM model fits the data at best inconsistently. Few studies of supernova data on acceleration examine the question of curvature in depth.

The majority of SNe data is collected from a single patch of sky: the field bounded by RA 22:00, RA 04:00, Dec $+1°15'$ and Dec $-10°00'$ (the "highly-observed field"). This area comprises 1350 square degrees, or only 2.1% of the sky. Surveys taken in small fields outside the highly-observed field, such as the Riess "gold" dataset, have high internal consistency, while surveys covering larger areas of sky have much lower consistency; the "gold" dataset contains the same number of SNe as [1] but has a standard error less than a tenth the size. It is also telling that the four SNLS fields produced results that correlated poorly (37% confidence) with one another [50], where two of the SNLS survey regions are well outside the highly-observed field. Compounding cosmographic bias, many of the remaining SNe observations are located in a region of sky antipodal from the highly-observed field; any vector or tensor contribution to cosmic dynamics will be dominated by dipole and quadrupole terms, and as such be seen with equal or opposite magnitude in the antipodal direction (that is, if we observe a change in Q of ΔQ along the x^idirection, we should expect a change of $-\Delta Q$ in the event of a vector contribution, or ΔQ in the event of a tensor contribution, along the $-x^i$ direction).

There is, furthermore, no SNe data whatsoever from above Dec $+62°$or below Dec $-37°$. The authors of [58] note a variation in the apparent Hubble parameter for galaxies in this equatorial band (no WiggleZ region lies further north than Dec $+8°$ or Dec $-19°$); variation to the Hubble flow could potentially be even greater outside this region. There is also no evaluation of whether the Hubble flow remains isotropic beyond $z = 0.3$ [77].

Indeed, Zehavi *et al.* comment [89] on the lack of sky coverage in their analysis of local Hubble flows, noting that "sparse sampling and the incomplete sky coverage (especially at low Galactic latitudes) may introduce a bias in the peculiar monopole due to its covariance with higher multipoles". While the fact of greater redshift in the range where acceleration can be measured should overcome the peculiar velocities of galaxies, the data problem remains.

2.4 Conclusions

Many reasonable constraints prevent a full-sky survey of supernovae. In the optical band, much of the sky is obscured by the "zone of avoidance" created by the plane of our own galaxy [63]. The so-called "Great Attractor", certain to be a region of particularly high peculiar velocities and therefore great shifts in the apparent Hubble parameter, lies in this zone [64]. Furthermore, with only a single space-based optical observatory (the *Hubble Space Telescope*) operating, detailed observation of the sky is restricted to those latitudes accessible by ground-based observatories, none of which are located in Arctic latitudes. However, the directional deficit of SNe surveys, aggregated together, cannot be ignored.

In the light of Tegmark *et al.*'s discovery [28] of a preferred axis to the CMB quadrupole, and Land & Maguiejo's subsequent observation [32] of a preferred axis in higher multipole moments aligned with the the quadrupole (the so-called "Axis of Evil"), the default assumption should be that anisotropic acceleration is not ruled out. Indeed, the prominent CMB "Cold Spot" [34] falls within the highly-observed field, although no surveys or SNe are located exactly in its direction.

As such, Wagoner's assertion of the cosmological principle as "statistically valid" [45] has been misapplied by analysts of acceleration data. A tensorial theory of cosmic acceleration would preserve homogeneity, in the sense that every observer sees "the same version of cosmic history" [21], at the expense of isotropy in the form of spherical symmetry.

More fundamentally, most studies of cosmic acceleration to date operate on the assumption that acceleration is isotropic, that is, that the acceleration field is equal in every direction, and therefore must be explained either by a cosmological constant or a scalar field. As Mörtsell and Clarkson note, "[a]t best this gives a small error to all our considerations; at worst, many of our conclusions might be wrong" [61]. In particular, the data as presented cannot distinguish between a scalar-field theory of acceleration, a vector-field theory of acceleration, a cosmological constant theory of acceleration, and a time-dependent tensor-field theory of acceleration.

Meanwhile, the simplest theory of acceleration, a cosmological constant, is challenged on two fronts: not only is Ω_X's value far out of line with that predicted for Ω_Λ by theory [72], but while its equation of state is close to $w_X = -1$ measurements have tended to favor a value slightly smaller than -1.

It is interesting to note that when Ω_K is evaluated, supernova data favor a closed universe (although always in a manner compatible with a flat universe); this conclusion is consistent with the curvature parameter evaluated by WMAP [62].

2.4.1 Recommendations

In light of these weaknesses of the current information on cosmic acceleration, the following program is recommended:

Analyses of SNe data should always consider the possibility of a closed or open universe as well as a flat one.

Additional SNe surveys for redshifts $.15 < z < 2$ should be carried out in unexamined areas of sky not obscured by the plane of the galaxy, such as for example the celestial north and south poles. The optimal region for these surveys is in rings located 90° from the center of the highly-observed field, which will maximize the difference in the event of a tensor-field (that is, gravitational-wave) acceleration.

In light of this need and the lack of ground-based observatories, as well as the infrared transparency of the Zone of Avoidance, priority should be given to the Wide Field Infrared Survey Telescope ("WFIRST") project [65], which incorporates the Super Nova/Acceleration Probe [66, 67] and Joint Dark Energy Mission [68, 69]. This telescope is currently scheduled to be launched in 2016.

As WiggleZ continues, its data on galactic redshifts should be examined for angular dependence as well. The completion of WiggleZ will provide invaluable information on baryon acoustic oscillations which will make possible the charting of the history of H and Q at much higher redshifts than is possible through the examination of supernova data.

Zhao *et al.* have also noted the possibility of using the Einstein telescope as an instrument for examining dark energy through the use of gravitational wave emissions from colliding binary objects as a "standard siren" analogous to the standard candle of type Ia SNe [111].

Cooray and Caldwell [102], implicitly identifying the same problem of lack of angular coverage as we note herein, propose a program of near-redshift surveys covering a large but practical area of sky which could also provide the relevant information with existing facilities. Some efforts have been made ([147, 148, 149, 150]) to re-evaluate the data in hand to look for signs of angular dependence in the Hubble parameter; these efforts have not produced conclusive results. The rejection of certain SNe in [51] should be re-evaluated in light of possible inadvertent obscuring of evidence for angular dependence in H.

Overall, the need is underscored for new theories of acceleration, particularly ones that attempt to explain acceleration through the action of tensor perturbations in a closed universe. Wagoner's formula (EQUATION 2.1) and its generalizations must be generalized further to take into account the possibility of anisotropic fields as the cause of anisotropic cosmic acceleration.

Chapter 3

Constraining the parameters of binary systems through time-dependent light deflection[1]

3.1 Introduction

Zhao *et al.*'s suggestion [111] that gravitational-wave emissions from merging neutron star binaries may be used as a "standard siren" for determining cosmological parameters, with gravitational waves traveling undisturbed by interstellar dust or the galactic foreground, opens up the possibility of gravitational astronomy providing a hugely important source of whole-sky observational data when the first generation of practical gravitational telescopes comes online. Of critical importance

[1] Portions of this chapter have been previously published as [152] as part of this research.

to establishing such a "standard siren" is the typification of gravitational wave sources before the catastrophic events which cause them to emit large amounts of gravitational radiation. Given the extragalactic nature of gamma-ray bursts this is difficult. However, very large extragalactic binaries close to merger may be constrainable with current technology.

The deflection of light by gravity is the oldest experimentally-verified test of the theory of general relativity [116]. With the continued improvement in observational resolution in astronomy, particularly through very-long-baseline interferometry (VLBI), the detection of more subtle effects of this light deflection becomes practical. Consequently, light deflection can be used to measure the properties of distant systems. This work supplies a theory for using time-variable light deflection to measure or constrain the parameters of binary systems. Specifically, the deflection angle of a light ray from a distant source is related to the configuration and motion of a binary system located in a distant galaxy somewhere between the point of emission of the light ray and its observation.

Super-massive black hole binaries (SMBHBs) are thought to form the cores and primary energy sources of the broad class of galaxies termed "active galaxies", "blazars", or "quasars". However, a combination of distance, radio noise, and optical thickness makes direct observation of presumed SMBHBs impractical. Observing a time-dependent motion in the image of the galaxy can provide information on the mass and orbital parameters of an SMBHB candidate.

Work by Damour and Esposito-Farese [120] and by Kopeikin *et al.* [119] establishes a theory of time-dependent light deflection by describing the time-dependent part of the deflection through the quadrupole term, which is the lowest-order term resulting from the mass distribution whose effects are practical to evaluate using current astronomical observational techniques. The work of Mashhoon and Kopeikin [139] in examining gravitomagnetic effects furthermore provides a theory for evaluating the contribution of the spin dipole of such systems and complements the work of Einstein [140] in providing a complete theory for stating the location of the deflected image in the weak field limit. We generalise these theories to a stronger-field regime and put constraints on the theory's applicability in this regime.

As a case study of an active galaxy, the theory is applied to the galaxy 3C66B, a nearby active galaxy with a candidate SMBHB core [127], and theoretical constraints on 3C66B's parameters from a light deflection experiment are compared to the constraints claimed by Jenet *et al.* [117].

3.2 Theory

3.2.1 Notations, definitions & assumptions

We assume that Einstein's theory of general relativity is true to the limits of our ability to observe and applicable to the systems under examination. We do not address MOND or other post-Einsteinian models.

Throughout this chapter, "emitter" refers to the source of light rays being observed;

"deflector" refers to the mass distribution causing a change in the metric of space-time from flatness; and "observer" refers to the point where the light rays produced by the emitter are observed.

We also make use of a coordinate system derived from the Cartesian system, defined thus: in a space that is asymptotically Cartesian let a line be described by

$$x^i\left(t\right) = k^i\left(t - t_0\right) + x_0^i. \tag{3.1}$$

Let t^* be the time associated with the line's closest approach to the origin of the Cartesian system. Let $\tau = t - t^*$ denote a new time coordinate (that is, at $\tau = 0$ the line reaches the closest point to the origin of both the Cartesian and projected systems). Space coordinates are projected onto a plane passing through the origin of the coordinate system and perpendicular to a line from the observer to the origin of the coordinate system; these new space coordinates are denoted $\xi^i = \Pi^{ij} x^j\left(t^*\right)$ where the projection operator is defined $\Pi^{ij} \equiv \delta^{ij} - k^i k^j$. In the projected coordinate system, the index 0 refers to τ and the indices i denote coordinates ξ^i.

For a trajectory described by (3.1) let $\xi^j \equiv \Pi^{ij}\left. x^i\left(\tau\right)\right|_{\tau=0}$ be the "vector impact parameter" of the trajectory and let $d \equiv \left|\xi^i\right|$ be the "scalar impact parameter" of the trajectory. Since the space is asymptotically flat, d is also the ratio of the magnitudes of the angular and linear momenta of the light ray. Note then that for the trajectory described by $x^i\left(\tau\right)$, $r\left(\tau\right) = \sqrt{d^2 + \tau^2}^2$. Let the unit vector $n^i \equiv \xi^i / d$.

We assume that the wavelengths of all light rays observed are much shorter than

[2]In this chapter we use the convention $G = c = 1$ to simplify our equations.

the longest wavelength of gravitational radiation emitted by the deflecting system.

3.2.2 General theory

Background

Consider a photon emitted at some distant point x_0^i at some time in the distant past t_0. This beam of light in asymptotically flat space follows a path k^i such that the coordinate x^i of the photon is given by the relation (3.1); therefore, $k^i = \left.\frac{\partial x^i}{\partial t}\right|_{t=-\infty}$. Let k^i be normalized such that $k^i k_i = 1$; then the vector $k^\alpha = (1, k^i)$ is parallel to the four-momentum of the photon in flat space.

Let an asymptotically-flat metric $g_{\alpha\beta}$[3] be a function of some affine parameter λ. Let $K^\alpha \equiv k^\alpha + \kappa^\alpha(\lambda) + \Xi^\alpha(\lambda)$ be the trajectory of a photon moving in this metric space, where κ^α describes the part of the trajectory arising from the spherically-symmetric non-flat part of the metric and Ξ^α describes the trajectory arising from a perturbation to the metric. Then, we have the geodesic equation [4, equation 87.3]

$$\frac{d\left(\kappa^\alpha + \Xi^\alpha\right)}{d\lambda} + \Gamma^\alpha_{\beta\gamma} K^\beta K^\gamma = 0. \tag{3.2}$$

The quantity $d\left(\kappa^\alpha + \Xi^\alpha\right)/d\lambda$ corresponds to the change in momentum of the light ray in space, which when projected onto a plane of observation corresponds to the

[3] All metrics $g_{\mu\nu}$ in this chapter are stated using the harmonic gauge condition, that is, $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$. The Minkowski metric in Cartesian coordinates is chosen with signature $(-,+,+,+)$ and is denoted $\eta_{\mu\nu}$, and we make use of the Einstein summation convention.

angular deflection of the light ray. We define this deflection vector by [119, 120]

$$\alpha^i\left(t, \xi^i\right) \equiv \Pi^i_j \left[\kappa^j + \Xi^j\right]_{\text{observer}} + \Delta\alpha^i \tag{3.3}$$

where the term $\Delta\alpha^i$ corresponds to corrections arising from any contribution to deflection other than our deflector.

In the case of Eddington's experiment [116] on solar deflection, the "true" position of the emitter – that is, the position of the emitter observed in the limit of intervening deflection going to zero – was known. In the case of deflectors with small proper motion, in this case extragalactic or otherwise distant objects, where the emitter would be seen without the intervening deflector may not be known; therefore, the periastron of the light ray must be determined by other means. Let P be the periastron of the light ray's trajectory about the deflector; in such cases, the time delay between the deflection and the motion of the deflector is related to the periastron by

$$P = t_{\text{peak deflection}} - t_{\text{alignment}} \tag{3.4}$$

where $t_{\text{peak deflection}}$ is the time when the image of the source is observed to be deflected most from the position of the deflector and $t_{\text{alignment}}$ is the time when the projected components of the system and the deflected image fall into a line, assuming that $P < \frac{p}{2}$ and that the change in the gravitational field propagates at the speed of light.

Description of deflector

Our deflector of interest is as follows: two objects are denoted with the indices 1 and 2. The mass of object 1 $m_1 \geq m_2$. The objects have positions $x_1^i(t)$ and $x_2^i(t)$ and velocities $v_1^i(t)$ and $v_1^i(t)$[Figure 3.1]. Then our source has density distribution

$$\rho\left(t, x^i\right) = m_1 \delta\left(x^i - x_1^i(t)\right) + m_2 \delta\left(x^i - x_2^i(t)\right) \tag{3.5}$$

and velocity distribution

$$v^i\left(t, x^j\right) = v_1^i \delta\left(x^j - x_1^j\right) + v_2^i \delta\left(x^j - x_2^j\right) \tag{3.6}$$

where $\delta\left(x^i\right)$ is the three-dimensional Dirac delta distribution.

Our metric has the form $g_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu} + h_{\mu\nu}$ where $s_{\mu\nu}$ is the non-Minkowski part of the Schwarzschild metric and $h_{\mu\nu}$ is a small perturbation. Let $h^Q_{\mu\nu}$ be the perturbation resulting from the quadrupole moment of the mass distribution and let $h^S_{\mu\nu}$ be the perturbation resulting from the spin dipole of the mass distribution. Let the variable $s = t - r$. Then explicitly, the metric is given by [120, 139, 142, 4, p 181],

$$s_{00} = 2m/r \tag{3.7}$$

$$s_{0i} = 0 \tag{3.8}$$

$$s_{ij} = \left[\left(1-\frac{2m}{r}\right)^{-1} - 1\right]\frac{x^i}{r^2}\delta_{ij} \tag{3.9}$$

$$h^Q_{00} = \frac{\partial^2}{\partial x^i \partial x^j}\frac{\mathcal{Q}_{ij}(s)}{r} \tag{3.10}$$

$$h^Q_{0i} = -2\frac{\partial^2}{\partial x^i \partial t}\frac{\mathcal{Q}_{ij}(s)}{r} \tag{3.11}$$

$$h^Q_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}\frac{\mathcal{Q}_{ij}(s)}{r}\delta_{ij} + 2\frac{\partial^2}{\partial t^2}\frac{\mathcal{Q}_{ij}(s)}{r} \tag{3.12}$$

$$h^S_{0i} = 2\frac{S_j x_k \epsilon_i^{jk}}{r^3} \tag{3.13}$$

$$h^S_{00} = h^S_{ij} = 0 \tag{3.14}$$

where the vector $S^i \equiv (J^{23}, J^{31}, J^{12})$ and $J^{ij} \equiv \int (x^i T^{j0} - x^j T^{i0})\, dV$ [4, chapter 2.9].

Let the objects orbit one another with a known period p. Let our coordinate system origin be located at the center of mass of the binary and let $m = m_1 + m_2$. Let $a^i \equiv x_1^i - x_2^i$ be a vector denoting the spatial separation of the two masses and $l \equiv |a^i|$. Let the mass ratio $b \equiv \frac{m_2}{m_1} \leq 1$.

By our choice of coordinates, the dipole term of the deflector's mass distribution is zero.

Using the Landau-Lifschitz definition of the transverse traceless quadrupole [2, equation 41.3], the quadrupole moment of the deflector is:

$$\mathcal{Q}_{ij}(t) = \int \rho(\mathbf{x}, t) \left[3x_i x_j - r^2 \delta_{ij}\right] dV = \frac{mb}{(1+b)^2} \left[3a_i a_j - l^2 \delta_{ij}\right]. \tag{3.15}$$

In the case that the masses travel in almost circular orbits about their common center of mass, then in a primed coordinate system related to our chosen system only by unitary rotations,

$$a'^i(t) = l \begin{pmatrix} \sin\left(\frac{2\pi t}{p} + \phi'\right) \\ 0 \\ \cos\left(\frac{2\pi t}{p} + \phi'\right) \end{pmatrix} + \delta a'^i(t) \tag{3.16}$$

where ϕ' represents a constant phase term , and where $\delta a'^i$ is small. Rotating from the primed system first about the y-axis, then the x-axis, then the z-axis, we have

$$a^i(t) = l \begin{pmatrix} \cos\Psi \sin\left(\frac{2\pi t}{p} + \phi\right) + \sin\Psi \sin\Theta \cos\left(\frac{2\pi t}{p} + \phi\right) \\ -\sin\Psi \sin\left(\frac{2\pi t}{p} + \phi\right) + \cos\Psi \sin\Theta \cos\left(\frac{2\pi t}{p} + \phi\right) \\ \cos\Theta \cos\left(\frac{2\pi t}{p} + \phi\right) \end{pmatrix} + \delta a^i(t) \tag{3.17}$$

where ϕ subsumes rotation about the y-axis with ϕ' and where Θ and Ψ are the angles of rotation of the plane of motion away from the xz-plane about the x- and z-axes respectively.

Solution to the geodesic equation

The theory of the effects of small perturbations to the metric on light propagation in the weak-field limit is already developed [119, 120]. However, since the effects

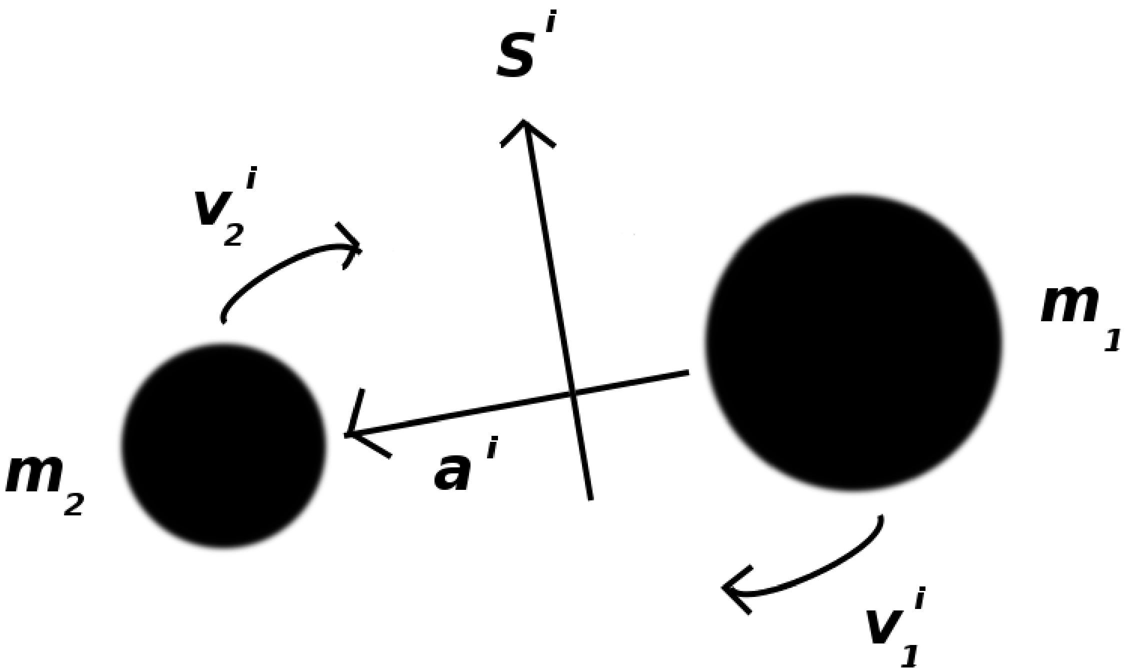

Figure 3.1: Configuration of a binary deflector
Object 1 has mass m_1, velocity v_1^i and is located as position x_1^i; object 2 has corresponding m_2, v_2^i and x_2^i. $x_1^i - x_2^i = a^i$ and the spin vector S^i where $S^i \equiv (J^{23}, J^{31}, J^{12})$ and $J^{ij} \equiv \int (x^i T^{j0} - x^j T^{i0})\, dV$ is perpendicular to a^i, v_1^i and v_2^i.

of a quadrupolar perturbation fall off as d^3, it is desirable to expand the theory to be applicable to regions of stronger fields. We note in particular that for a closely-orbiting compact binary system, such as an evolved SMBHB, then m and l will be of similar magnitude; therefore, we extend the first-order theory of light deflection to order $\mathcal{O}\left(m/d\right)^3$.

First, note that all terms in (3.8) are $\mathcal{O}\left(m/r\right)$ or higher and that all terms in (3.11) are of $\mathcal{O}\left(ml^2/r^3\right) \leq \mathcal{O}\left(m^3/r^3\right)$. Let $\mathcal{O}\left(m^3/r^3\right)$ be small such that all higher orders are negligible. Then, suppressing negligible terms,

$$\Gamma^{\alpha}_{\beta\gamma} = -\frac{1}{2}\left(\eta^{\alpha\delta} + s^{\alpha\delta}\right)\left(s_{\beta\delta,\gamma} + s_{\gamma\delta,\beta} - s_{\beta\gamma,\delta}\right) - \frac{1}{2}\left(\eta^{\alpha\delta}\right)\left(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta}\right). \tag{3.18}$$

Let the Christoffel symbol associated with the Schwarzschild metric $\Gamma^{\alpha(\mathrm{S})}_{\beta\gamma} \equiv -\frac{1}{2}\left(\eta^{\alpha\delta} + s^{\alpha\delta}\right)\left(s_{\beta\delta,\gamma}\right.$

and the remaining part resulting from the perturbation $\Gamma^{\alpha(\mathrm{h})}_{\beta\gamma} \equiv -\frac{1}{2}\left(\eta^{\alpha\delta}\right)\left(h_{\beta\delta,\gamma}+h_{\gamma\delta,\beta}-h_{\beta\gamma,\delta}\right)$. Then (3.2) becomes

$$\dot{\kappa}^{\alpha}+\dot{\Xi}^{\alpha}+\left(\Gamma^{\alpha(\mathrm{S})}_{\beta\gamma}+\Gamma^{\alpha(\mathrm{h})}_{\beta\gamma}\right)\left(k^{\beta}+\kappa^{\beta}+\Xi^{\beta}\right)\left(k^{\gamma}+\kappa^{\gamma}+\Xi^{\gamma}\right)=0. \tag{3.19}$$

Since all $\Gamma^{\alpha(\mathrm{S})}_{\beta\gamma}$ and all components of κ^{α} must be at least of $\mathcal{O}\left(m/r\right)$ or zero, (3.19) expands, again suppressing negligible terms, to

$$\dot{\kappa}^{\alpha}+\dot{\Xi}^{\alpha}+\Gamma^{\alpha(\mathrm{S})}_{\beta\gamma}\left(k^{\beta}+\kappa^{\beta}\right)\left(k^{\gamma}+\kappa^{\gamma}\right)+\Gamma^{\alpha(\mathrm{h})}_{\beta\gamma}k^{\beta}k^{\gamma}=0. \tag{3.20}$$

Since

$$\dot{\kappa}^{\alpha}+\Gamma^{\alpha(\mathrm{S})}_{\beta\gamma}\left(k^{\beta}+\kappa^{\beta}\right)\left(k^{\gamma}+\kappa^{\gamma}\right)=0, \tag{3.21}$$

we conclude

$$\dot{\Xi}^{\alpha}+\Gamma^{\alpha(\mathrm{h})}_{\beta\gamma}k^{\beta}k^{\gamma}=0 \tag{3.22}$$

which is exactly the result for the weak-field approximation [119, 120].

Plugging (3.21) and (3.22) into (3.3) and choosing τ as our affine parameter, we can define the Schwarzschild and non-Schwarzschild parts of the deflection angle [FIGURE 3.2]:

$$\alpha^{i}_{M}\left(\xi^{i}\right) \equiv \Pi^{i}_{j}\kappa^{j} \tag{3.23}$$

$$\alpha^{i}_{h}\left(t,\xi^{i}\right) \equiv \Pi^{i}_{j}\Xi^{j} = -\frac{1}{2}\Pi^{ij}\int_{-\infty}^{\infty}\left(h_{\beta\delta,j}+h_{j\delta,\beta}-h_{\beta j,\delta}\right)k^{\beta}k^{\gamma}d\tau. \tag{3.24}$$

The monopole term $\alpha^{i}_{M}\left(\xi^{i}\right)$ of the deflection produced by the core is static and unique, regardless of changes of configuration within the core [118, 116]. We can use the general, exact solution for κ^{α} provided by Darwin [130]:

Choose spherical coordinates. By the symmetry of the monopole term, this part of the trajectory of the light ray must lie in a plane, so we can choose the coordinate θ as an affine parameter and the coordinate ϕ as constant. Then we obtain an equation of motion

$$-\frac{r-2m}{r}\left(\frac{dt}{d\theta}\right)^2+\frac{r}{r-2m}\left(\frac{dr}{d\theta}\right)^2+r^2=0. \tag{3.25}$$

Identifying the impact parameter with a conserved quantity in the system $\frac{r^3}{r-2m}\frac{dt}{d\theta}=d$ and substituting in yields three solutions; we discard the two where the light ray never reaches a distant observer and take the remaining one,

$$\frac{1}{r\left(\theta\right)}=-\frac{V-U+2m}{4mU}+\frac{V-U+6m}{4mU}\mathrm{sn}^2\zeta\left(\theta\right) \tag{3.26}$$

where the constant V is defined by $V^2\equiv\left(U-2m\right)\left(U+6m\right)$, the periastron and impact parameter are related by $d^2\equiv U^3/\left(U-2m\right)$ and $\zeta\left(\theta\right)\equiv\sqrt{\frac{V}{U}}\left(\theta+\theta_0\right)$, and $\mathrm{sn}\zeta$ is the Jacobi elliptic sn function [143, 16.1.5]. In the limit of $U\gg m$, inverting this relationship and taking its asymptotic limits at large r leads to the well-known relationship

$$\alpha^i_{M,\text{weak field}}\left(\xi^i\right)=\frac{4m}{d}n^i. \tag{3.27}$$

As $U\to 3m$, however, the deflection becomes [141]

$$\mu\left(\xi^i\right)=\left(\ln\frac{m}{d}+\ln\left[648\left(7\sqrt{3}-12\right)\right]-\pi\right)\approx\left(\ln\frac{m}{d}+1.248\right) \tag{3.28}$$

where μ is the angle of deflection about the apse of the trajectory, rather than the deflection seen by a distant observer; the angles involved are no longer necessarily small so we cannot approximate $\alpha_M=\mu$. In the case of an impact parameter comparable to $3m$, it is no longer observationally useful to consider the monopolar displacement in and of itself as small differences in impact parameter cause great changes in deflection angle, and multiple images of a source may be detectable, some of which may result from geodesics which travel several times around the deflector. Our consideration therefore must focus not on the static deflection but on time-dependent deflections arising from higher multipole moments of the deflector.

Kopeikin and Mashhoon [139] develop the effect of the rotation of a system on that system's deflection of light, in the weak field approximation. Investigation of this effect is useful for the system as described in that every practical case of an astronomical binary will display orbital motion. However, the theory developed by Kopeikin and Mashhoon is only sometimes compatible with the strong-field approximation presented herein.

The integration of (3.13) is trivial. Let $\alpha^i_S\left(\xi^i\right)$ be that part of α^i_h determined by $h^S_{\mu\nu}$. when the deflector is stationary relative to the observer, the resulting contribution is given by

$$\alpha^i_S\left(\xi^i\right) = \tfrac{2}{d^2} \left[2S^j k^k n^l \delta^m_j \epsilon_{klm} n^i + k^j S^k \epsilon^i_{jk}\right]. \tag{3.29}$$

Calculating S^i with (3.17) for the case of a binary whose components are in almost-circular orbits,

$$S^i = -m\frac{b}{1+b}\frac{2\pi l^2}{p}\left(\frac{b}{\left(1+b\right)^2 - b^2\left(2\pi l/p\right)^2} + \frac{1}{\left(1+b\right)^2 - \left(2\pi l/p\right)^2}\right)\begin{pmatrix}\sin\Psi\cos\Theta \\ \cos\Psi\cos\Theta \\ -\sin\Theta\end{pmatrix}. \tag{3.30}$$

We must emphasize that (3.29) is compatible with the $\mathcal{O}\left(m^3/r^3\right)$ generalization above only when $\mathcal{O}\left(ml^2/d^2p\right) \geq \mathcal{O}\left(m^3/d^3\right)$; in particular, when $l \approx p$ the system's motion is no longer slow. We draw attention to this contribution to emphasize the difficulty in associating an image with a particular source and to underscore the utility of time-dependent deflection versus time-independent deflection in parametrizing a system.

Let $\alpha_Q^i\left(t,\xi^i\right)$ be that part of α_h^i determined by $h_{\mu\nu}^Q$. α_Q^i is determined by plugging (3.11) into (3.24); while [120] uses the method of Fourier transforms, the form of (3.11) allows direct integration of a Fourier series decomposition as well. Either way, the result is the following deflection [119, 120][4]:

$$\alpha_Q^i\left(t,\xi^i\right)=\frac{12}{d^3}\frac{mb}{\left(1+b\right)^2}\left[\left(a_2^2\left(s\right)-a_1^2\left(s\right)\right)n^i-a_1\left(s\right)a_2\left(s\right)\epsilon^i_{\ jk}k^jn^k\right] \tag{3.31}$$

for which we reiterate the following properties: firstly, in contrast to the monopolar case where α_M^i always points along ξ^i, the quadrupolar deflection has a contribution parallel toξ^i, $\alpha_{Q\|}$, and also a contribution perpendicular to ξ^i, $\alpha_{Q\perp}$, which vanishes only in the case that a component of the projected quadrupole moment vanishes, that is, only if the axis of rotation of the deflector is perpendicular to our line of sight; and secondly, the deflection depends only on the configuration of the deflector at the time of the light ray's closest approach to the center of mass, $t=t^*$.

[4]The symbol ϵ_{ijk} represents the Levi-Civita permutation symbol defined such that $\epsilon_{123}=1$.

In the case of almost-circular motion, inserting (3.17) into (3.31) leads to

$$\alpha^i_{Q\parallel}\left(t,\xi^i\right) = \frac{12l^2}{d^3}\frac{mb}{\left(1+b\right)^2}\left\{\begin{array}{c}\frac{1}{2}\cos 2\Psi\left[\begin{array}{c}\left(1+\sin^2\Theta\right)\cos\left(\frac{4\pi s}{p}+2\phi\right)\\ +\sin^2\Theta-1\end{array}\right]\\ -\sin 2\Psi\sin\Theta\sin\left(\frac{4\pi s}{p}+2\phi\right)\end{array}\right\}n^i \tag{3.32}$$

$$\alpha^i_{Q\perp}\left(t,\xi^i\right) = -\frac{6l^2}{d^3}\frac{mb}{\left(1+b\right)^2}\times \tag{3.33}$$

$$\times\left\{\begin{array}{c}\frac{1}{2}\sin 2\Psi\left[\begin{array}{c}\left(-1+\sin^2\Theta\right)\\ +\left(1+\sin^2\Theta\right)\cos\left(\frac{4\pi s}{p}+2\phi\right)\end{array}\right]\\ +\cos 2\Psi\sin\Theta\sin\left(\frac{4\pi s}{p}+2\phi\right)\end{array}\right\}\epsilon^i_{jk}k^j n^k.$$

The relationship (3.33) is original to this work and has not previously appeared. From this relationship it is easy to see that the time-dependent deflection of the emitter's image is periodic, with a period half that of the orbit of the core's components.

The greatest time-dependent deflection is observed when the emitter lies on the line of the semi-major axis of the apparent motion; when $b = 1$; and when the plane of the system lies perpendicular to the plane of observation. In this case, (3.33) reduces to

$$\alpha_Q\left(t,d\right) \leq \frac{3l^2}{2d^3}m\left[\cos\left(\frac{4\pi s}{p}+2\phi\right)-1\right] \tag{3.34}$$

so the total quadrupolar deflection seen over one half-period of the deflector's motion is

$$\Delta\alpha_Q\left(d\right) \leq -\frac{3l^2}{d^3}m. \tag{3.35}$$

Compared to the monopole deflection in the case of a large impact parameter,

$$\left|\frac{\Delta\alpha_Q}{\alpha_M}\right| \leq \frac{3l^2}{d^2}. \tag{3.36}$$

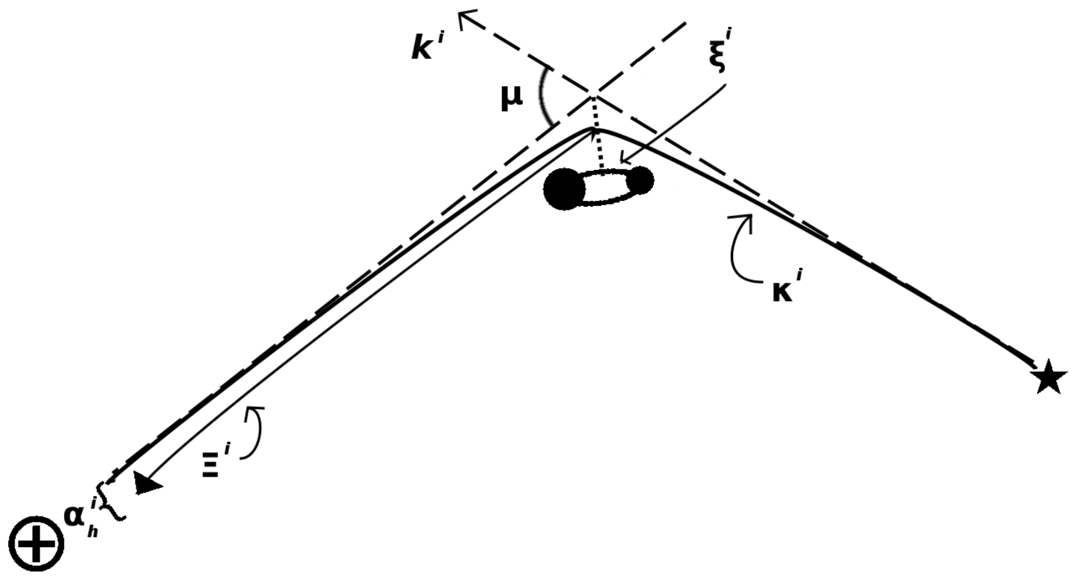

Figure 3.2: Light deflection by a binary system
A light ray produced by the emitter initially follows trajectory k^i, which has its closest approach to the origin of the coordinate system at ξ^i. In a pure Schwarzschild space, the light ray follows trajectory $k^i + \kappa^i(\lambda)$ and is deflected about the apse of its trajectory by angle μ; in a perturbed Schwarzschild space, it follows trajectory $k^i + \kappa^i(\lambda) + \Xi^i(\lambda)$ and the observer on Earth ($\oplus$) records an additional deflection of α^i_h.

Other contributions to the deflection angle

If the path of the light ray after its closest approach to the deflector but far from the deflector is nearly occulted (for example the Sun or another star), then deflection from this intermediate deflector, $\alpha^i_{\text{intermediate}}\left(\xi^i_{n,\text{int}}\right)$, must be taken into account as well. Where$\xi^i_{n,\mathrm{int}}$ refer to the vector impact parameters of light relative to these intermediate deflectors, $d_n \equiv \left|\xi^i_{n,\mathrm{int}}\right|$, and m_n are the masses of these deflectors, and where m_n/d_n is small for all n,

$$\alpha^i_{\mathrm{intermediate}}\left(\xi^i_n\right) = -\frac{4m_n}{d_n}\frac{\xi^i_{n,\mathrm{int}}}{d_n}. \tag{3.37}$$

In linear approximation and in the harmonic gauge, the various deflections can be superposed linearly. The total deflection of the light ray from our source, therefore, is given by

$$\alpha^i\left(t,\xi^i\right) = \alpha^i_Q\left(t,\xi^i\right) + \alpha^i_M\left(\xi^i\right) + \alpha^i_S\left(\xi^i\right) + \alpha^i_{\mathrm{intermediate}}\left(\xi^i_{n,\mathrm{int}}\right). \tag{3.38}$$

3.2.3 Application to 3C66B

3C66B, also known as 0220+43, is a radio galaxy [123] with $z = .0215$ [124], approximately 91 Mpc distant from the Milky Way[5]. 3C66B exhibits jets emerging from its core, making it a good candidate for the location of a SMBHB [126]. Thus far, no other candidate SMBHB has emerged with an orbital period as short as 3C66B's [144] and the system is estimated to have an inspiral time on the order of centuries [145].

[5]We use a value of 71 km/s/Mpc for the Hubble constant H_0 for all distance calculations.[125]

Parameters of the system

Sudou *et al.* [127] give upper-bound estimates by direct radio observation for 3C66B's core, including an upper limit on m, a period, and an orientation of the core's motion. Sudou also reports a limit on the minimum impact parameter available for determining the parameters of the system using a first-order approximation theory, corresponding to the limit of optical transparency at VLBI's higher operating frequency. The upper limits Sudou gives are:

$$m \leq 4.4\,(1+b)^2 \times 10^{10}\text{solar mass} = 6.5\,(1+b)^2 \times 10^{15}\text{cm} \tag{3.39}$$

$$l \leq 5.1\,(1+b) \times 10^{16}\text{cm} \tag{3.40}$$

$$U \geq 23\mu\text{as} = 3.1 \times 10^{16}\text{cm} \tag{3.41}$$

$$d \geq 3.7 \times 10^{16}\text{cm} \tag{3.42}$$

$$p = 1.05 \pm .03\text{years} \tag{3.43}$$

$$\Theta = 15^\circ \pm 7^\circ \tag{3.44}$$

where P is constrained by the limit of the core's opacity in the radio spectrum and Θ is derived from the apparent eccentricities of the elliptical boundaries of radio opacity. From l and P we can furthermore conclude that in the case of maximized l, under Sudou's estimates $q \leq .20$.

Iguchi *et al.* have recently published [145] new estimates constraining 3C66B:

$$m \approx 1.9 \times 10^9 \text{solar mass} = 5.6 \times 10^{14}\text{cm} \tag{3.45}$$

$$l \approx 1.9 \times 10^{16}\text{cm} \tag{3.46}$$

$$U \approx 1.2 \times 10^{16}\text{cm} \tag{3.47}$$

$$d \gtrsim 3.7 \times 10^{16}\text{cm} \tag{3.48}$$

$$p = 1.05 \pm .03\text{years} \tag{3.49}$$

$$\Theta = 15^\circ \pm 7^\circ \tag{3.50}$$

$$b \approx 0.58. \tag{3.51}$$

Estimates for distant emitters

Although highly eccentric motion in 3C66B is not ruled out [128], the age of the presumed binary is great enough to have circularized the orbit through gravitational radiation under most conditions [129]. We present the case of circular motion as an upper limit on the time-dependent deflection angle, noting that if all other parameters are constant then in the case of eccentric motion any time-dependent separation of the masses must have l as an upper bound.

Using the maximal figure for mass and the minimal figure for impact parameter in (3.39-3.44) and applying (3.28), the ratio $m/l = .30$, placing our proposed system in the regime of strong deflection. We find a monopolar deflection of:

$$\mu = \left(\ln \left(\frac{6.5 \times 10^{15}\text{cm}}{3.7 \times 10^{16}\text{cm}} (1.44) \right) + 1.248 \right) = .13\text{radian} = 7.2^\circ. \tag{3.52}$$

The components of the system as proposed by Sudou have $\frac{2\pi l}{p} \leq .39$. Therefore it is not reasonable to apply (3.29) to 3C66B in the regime where deflection from the quadrupole moment will be detectable.

Deflected images lying along the major axis of the core with the system as constrained in (3.39-3.44) will have time-dependent deflections in the following amounts:

$$\begin{aligned} \Delta\alpha_{Q\parallel}\left(d\right) \leq& \frac{12l^2}{d^3}\frac{mb}{\left(1+b\right)^2}\left(1.07\right) \\ \leq& \frac{12\left(5.1\times10^{16}\text{cm}\right)^2\left(1.2\right)^2}{\left(3.7\times10^{16}\text{cm}\right)^3}\frac{\left(6.5\times10^{15}\text{cm}\right)\left(1.2\right)^2\left(.2\right)}{\left(1.2\right)^2}\left(1.07\right) \\ \leq& 5.8\times10^{-5}\text{arcsecond} \end{aligned} \tag{3.53}$$

$$\begin{aligned} \Delta\alpha_{Q\perp}\left(d\right) \leq& \frac{24l^2}{d^3}\frac{mb}{\left(1+b\right)^2}\left(.26\right) \\ \leq& \frac{24\left(5.1\times10^{16}\text{cm}\right)^2\left(1.2\right)^2}{\left(3.7\times10^{16}\text{cm}\right)^3}\frac{\left(6.5\times10^{15}\text{cm}\right)\left(1.2\right)^2\left(.2\right)}{\left(1.2\right)^2}\left(.26\right) \\ \leq& 1.4\times10^{-5}\text{arcsecond} \end{aligned} \tag{3.54}$$

with a period of $p/2 = .53 \pm .02$years for each component of the deflection.

Under Iguchi *et al.*'s new estimates, the deflections take on the following values:

$$\Delta\alpha^i_{Q\parallel}\left(t,\xi^i\right) \approx 3.7\times10^{-7}\text{arcsecond} \tag{3.55}$$

$$\Delta\alpha^i_{Q\perp}\left(t,\xi^i\right) \approx 8.8\times10^{-8}\text{arcsecond.} \tag{3.56}$$

3.3 Observational techniques

3.3.1 Interferometry

Electromagnetic interferometry provides the best currently-available techniques for high-resolution astronomy. The use of space-based interferometry and improvements in equipment allowing for higher frequencies of observation continue to steadily improve resolution capabilities. The current most powerful technique available is VLBI, which Sudou *et al.* used to determine the motion in the core of 3C66B [127].

VLBA, the Very Long Baseline Array, is an array of ten radio telescopes [133] operating in wavelengths as short as 3mm operating as a single large interferometer. The current best available resolution is 1.7×10^{-5}arcsecond [134], making VLBA currently capable of constraining the parameters of 3C66B further through direct observation as well as the Jenet pulsar timing experiment described below accomplishes indirectly. The launch of the space-based ASTRO-G satellite [135] will extend the resolution capabilities further.

The SIM PlanetQuest mission (formerly Space Interferometry Mission), currently scheduled for launch in 2015 [136], is expected to have a resolution capability of 4×10^{-6}arcsecond [137]. SIM will operate in the optical band and quasar observation is part of the planned mission.

Farther into the future, the MAXIM (Micro-Arcsecond X-ray Interferometry Mission) satellite array currently in development [138] is expected to give resolutions on the order of 10^{-7}arcsecond in the x-ray band, and is explicitly designed with the observation of black holes in mind.

3.3.2 Pulsar timing

Jenet *et al.* [117] examined the period of the pulsar PSR B1855+09 for changes in its period over several years, motivated by the idea that as gravitational waves generated by the core of 3C66B pass near the pulsar, the pulsar's signal should be modulated with a period related to the period of the proposed 3C66B SMBHB. The distance between the Solar System and the pulsar furthermore give the advantage that the signals observed modulating the pulsar are some 4000 years older than the motion observed in the 3C66B core. However, Jenet's experiment produced a null result.

The experiment's analysis involved examining the frequency space of the pulsar's signal for components in a range from $1/27.8\text{yr}^{-1}$ to 19.5yr^{-1}, then subtracting out the one-year and six-month components resulting from geodetic effects. The results are described as showing no signal distinguishable from noise other than the already-known main oscillation frequencies of the pulsar. Therefore the magnitude of gravitational waves generated by 3C66B, and consequently the parameters of its core, can be further constrained.

Jenet *et al.* assert that a system with $m\left(\frac{b}{(1+b)_t^2}\right)^{3/.5} \geq .7 \times 10^{10}$solar mass can be ruled out by the observed null result in the change in pulsar periods over seven years; this corresponds in the optimal case of $q = .2$ to a system with $m = 2.3 \times 10^{10}$solar mass $= 3.4 \times 10^{15}$cm. For a system under these new constraints, we estimate optimal peak deflections:

$$\Delta\alpha_{Q\parallel}(d) \leq \frac{12\,(5.1\times10^{16}\text{cm})^2\,(1.2)^2}{(3.1\times10^{16}\text{cm})^3}\left(3.4\times10^{15}\text{cm}\right)\frac{(.2)}{(1.2)^2}(1.07) = \quad 2.1\times10^{-5}\text{arcsecond} \tag{3.57}$$

$$\Delta\alpha_{Q\perp}(d) \leq \frac{12\,(5.1\times10^{16}\text{cm})^2\,(1.2)^2}{(3.1\times10^{16}\text{cm})^3}\left(3.4\times10^{15}\text{cm}\right)\frac{(.2)}{(1.2)^2}(.26) = \quad 5.0\times10^{-6}\text{arcsecond} \tag{3.58}$$

which remain within the detection limit of VLBA as currently configured.

3.4 Conclusions

A theory of light deflection by time-dependent distributions of matter has been presented for metrics which are perturbations of the Schwarzschild metric, accounting for deflection resulting from time-independent and time-dependent terms in the metric. To order m^3/r^3, deflections originating from the quadrupole moment of the mass distribution and, with some constraints, the dipole moment of the system's spin can be linearly superposed on the system as if in a weak-field approximation. The theory can be practically evaluated for and applied to a model of the core of an active galaxy, but the theory of light deflection from the spin of the deflector needs further development for applicability in the regime of strong deflection.

The examination of time-dependent light deflection is a feasible technique for the evaluation of proposed SMBHB systems, under idealized circumstances. In the event that a suitable emitter exists, examination of light deflection can be used to constrain the parameters of the proposed SMBHB in the core of 3C66B. We emphasize that while the existence of an identifiable suitable emitter in the case of 3C66B is unlikely, the theory can be applied equally well to any other SMBHB candidate, any of which may have a suitable source; in particular, active galaxies displaying Einstein rings or other artifacts of strong gravitational lensing, especially multiple images, should be examined. The theory can be equally well applied to intragalactic objects, although nearer objects will require further corrections for proper motion.

The quadrupolar motion in the core of 3C66B can be examined and evaluated by the observation of deflected images in the region of the sky near the core of the galaxy, if found; the time-dependent part of the deflection has a magnitude of up to 58 microarcseconds parallel to the impact parameter of the emitter and up to 28 microarcseconds perpendicular to the impact parameter.

For the case of 3C66B, for most emitters pulsar timing can constrain the parameters of the deflecting system better than time-dependent light deflection can. VLBA in its current configuration is capable of constraining the parameters of the core of 3C66B under ideal circumstances, but newer estimates of the parameters of the system show a change in angular light deflection considerably smaller than what VLBA could resolve. Anticipated interferometers will have resolutions up to two orders of magnitude greater and will be capable of evaluating the parameters of the system while examining it in a wide range of frequencies, and may make the observation of time-dependent light deflection resulting from motion in the core of 3C66B more practical.

Chapter 4

The Bianchi IX cosmology

In pursuit of a theory within the context of unmodified general relativity which can explain cosmic acceleration while remaining compatible with the cosmic microwave background, we wish to relax as few constraints on our cosmological model as necessary. Therefore while having sacrificed the requirement of isotropy in the sense of spherical symmetry in the dark energy field, we wish to retain a stronger [2, ss. 116] condition of the Copernican principle on our space, that of homogeneity [4, Chap. 13 sec. 1]. It is also desirable to have a model whose limiting case is a Friedmann cosmology, in order to explain the almost-isotropic (that is, almost-Friedmannian) character of the CMB. Furthermore, a model which is spatially closed, in order to match models favored by CMB and SNe data, is desirable; such a model would, if complying with all other conditions, have a flat universe as a limiting case in the limit of an infinitely large radius of curvature.

Bianchi showed [70] that there exists exactly one homogeneous[1] space with a closed Friedmann universe as a limiting case: the Bianchi type IX cosmology, for short "Bianchi IX".

4.1 The Bianchi classification scheme

Bianchi observed that all three-dimensional homogeneous spaces could be classified into nine types, based on categorization of the symmetries, that is the Killing field, in each space. Behr noted [71] that this categorization scheme could be simplified to filling a parameter space of four parameters: one running over the real numbers and three reducible to the sign function $\mathrm{sgn}\,(x)$.

Consider some space with metric $ds^2 = dt^2 - g_{ij}dx^i dx^j$ (that is, a space in Gaussian coordinates) where $g_{ij} = g_{ij}\,(t, x^i)$. If the sub-space with metric tensor g_{ij} is homogeneous, then there exists a set of vectors that solve $\xi_{i;j} + \xi_{j;i} = 0$; these are the Killing vectors of the space[4]. In an homogeneous space, these vectors will (where $[\,,\,]$ is a commutator) obey the commutation relationship

$$[\xi_i, \xi_j] \equiv \xi_i \xi_j - \xi_j \xi_i = C^k_{ij} \xi_k \tag{4.1}$$

where in an homogeneous space, the object C^k_{ij} is a constant pseudo-tensor, the "structure constants" of an homogeneous space, with the antisymmetry property $C^k_{[ij]} = C^k_{ij}$ [2, ss. 116].

[1]A homogeneous space is a space such that for any two points in the space, there exists a geodesic, not necessarily of finite length, connecting those two points.

We always have the freedom to perform separation of variables the functions g_{ij}; let us do so by defining the matrix γ_{ab} such that

$$g_{ij}\left(t,x^k\right)=-\gamma_{ab}\left(t\right)e_i^{(a)}\left(x^k\right)e_j^{(b)}\left(x^k\right) \tag{4.2}$$

.[2] The 3×3 matrix $e_i^{(a)}\left(x^k\right)$ is a triad [2, 112, ss. 98][3] of vectors (“frame vectors”); in the language of linear algebra, the quantity $e_i^a dx^i$ is a one-form on a homogeneous space.

Furthermore, let us define the matrix $e^i_{(a)}$ such that $e^i_{(a)}e_j^{(a)}=\delta^i_j$; from this it follows that $e^i_{(a)}e_i^{(b)}=\delta^{(b)}_{(a)}$. From these relationships we can transform between any tensor and its decomposition into triads by saying that for some tensor $A^{i_1i_2i_3\ldots i_m}_{j_1j_2j_3\ldots j_n}$,

$$A^{i_1i_2i_3\ldots i_m}_{j_1j_2j_3\ldots j_n}=A^{(a)_1(a)_2(a)_3\ldots(a)_m}_{(b)_1(b)_2(b)_3\ldots(b)_n}\left(e^{i_1}_{(a)_1}e^{i_2}_{(a)_2}e^{i_3}_{(a)_3}\ldots e^{i_m}_{(a)_m}\right)\left(e^{(b)_1}_{j_1}e^{(b)_2}_{j_2}e^{(b)_3}_{j_3}\ldots e^{(b)_n}_{j_n}\right); \tag{4.3}$$

therefore in an homogeneous space we can perform separation of variables on the partial differential equations of general relativity and solve the time-dependent parts as ordinary differential equations.

The frame vectors obey the properties

$$e^{(a)}_{i,j}-e^{(a)}_{j,i}=C^a_{\ bc}e_i^{(b)}e_j^{(c)} \tag{4.4}$$

[2]Indices from the beginning of the Latin alphabet (a, b, c,...) denote triad indices; indices from the middle of the alphabet (i, j, k,...) denote regular indices. Where the two are mixed or the application is otherwise ambiguous, triad indices are enclosed in parentheses; in this work, this notation never means the tensor symmetrization operation.

[3]The widespread *Fourth Revised English Edition* of [2] contains numerous serious typographical errors in the section introducing the tetrad formalism. The Russian-language *Seventh Corrected Edition*[112] contains the correct formulas.

[7].

Grishchuk discusses two criteria for an homogeneous space, constrasting two competing definitions; one, originating from Bianchi [70], in which a space is termed "homogeneous" if it admits a group of motions G_3 operates continuously on a space composed of a set of hypersurfaces V_3; that is, if for every point x^i in the space, the operation $gx^i = y^i$ for $g \in G_3$ and y^i is another point in the space; and the other, from Zel'manov, which generalizes Bianchi's definition to three-spaces which are submanifolds of a four-dimensional space-time. Grishchuk finds these two definition to be compatible. The structure constants C^a_{bc} typify a homogeneous space and are given by the following rule [71]:

$$C^a_{bc} = \varepsilon_{bcd} n^{ad} + \delta^d_c a_b - \delta^d_b a_c \tag{4.5}$$

where the object n^{ab} is a diagonal matrix diag $\left(n^{(1)}, n^{(2)}, n^{(3)}\right)$ and a_a is the vector $(a, 0, 0)$, the values of this matrix and vector typifyied by the underlying cosmology (Table 4.1).

The cosmologies of Bianchi types I, V, VII$_0$, VII$_a$ and IX are of particular interest as they have isotropic spaces as limiting cases [151]; specifically, a universe with metric

$$ds^2 = dt^2 - a^2 \eta_{ab} e^{(a)}_i e^{(b)}_j dx^i dx^j \tag{4.6}$$

is a flat $K = 0$ universe for Bianchi type I or VII$_0$, an open $K = -1$ universe for

Bianchi type	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$
I	0	0	0	0
II	0	1	0	0
III	1	0	1	-1
IV	1	0	0	1
V	1	0	0	0
VI_0	0	0	1	-1
VI_a	a	0	1	-1
VII_0	0	1	1	0
VII_a	a	1	1	0
VIII	0	1	1	-1
IX	0	1	1	1

Table 4.1: The Bianchi classification scheme

Constants for the different homogeneous spaces of the Bianchi classification scheme[2, 21, 71, 104]. The quantity a runs over the real numbers. This parametrization is not unique (we could, for example, have chosen $(-1, -1, -1)$ for $\left(n^{(1)}, n^{(2)}, n^{(3)}\right)$ in the type IX space).

Bianchi types V or VII_a and some cases of Bianchi type III (type III is itself a particular case of Bianchi type VI_a [71]), and a closed $K = 1$ universe for Bianchi type IX [2, 10, 21]. Bianchi IX is the only homogeneous closed cosmological model in the context of general relativity [104, 151].

4.2 The Kasner universe

In order to illustrate the possible effects of an anisotropic but homogeneous cosmology on cosmic dynamics, we will consider a Bianchi type I cosmology that generalizes the Friedmann cosmology: the Kasner universe [76]; [2, ss. 117].

Let our metric read

$$ds^2 = dt^2 - t^{2p_1}\left(dx^1\right)^2 - t^{2p_2}\left(dx^2\right)^2 - t^{2p_3}\left(dx^3\right)^2 \tag{4.7}$$

where p_1, p_2, p_3 are constants. In a co-moving coordinate system we quickly arrive at the following set of Einstein equations:

$$\left[(p_1+p_2+p_3) - \left(p_1^2+p_2^2+p_3^2\right)\right] t^{-2} = \frac{1}{2}k\left(\epsilon+3p\right) \tag{4.8}$$

$$\left(p_1+p_2+p_3-1\right) p_1 t^{-2} = \frac{1}{2}k\left(p-\epsilon\right) \tag{4.9}$$

$$\left(p_1+p_2+p_3-1\right) p_2 t^{-2} = \frac{1}{2}k\left(p-\epsilon\right) \tag{4.10}$$

$$\left(p_1+p_2+p_3-1\right) p_3 t^{-2} = \frac{1}{2}k\left(p-\epsilon\right). \tag{4.11}$$

These equations necessitate either an isotropic but unusual ($p = \epsilon$) universe or a vacuum ($\epsilon = p = 0$) universe, in which we have either the trivial solution $p_1 = p_2 = p_3 = 0$ (Minkowski space) or the more interesting solution

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \tag{4.12}$$

This solution admits a parametrization of p_1, p_2, p_3 such that (if we choose $p_1 \leq p_2 \leq p_3$)

$$\begin{aligned} p_1 &= -u/\left(1+u+u^2\right) \\ p_2 &= \left(1+u\right)/\left(1+u+u^2\right) \\ p_3 &= u\left(1+u\right)/\left(1+u+u^2\right) \end{aligned} \tag{4.13}$$

where $u > 0$; these relations have the nice symmetry property that $p_i(u) = p_i(1/u)$.

An implication of (4.12) is that singularities in the Kasner universe fall into two classes: one-dimensional "spindle" singularities where expansion tends toward infinity in one direction while the universe collapses in two directions; and "pancake" singularities where expansion goes to infinity in two directions while the universe collapses in the third.

4.2.1 Scale factor

The scale factor a does not necessarily have an intrinsic meaning, but instead compares distances as a function of time. In an isotropic cosmology such as the Friedmann model a can be given a real geometric meaning; in an open or closed Friedmann universe, the scale factor appears simply in the Ricci curvature of space $R^i_j = (2K/a^2)\,\delta^i_j$. As such, the scale factor can be regarded as the radius of curvature of the universe. In particular, in a closed isotropic universe a can be considered to have the direct physical meaning of the radius of curvature of the spherical space, so in a closed isotropic universe one could meaningfully say "the radius of the universe is a".

When space is no longer isotropic, the definition of scale factor breaks down. It is, of course, possible to define any positive function as "the" scale factor. Grishchuk *et al.* [10, section 4], for example, use a metric

$$\begin{aligned}
\gamma_{11} &= \frac{1}{4}a^2 e^{2\alpha} \\
\gamma_{22} &= \frac{1}{4}a^2 e^{2\beta} \\
\gamma_{33} &= \frac{1}{4}a^2 e^{2\gamma}
\end{aligned} \tag{4.14}$$

and propose the definition

$$a^2 \equiv \frac{1}{12}\gamma_{ab}\eta^{ab} \tag{4.15}$$

in the context of a vacuum cosmology, motivated by the coincidence of this definition of the scale factor with one the authors introduce in separating the Bianchi IX metric into background and gravitational-wave parts. The authors also discuss a definition of scale factor such that

$$a^2 \equiv (\det \gamma_{ab})^{1/3}\,. \tag{4.16}$$

This definition has the advantage that it relates the scale factor to a definite physical quantity, a volume element, but it contains a deeper flaw: with such a definition in place the Einstein equations admit no solution other than the background solution at quadratic and higher orders. If we define the quantity

$$\delta \equiv \alpha + \beta + \gamma \tag{4.17}$$

then

$$a^2 \equiv (\det \gamma_{ab})^{1/3} \implies e^{\delta} = 1 \implies \delta = 0. \tag{4.18}$$

In either case, though, discussion of possible definitions of a attempt to solve a problem that does not exist. The question of what definition of scale factor to select is analogous to the question of which of the orthocenter, incenter or circumcenter of a triangle is the "true" center. Consequently, attempting to extract a single scale factor – and thus a single Hubble parameter or a single deceleration parameter – from anisotropic Einstein equations is a fool's errand.

We can, if we wish, split the metric (4.14) into isotropic and anisotropic parts by noting that the quantity $a_F e^{\delta}$ is isotropic and that any two of the quantities $\alpha - \beta$,

$\alpha - \gamma$ and $\beta - \gamma$ combined with $a_F e^{\delta}$ contain all the information needed to describe the metric [22]; pursuing this route would be a distraction from our main task, however.

Alternately, we could follow [2]'s Kasner-like approach to the Bianchi IX cosmology and deal with only the metric coefficients as $\gamma_{11} = a^2, \gamma_{22} = b^2, \gamma_{33} = c^2$ for functions a, b, c, ignoring the idea of a unique "scale factor" or Friedmann-like behavior. This approach will obscure the nature of the cosmology discussed below.

Instead, let the notion of scale factor a, Hubble parameter H and deceleration parameter Q be generalized. In a homogeneous cosmology with a diagonal metric, define the following matrices: the generalized scale factor,

$$a_{ab} \equiv \begin{pmatrix} (\gamma_{11})^{1/2} & 0 & 0 \\ 0 & (\gamma_{22})^{1/2} & 0 \\ 0 & 0 & (\gamma_{33})^{1/2} \end{pmatrix} \tag{4.19}$$

(recalling that non-integer powers of a matrix are not defined, so we could not simply say $a_{ab} \equiv (\gamma_{ab})^{1/2}$). In a Bianchi I cosmology only, from this definition we can then define the redshift matrix (in homogeneous cosmologies other than Bianchi I the geodesic equations are non-linear; see CHAPTER 5):

$$z_a^b \equiv a_{ac}(\eta_R)\, a^{bc}(\eta_E) - \delta_a^b = \begin{pmatrix} \frac{a_{11}(t_R)}{a_{11}(t_E)} - 1 & 0 & 0 \\ 0 & \frac{a_{22}(t_R)}{a_{22}(t_E)} - 1 & 0 \\ 0 & 0 & \frac{a_{33}(t_R)}{a_{33}(t_E)} - 1 \end{pmatrix} \tag{4.20}$$

where the subscript R denotes the function evaluated at the time of observation of light, and E denotes the function evaluated at the time of emission, and finally the generalized, anisotropic Hubble parameter and deceleration parameter:

$$H_{ab} \equiv \frac{1}{2}\frac{d}{dt}\ln\gamma_{ab} = \begin{pmatrix} \dot{a}_{11}/a_{11} & 0 & 0 \\ 0 & \dot{a}_{22}/a_{22} & 0 \\ 0 & 0 & \dot{a}_{33}/a_{33} \end{pmatrix} \tag{4.21}$$

$$Q_a^b \equiv \frac{d}{dt}H^{ac}\eta_{bc} - \delta_b^a = -\begin{pmatrix} \ddot{a}_{11}a_{11}/\left(\dot{a}_{11}\right)^2 & 0 & 0 \\ 0 & \ddot{a}_{22}a_{22}/\left(\dot{a}_{22}\right)^2 & 0 \\ 0 & 0 & \ddot{a}_{33}a_{33}/\left(\dot{a}_{33}\right)^2 \end{pmatrix}. \tag{4.22}$$

This approach is essentially a generalization of that developed by Barrow in [22]; the object (4.21) is closely related to the shear tensor [21, 26] which was adapted from fluid dynamics. The practical purpose of these definitions is to provide a mathematical description of observed quantities; let $\mathbf{e}^i$ be a unit vector pointing in the direction of observation. Then the redshift observed in the $\mathbf{e}^i$ direction is given by

$$z\left(e^i, t\right) = z_b^a e_{(b)}^i e_j^{(a)} \mathbf{e}^j \mathbf{e}_i \tag{4.23}$$

and similarly for other functions of the scale factor. Each of these functions can be averaged over the whole sky to extract a monopole value, these averages denoted by a bar:

$$\bar{a} \equiv \frac{\int a_{ab} e_i^{(b)} e_j^{(a)} \mathbf{e}^i \mathbf{e}^j dS}{\int \eta_{ij} \mathbf{e}^i \mathbf{e}^j dS} = \frac{1}{3} a_{ab}\eta^{ab} = \frac{1}{3}\left(a_{11} + a_{22} + a_{33}\right) \tag{4.24}$$

etc.; by "average" we mean, simply, the arithmetic mean of the function summed over the whole sky.

4.2.2 Dynamics in the Kasner universe

An observer in a Kasner universe will see the consequences of that universe's evolution. Examination of the observational consequences of the Kasner universe provides an illustrative example of potential consequences of anisotropy in other cosmologies.

Expansion

Misner, Thorne & Wheeler argue [76] that the Kasner universe is expanding, as the volume element is always increasing:

$$\frac{dV}{dt} = \frac{d}{dt}\sqrt{\|g_{ij}\|}dx^1dx^2dx^3 = \frac{d}{dt}\left(t^{p_1+p_2+p_3}\right)dx^1dx^2dx^3 = dx^1dx^2dx^3. \qquad (4.25)$$

However, as noted above there is no unique way to define the scale factor. In terms of the averaged quantity defined in (4.24) we have

$$\bar{a} = \frac{1}{3}\left(t^{p_1} + t^{p_2} + t^{p_3}\right) \qquad (4.26)$$

which, when we expand around $t = 1$, is approximately

$$\bar{a}\left(t \approx 1\right) = \frac{1}{3}\left(2 + t\right) + \mathcal{O}\left(t^3\right). \qquad (4.27)$$

But in the limit of t small, we have

$$\bar{a} \approx \frac{1}{3}t^{p_1}, \qquad (4.28)$$

which is clearly a decreasing function; so the Kasner universe is not unambiguously expanding and even so fundamental a property as expansion or contraction is a matter of the choice of definition.

Redshift

Redshift in a Kasner universe is given by

$$z^i_j = \begin{pmatrix} (t_R/t_E)^{p_1} - 1 & 0 & 0 \\ 0 & (t_R/t_E)^{p_2} - 1 & 0 \\ 0 & 0 & (t_R/t_E)^{p_3} - 1 \end{pmatrix} \tag{4.29}$$

$$\bar{z} = \frac{1}{3}\left[\left(\frac{t_R}{t_E} - 1\right)^{p_1} + \left(\frac{t_R}{t_E} - 1\right)^{p_2} + \left(\frac{t_R}{t_E} - 1\right)^{p_3}\right]. \tag{4.30}$$

In the circumstance when $t_R \gg t_E$,

$$\bar{z} \approx \frac{1}{3}\left(\frac{t_R}{t_E}\right)^{p_3}. \tag{4.31}$$

Of particular interest is the quantity $\Delta T/T_R$, the variation in CMB temperature from the average (accepting for the moment that the vacuum Kasner universe approximates a matter-filled one at a sufficiently young age), which is given approximately by

$$\frac{\Delta T}{T_R} \approx \left[3\left(\frac{t_R}{t_E}\right)^{-p_3} \begin{pmatrix} (t_R/t_E)^{p_1} & 0 & 0 \\ 0 & (t_R/t_E)^{p_2} & 0 \\ 0 & 0 & (t_R/t_E)^{p_3} \end{pmatrix} - \eta_{ab} \right] \mathbf{e}^i \mathbf{e}^j =$$
$$= \begin{pmatrix} 3\,(t_R/t_E)^{p_1-p_3} - 1 & 0 & 0 \\ 0 & 3\,(t_R/t_E)^{p_2-p_3} - 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{e}^i \mathbf{e}^j \approx \tag{4.32}$$
$$\approx \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{e}^i \mathbf{e}^j$$

(except in the case when $p_2 = p_3 = 2/3$, in which event the (2,2) entry in (4.32) will read 2). The CMB in a mature Kasner universe has a pronounced anisotropy, with the observed temperature matching the average temperature only in a circle around the axis of anisotropy. Notably, the primary axis of the anisotropy is at a right angle to the axis along which the Kasner universe is contracting – not on a parallel axis!

Hubble flow & deceleration parameter

The Kasner universe has Hubble flow

$$H_{ab} = \frac{1}{t} \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{4.33}$$

$$\bar{H} = \frac{1}{3}t^{-1} \tag{4.34}$$

and deceleration parameter

$$Q_a^b = \begin{pmatrix} (1-p_1)/p_1 & 0 & 0 \\ 0 & (1-p_2)/p_2 & 0 \\ 0 & 0 & (1-p_3)/p_3 \end{pmatrix} \tag{4.35}$$

$$\bar{Q} = -1. \tag{4.36}$$

which are necessarily anisotropic; on average a Kasner universe will appear to be accelerating, when the average taken is the parameter $\bar{Q}$. The use of $\bar{Q}$ contrasts with q in that q is defined with the assumption of isotropic deceleration already made (q is defined as a function of a). In the limit that the parameter $u \to \infty$ an observer in a Kasner universe would see a universe with a positive Hubble flow (redshift) over most of the sky, but see blueshift in a third direction. An observer looking only at averages, though, would not be able to distinguish between an isotropic universe and a Kasner universe merely by examining the Hubble flow; only with a complete picture of the sky is such a test possible. The Hubble flow in the case of minimal anisotropy has the form

$$H_{ab}(u=1) = \frac{1}{t}\begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \tag{4.37}$$

– appearing like a Friedmannian matter-dominated universe in two directions –

and in the case of maximal isotropy

$$\lim_{u\to 0} H_{ab} = \frac{1}{t}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.38}$$

Similarly, an observer looking only at the averaged deceleration parameter sees a universe accelerating as though driven by a cosmological constant; only with good enough information will the observer notice a strong angular dependence in the acceleration field, which in the case of minimal anisotropy has the form

$$Q_a^b\,(u=1) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{4.39}$$

– decelerating like a Friedmann cosmology in two directions – and in the case of maximal anisotropy has the form

$$\lim_{u\to\infty} Q_a^b = \begin{pmatrix} -\infty & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.40}$$

Moreover, even though acceleration along two axes is negative in the least-anisotropic Kasner universe, the impact of the positive-accelerating direction is such that the isepitach[4] of zero acceleration, the boundary an observer sees on the sky between regions where objects accelerate and objects decelerate, is a circle 83° from the axis of acceleration; only less than 8% of the sky appears close to "normal" to an

[4] A neologism denoting a path of constant acceleration, similar to "isobar" or "isochor", from Greek *"epitachounse"*, acceleration.

observer expecting to record a Friedmann universe!

While the vacuum Kasner universe is ruled out as a possible approximate cosmology both for reasons of the CMB, which appears isotropic to a high degree [16], and due to the Hubble flow, which appears almost isotropic below $z = 0.3$ [77], the surprising difficulties in distinguishing between its dynamics and that of a Friedmann universe serve as a reminder that sampling of cosmological parameters must be done in an unbiased fashion and that isotropy must be tested rather than assumed. The Kasner universe also has an application as a limiting case of the BKL universe [74] discussed below, to which it appears identical for observers looking over a period of time that is small compared to the radius of curvature of the universe. Finally, the anisotropic Kasner universe serves as a limiting case to some types of cosmology described by the more general Bianchi IX model.

4.3 Gravitational wave nature of Bianchi IX

The Bianchi IX has been considered by cosmologists repeatedly since the establishment of general relativity to provide possible explanations for cosmological phenomena.

Belinsky, Khalatnikov and Lifshitz discussed [74] a Bianchi IX cosmology (the “BKL cosmology”) which undergoes several “bounces” as it evolves – rather than expanding from or converging to a point, it contracts along one axis while expanding along two others until the smallest metric component reaches a minimum value, at which point the axes swap roles. Misner [73] refers to a Bianchi IX universe as the “mixmaster universe”, pursuing an resolution to the horizon problem through the non-linearity of the Bianchi IX cosmology; through the mechanism of bounces, all parts of the universe may be brought into causal connection. Bouncing vacuum cosmologies are, like the vacuum Kasner universe, intrinsically highly anisotropic; while in the long run they tend to act isotropically due to the back-reaction of matter [75, 76] they will still exhibit strong CMB anisotropy [19]. Supernova data ([1, 48] *etc.*) and CMB data on the value of Ω_M ([31] *etc.*) coupled with the existence of high-redshift objects [72] rule out bouncing cosmologies, or at least bouncing cosmologies with a period of at most a few billion years, to a high degree of confidence.

The BKL cosmology undergoes anisotropic acceleration (see SECTION 4.2.2). Meanwhile, numerical modeling has suggested [98, 146] that a matter-filled Bianchi IX universe will also undergo periods of acceleration. Therefore, we have good reason to suppose that a property of Bianchi IX may be to generate anisotropic acceleration, and that consequences of the Bianchi IX cosmology may reveal a dark energy candidate with none of the failings of scalar-field or exotic models.

Wheeler showed [78] that an almost-isotropic Bianchi IX universe admitted a weak tensorial perturbation that took the form of a wave (that is, solving an equation of the form $\ddot{f} + nf(t) = g(t)$). Grishchuk *et al.* were able to generalize this result [10]:

The Bianchi IX space has frame vectors

$$\begin{aligned} e^1_i &= \left(\cos x^3,\ \sin x^1 \sin x^3, 0\right) \\ e^2_i &= \left(-\sin x^3,\ \sin x^1 \cos x^3, 0\right) \\ e^3_i &= (0,\ \cos x^1, 1)\,. \end{aligned} \tag{4.41}$$

Consider the metric of a Bianchi IX cosmology:

$$ds^2 = dt^2 - \gamma_{ab} e^a_i e^b_j dx^i dx^j. \tag{4.42}$$

When the matrix $\gamma_{ab} \propto \eta_{ab}$ we recover the closed Friedmann cosmology. We can split the more general metric up into an isotropic (Friedmannian) part and a non-Friedmannian part:

$$\begin{aligned} ds^2 =& dt^2 - a_F^2 \eta_{ab} e^a_i e^b_j dx^i dx^j - \left(\gamma_{ab} - a_F^2 \eta_{ab}\right) e^a_i e^b_j dx^i dx^j = \\ =& ds_0^2 - \left(\gamma_{ab} - a_F^2 \eta_{ab}\right) e^a_i e^b_j dx^i dx^j \end{aligned} \tag{4.43}$$

where the background metric $ds_0^2 \equiv a_F^2 \eta_{ab} e^a_i e^b_j dx^i dx^j$. Grishchuk, Doroshkevich & Iudin showed that the object describing the space part of the anisotropic part of the metric at some moment in time,[5]

$$G^{ab}_{ij} \equiv 2\left(e^a_i e^b_j + e^b_i e^a_j\right) - \frac{4}{3}\eta^{ab}\eta_{cd} e^c_i e^d_j, \tag{4.44}$$

obeys the property

$$\left(G^{ab}_{ij}\right)^{;k}_{;k} = -\left(n^2 - 3K\right) G^{ab}_{ij} \tag{4.45}$$

for $n = 3$ and $K = 1$; that is, G^{ab}_{ij} is a tensor eigenfunction of the Laplace operator in a Bianchi IX space for waves with wavenumber $n = 3$. A similar property for

[5] a_F has been scaled here to equal 1

open spaces is true of the Bianchi type VII_a models [19].[6]

Lifshitz, in his development of the theory of cosmological perturbations [75, 79, 2, ss. 115], claims that tensorial perturbations, including gravitational waves, can only have a diminishing effect over time. Lifshitz is, however, considering only the class of *local* tensorial perturbations.

In contrast, the gravitational waves in Bianchi IX will have wavelengths comparable to the radius of curvature of the universe. Kristian and Sachs note [25] that the wavelength of cosmic shear (and thus, if anisotropy is present among all principle axes of the space, of cosmological gravitational waves) must be at least 2×10^{10} years – longer than the Hubble radius [16] – and could potentially be far longer (see SECTION 6.1).

We will consider first the regime of weak gravitational waves in an almost-isotropic universe and then "quasi-isotropic" waves; that is, the regime in which components of the metric evolve at equal powers of t.

[6] We could also choose to interpret Bianchi I as the degenerate case of a flat universe containing gravitational waves of infinite wavelength with $n = 0$. The Kasner universe, however, is *not* such a universe: all the anisotropy is governed by a single parameter, u, so the system has an insufficient number of degrees of freedom. The Kasner universe is more like the Taub universe [108].

4.3.1 Einstein equations in the tetrad formalism

For a metric $g_{\alpha\beta}$, let the space-space part of the metric be decomposed as in (4.2). Similarly, the tensors

$$R_{ij} = R_{ab} e_i^a e_j^b \tag{4.46}$$

$$T_{ij} = T_{ab} e_i^a e_j^b \tag{4.47}$$

with all space dependence in the frame vectors. Assume the energy-momentum tensor describes a perfect fluid. Then the Einstein equations can be rewritten:

$$R_{00} = kT_{00} - \frac{1}{2} kT g_{00} \tag{4.48}$$

$$R_{0i} = kT_{0i} - \frac{1}{2} kT g_{0i} \tag{4.49}$$

$$R_{ab} = k \left(T_{ab} - \frac{1}{2} T \gamma_{ab} \right). \tag{4.50}$$

If we have energy-momentum tensor

$$T_{\mu\nu} = (p + \epsilon) u_\mu u_\nu - p g_{\mu\nu} \tag{4.51}$$

$$T = \epsilon - 3p \tag{4.52}$$

then

$$T_{00} = (p + \epsilon) u_0 u_0 - p g_{00} \tag{4.53}$$

$$T_{0i} = (p + \epsilon) u_0 u_i - p g_{0i} \tag{4.54}$$

$$T_{ab} = (p + \epsilon) u_a u_b - p \gamma_{ab}. \tag{4.55}$$

If we then choose a synchronous Gaussian reference system, as we always have the freedom to do,

$$g_{00} = 1 \tag{4.56}$$

$$g_{0i} = 0 \tag{4.57}$$

so the Einstein equations read

$$R_{00} = k\,(p+\epsilon)\,u_0 u_i k\,(p+\epsilon)\,u_0 u_0 - kpg_{00} - \frac{1}{2}k\,(\epsilon - 3p) \tag{4.58}$$

$$R_{0i} = k\,(p+\epsilon)\,u_0 u_i \tag{4.59}$$

$$R_{ab} = k\,(p+\epsilon)\,u_a u_b - kp\gamma_{ab} - \frac{1}{2}k\,(\epsilon - 3p)\,\gamma_{ab}. \tag{4.60}$$

If we then demand that our coordinate system be co-moving with matter,

$$u^0 = 1 \tag{4.61}$$

$$u^i = 0 \tag{4.62}$$

then

$$R_{00} = \frac{1}{2}k\,(\epsilon + 3p) \tag{4.63}$$

$$R_{0i} = 0 \tag{4.64}$$

$$R_{ab} = \frac{1}{2}k\,(p - \epsilon)\,\gamma_{ab}. \tag{4.65}$$

Let

$$d_{ab} \equiv \frac{1}{2}\frac{\partial}{\partial t} g_{ij} e^i_a e^j_b = \frac{1}{2}\frac{d}{dt}\gamma_{ab} \tag{4.66}$$

and

$$d \equiv d_{ab}\gamma^{ab}. \tag{4.67}$$

The Christoffel symbols associated with our metric then become [2, ss. 97]

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^i_{00} = 0 \tag{4.68}$$

$$\Gamma^0_{ij} = d_{ij} \tag{4.69}$$

$$\Gamma^i_{0j} = d^i_j \tag{4.70}$$

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} \tag{4.71}$$

where $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols associated with the three-dimensional metric tensor $-g_{ij}$. The Ricci tensor can then be written as [2, ss. 97]:

$$R_{00} = -\dot{d} - d^b_a d^a_b \tag{4.72}$$

$$R_{0i} = 0 \tag{4.73}$$

$$R_{ab} = \dot{d}_{ab} + d d_{ab} - 2 d_{ac} d^c_b - P_{ab} \tag{4.74}$$

or explicitly [10]

$$\dot{d} + d^b_a d^a_b = -\frac{1}{2}k\left(\epsilon + 3p\right) \tag{4.75}$$

$$\dot{d}_{ab} + d d_{ab} - 2 d_{ac} d^c_b - P_{ab} = \frac{1}{2}k\left(\epsilon - p\right)\gamma_{ab} \tag{4.76}$$

$$d^b_a C^{\ a}_{bc} = 0 \tag{4.77}$$

where P_{ij} is the three-dimensional Ricci tensor constructed from $\tilde{\Gamma}^i_{jk}$.

4.3.2 The curvature tensor for Bianchi IX

Grishchuk explicitly gives the curvature tensors for all Bianchi types, and a general method for easily deriving them, in [7]. These tensors can be stated in removable and non-removable parts, with the removable parts corresponding to time-dependent rotations of the space. Let the symbol

$$\gamma_{abc} \equiv \gamma_{ad} C^{a}_{bc}. \tag{4.78}$$

Then where

$$\Gamma^{c}_{ab} \equiv \frac{1}{2}\gamma^{cd}\left(\gamma_{abd} + \gamma_{dab} - \gamma_{bda}\right) \tag{4.79}$$

(these are analogous to the Christoffel symbols of the full space, but with different symmetry properties) the non-removable part of the curvature tensor is given by

$$L_{ab} \equiv -2\Gamma^{c}_{a[b,c]} + 2\Gamma^{c}_{d[b}\Gamma^{d}_{|a|c]} + 2\Gamma^{c}_{ad}\Gamma^{d}_{[bc]} \tag{4.80}$$

where square brackets around the indices indicate the antisymmetric part of the tensor; the removable part is given by

$$b_{ab} \equiv \frac{1}{2}v_c C^{c}_{ba} + \frac{1}{2}\left(f_a v_b - f_b v_a\right) \tag{4.81}$$

and finally the curvature tensor

$$P_{ab} = L_{ab} - b_{bc}d^{c}_{a} - b_{ac}d^{c}_{b} - b_{ba}d. \tag{4.82}$$

In the co-moving case that $v_a = 0$ we can simply state $P_{ab} = H_{ab}$. For the particular case of Bianchi IX (the frame vectors (4.41)) and the curvature tensor when $v_a = 0$

reads, for diagonal components:

$$P_a^b = \left[\frac{\left(\gamma_{fg}\eta^{fg}\right)^2}{2\,\|\gamma_{cd}\|} - \gamma^{fg}\eta_{fg} \right] \delta_a^b - \gamma^{bc}\eta_{ac} - \frac{\gamma_{af}\gamma_{gh}\eta^{fg}\eta^{bh}}{\|\gamma_{cd}\|} \tag{4.83}$$

and for non-diagonal components:

$$P_a^b = -2\gamma^{cb}\eta_{ac} - \frac{1}{\|\gamma_{df}\|}\gamma_{ac}\gamma_{df}\eta^{bc}\eta^{df} \tag{4.84}$$

where $\|\gamma_{ab}\|$ is defined as the determinant of γ_{ab}. The Einstein equations show that when $v_a = 0$ the non-diagonal components of γ_{ab} must be zero, so as a consequence of our Gaussian choice of coordinate system we can without loss of generality, write the metric for Bianchi IX

$$\begin{aligned} \gamma_{11} &= a_F^2 e^{2\alpha} \\ \gamma_{22} &= a_F^2 e^{2\beta} \\ \gamma_{33} &= a_F^2 e^{2\gamma} \end{aligned} \tag{4.85}$$

with all other space-space components zero, so explicitly the the curvature tensor P_{ab} for Bianchi IX reads

$$P_{11} = \frac{1}{2}e^{-2\delta}\left(-e^{4\alpha} + \left(e^{2\beta} - e^{2\gamma}\right)^2\right)e^{2\alpha} \tag{4.86}$$

$$P_{22} = \frac{1}{2}e^{-2\delta}\left(-e^{4\beta} + \left(e^{2\gamma} - e^{2\alpha}\right)^2\right)e^{2\beta} \tag{4.87}$$

$$P_{33} = \frac{1}{2}e^{-2\delta}\left(-e^{4\gamma} + \left(e^{2\alpha} - e^{2\beta}\right)^2\right)e^{2\gamma} \tag{4.88}$$

$$P_{ab} = 0, \qquad a \neq b \tag{4.89}$$

and the contracted curvature scalar

$$P_{ab}\gamma^{ab} = 2a_F^{-2}e^{-2\delta}\left[e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2\left(e^{2\alpha+2\beta} + e^{2\beta+2\gamma} + e^{2\alpha+2\gamma}\right)\right]. \tag{4.90}$$

The background, Friedmannian universe is recovered in the case that $\alpha = \beta = \gamma = 0$.

4.4 Einstein equations for Bianchi IX

4.4.1 Exact equations

Let the symbol $\delta \equiv \alpha + \beta + \gamma$ for convenience as in (4.17). For our chosen metric, we have the auxiliary quantities

$$d_{11} = \left(a\dot{a} + a^2\dot{\alpha}\right) e^{2\alpha} \tag{4.91}$$

$$\dot{d}_{11} = \left(\dot{a}^2 + a\ddot{a} + 4a\dot{a}\dot{\alpha} + a^2\ddot{\alpha} + 2a^2\dot{\alpha}^2\right) e^{2\alpha} \tag{4.92}$$

$$d^1_1 = H + \dot{\alpha} \tag{4.93}$$

$$d = 3H + \dot{\delta} \tag{4.94}$$

and cyclic permutations in α, β, γ thereof for 22- and 33-quantities. The full Einstein equations for Bianchi IX read[7]

$$\left\{ \begin{array}{c} \frac{3}{a_F^2}\left(\dot{a}_F^2+1\right)+\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}+2\frac{\dot{a}_F}{a_F}\dot{\delta}+ \\ +a_F^{-2}e^{-2\delta}\left[\begin{array}{c} 2\left(e^{2\alpha+2\beta}+e^{2\alpha+2\gamma}+e^{2\beta+2\gamma}\right)- \\ -e^{4\alpha}-e^{4\beta}-e^{4\gamma}-3e^{2\delta} \end{array} \right] \end{array} \right\} = k\epsilon \tag{4.95}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\alpha}+\frac{\dot{a}_F}{a_F}\left(3\dot{\alpha}+\dot{\delta}\right)+\dot{\alpha}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\alpha}-\left(e^{2\beta}-e^{2\gamma}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(1)}\right) \tag{4.96}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\beta}+\frac{\dot{a}_F}{a_F}\left(3\dot{\beta}+\dot{\delta}\right)+\dot{\beta}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\beta}-\left(e^{2\gamma}-e^{2\alpha}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(2)}\right) \tag{4.97}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\gamma}+\frac{\dot{a}_F}{a_F}\left(3\dot{\gamma}+\dot{\delta}\right)+\dot{\gamma}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\gamma}-\left(e^{2\alpha}-e^{2\beta}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(3)}\right). \tag{4.98}$$

[7] These equations are a trivial generalization of those found in the vacuum cosmology described in [10].

We can also define quantities as components of a gravitational effective energy-momentum tensor:

$$k\epsilon_g \equiv -\left\{ \begin{array}{c} \dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma} + 2\frac{\dot{a}_F}{a_F}\dot{\delta} + \\ +a_F^{-2}e^{-2\delta} \left[\begin{array}{c} 2\left(e^{2\alpha+2\beta} + e^{2\alpha+2\gamma} + e^{2\beta+2\gamma}\right) - \\ -e^{4\alpha} - e^{4\beta} - e^{4\gamma} - 3e^{2\delta} \end{array} \right] \end{array} \right\} \tag{4.99}$$

$$\frac{1}{2}k\left(\epsilon_g - p_g^{(1)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\alpha} + \frac{\dot{a}_F}{a_F}\left(3\dot{\alpha} + \dot{\delta}\right) + \dot{\alpha}\dot{\delta} + \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\alpha} - \left(e^{2\beta} - e^{2\gamma}\right)^2 - e^{2\delta}\right] \end{array} \right\} \tag{4.100}$$

$$\frac{1}{2}k\left(\epsilon_g - p_g^{(2)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\beta} + \frac{\dot{a}_F}{a_F}\left(3\dot{\beta} + \dot{\delta}\right) + \dot{\beta}\dot{\delta} + \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\beta} - \left(e^{2\gamma} - e^{2\alpha}\right)^2 - e^{2\delta}\right] \end{array} \right\} \tag{4.101}$$

$$\frac{1}{2}k\left(\epsilon_g - p_g^{(3)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\gamma} + \frac{\dot{a}_F}{a_F}\left(3\dot{\gamma} + \dot{\delta}\right) + \dot{\gamma}\dot{\delta} + \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\gamma} - \left(e^{2\alpha} - e^{2\beta}\right)^2 - e^{2\delta}\right] \end{array} \right\} \tag{4.102}$$

$$kp_g^{(1)} \equiv \left[\begin{array}{c} 2\ddot{\alpha} + 6\frac{\dot{a}_F}{a_F}\dot{\alpha} + 2\dot{\alpha}^2 + \dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} - \dot{\beta}\dot{\gamma} + \\ +a_F^{-2} \left(\begin{array}{c} 5e^{2(\alpha-\beta-\gamma)} - 3e^{2(\beta-\alpha-\gamma)} - 3e^{2(\gamma-\alpha-\beta)} + \\ +6e^{-2\alpha} - 2e^{-2\gamma} - 2e^{-2\beta} - 1 \end{array} \right) \end{array} \right] \tag{4.103}$$

$$kp_g^{(2)} \equiv \left[\begin{array}{c} 2\ddot{\beta} + 6\frac{\dot{a}_F}{a_F}\dot{\beta} + 2\dot{\beta}^2 + \dot{\alpha}\dot{\beta} - \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma} + \\ +a_F^{-2} \left(\begin{array}{c} 5e^{2(\beta-\alpha-\gamma)} - 3e^{2(\gamma-\beta-\alpha)} - 3e^{2(\alpha-\beta-\gamma)} \\ +6e^{-2\beta} - 2e^{-2\alpha} - 2e^{-2\gamma} - 1 \end{array} \right) \end{array} \right] \tag{4.104}$$

$$kp_g^{(3)} \equiv \left[\begin{array}{c} 2\ddot{\gamma} + 6\frac{\dot{a}_F}{a_F}\dot{\gamma} + 2\dot{\gamma}^2 - \dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma} + \\ +a_F^{-2} \left(\begin{array}{c} 5e^{2(\gamma-\beta-\alpha)} - 3e^{2(\alpha-\gamma-\beta)} - 3e^{2(\beta-\gamma-\alpha)} \\ +6e^{-2\gamma} - 2e^{-2\beta} - 2e^{-2\alpha} - 1 \end{array} \right) \end{array} \right] \tag{4.105}$$

(all of which are zero when $\alpha = \beta = \gamma = 0$). The Bianchi identity $T^{\nu}_{\mu,\nu}$ demands $p_g^{(1)} = p_g^{(2)} = p_g^{(3)}$ so define the averaged gravitational pressure

$$kp_g \equiv \frac{1}{3}k\left(p_g^{(1)} + p_g^{(2)} + p_g^{(3)}\right) = \tag{4.106}$$
$$\equiv \left[\begin{array}{c} 2\ddot{\delta} + 6\frac{\dot{a}_F}{a_F}\dot{\delta} + 2\left(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2\right) + \\ +2\left(\dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma}\right) + \\ +a_F^{-2}\left(\begin{array}{c} -e^{2(\alpha-\beta-\gamma)} - e^{2(\beta-\alpha-\gamma)} - e^{2(\gamma-\alpha-\beta)} \\ +2e^{-2\alpha} + 2e^{-2\gamma} + 2e^{-2\beta} - 3 \end{array}\right) \end{array}\right].$$

Finally,

$$k\left(\epsilon_g + 3p_g\right) = 2\ddot{\delta} + 4\frac{\dot{a}_F}{a_F}\dot{\delta} + 2\left(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2\right) \tag{4.107}$$

[8]. Define a pseudo-conformal time coordinate η by $cdt \equiv a_F d\eta$; note that this fixes the relationship between t and η up to the level of the characteristic length a_i and a constant which can be set to zero. Given the impossibility of selecting a unique and objective definition for the scale factor, we do *not* define the conformal time using such a function. Define a correction term q to the matter energy density such that

$$\epsilon = \epsilon_F\left(1 + q\right). \tag{4.108}$$

[8]Equation (4.107) corrects an error of sign in [10, equation (27)].

In η-time, the Einstein equations for Bianchi IX, subtracting background terms on both sides, read:

$$\left\{ \begin{array}{c} \alpha'\beta' + \alpha'\gamma' + \beta'\gamma' + 2\frac{a'_F}{a_F}\delta' + \\ +e^{-2\delta} \left[\begin{array}{c} 2\left(e^{2\alpha+2\beta} + e^{2\alpha+2\gamma} + e^{2\beta+2\gamma}\right) - \\ -e^{4\alpha} - e^{4\beta} - e^{4\gamma} - 3e^{2\delta} \end{array} \right] \end{array} \right\} = a_F^2 k\epsilon_F q \tag{4.109}$$

$$\left\{ \begin{array}{c} \alpha'' + \frac{a'_F}{a_F}\left(2\alpha' + \delta'\right) + \alpha'\delta' + \\ +2e^{-2\delta}\left[e^{4\alpha} - \left(e^{2\beta} - e^{2\gamma}\right)^2 - e^{2\delta}\right] \end{array} \right\} = \frac{1-w}{2} a_F^2 k\epsilon_F q \tag{4.110}$$

$$\left\{ \begin{array}{c} \beta'' + \frac{a'_F}{a_F}\left(2\beta' + \delta'\right) + \beta'\delta' + \\ +2e^{-2\delta}\left[e^{4\beta} - \left(e^{2\gamma} - e^{2\alpha}\right)^2 - e^{2\delta}\right] \end{array} \right\} = \frac{1-w}{2} a_F^2 k\epsilon_F q \tag{4.111}$$

$$\left\{ \begin{array}{c} \gamma'' + \frac{a'_F}{a_F}\left(2\gamma' + \delta'\right) + \gamma'\delta' + \\ +2e^{-2\delta}\left[e^{4\gamma} - \left(e^{2\alpha} - e^{2\beta}\right)^2 - e^{2\delta}\right] \end{array} \right\} = \frac{1-w}{2} a_F^2 k\epsilon_F q. \tag{4.112}$$

We also note the Einstein equations have an exact formal solution

$$k\epsilon = \left(S a_F^{-3} e^{-\delta}\right)^{1+w} \tag{4.113}$$

where S is a constant of proportionality such that S^{1+w} has dimensionality of length to the $1+3w$ power. Finally the Einstein equations can be read as

$$kp_g^{(1)} + wa_F^2 k\epsilon_F q = kp_g^{(2)} + wa_F^2 k\epsilon_F q = kp_g^{(3)} + wa_F^2 k\epsilon_F q = a_F^2 k\epsilon_F q + k\epsilon_g = 0. \tag{4.114}$$

In other words, the effective energy-momentum tensor created by cosmological gravitational waves equals minus the back-reaction on matter energy density and pressure. Note that the quantity $k\epsilon_g/q$ is necessarily negative.

4.4.2 Solutions to the Einstein equations at zero order

For convenience, let us define the variable $x \equiv \frac{1+3w}{2}\eta$. Then at zero order the Einstein equations for a Bianchi IX universe have, for an arbitrary constant equation of state, the following solution and auxiliary quantities, which are identical to the solutions to the Einstein equations in the unperturbed closed Friedmann cosmology:

$$a_F = a_i (\sin x)^{\frac{2}{1+3w}} \tag{4.115}$$

$$a'_F = a_i (\sin x)^{\frac{1-3w}{1+3w}} \cos x \tag{4.116}$$

$$a''_F = \frac{1+3w}{2} a_i \left[\frac{1-3w}{1+3w} (\sin x)^{\frac{-6w}{1+3w}} \cos^2 x - (\sin x)^{\frac{2}{1+3w}} \right] \tag{4.117}$$

$$a'_F / a_F = \cot x \tag{4.118}$$

$$H_F = a_i^{-1} \cot x \csc x \tag{4.119}$$

$$Q_F = \frac{1+3w}{2} \sec^2 x. \tag{4.120}$$

The quantity a_i represents a characteristic scale for the universe and, in the background case, represents the radius of curvature of the universe at the extent of its maximum expansion. We treat a_i as an arbitrary constant for the time being.

4.4.3 Solutions at linear order

We approach perturbative solutions to the Einstein equations by letting the functions α, β, γ be small ($0 < |\alpha| \ll 1$ *etc.).* To first order, that is α, β, γ such that

$\alpha^2 \approx \beta^2 \approx \gamma^2 \approx 0$, the Einstein equations take the form:

$$2\frac{a_F'}{a_F}\delta_1' - 2\delta_1 = a_F^2 k\epsilon_F q_1 \tag{4.121}$$

$$\alpha_1'' + \frac{a_F'}{a_F}\left(2\alpha_1' + \delta_1'\right) + 8\alpha_1 - 4\delta_1 = \frac{1-w}{2}a_F^2 k\epsilon_F q_1 \tag{4.122}$$

$$\beta_1'' + \frac{a_F'}{a_F}\left(2\beta_1' + \delta_1'\right) + 8\beta_1 - 4\delta_1 = \frac{1-w}{2}a_F^2 k\epsilon_F q_1 \tag{4.123}$$

$$\gamma_1'' + \frac{a_F'}{a_F}\left(2\gamma_1' + \delta_1'\right) + 8\gamma_1 - 4\delta_1 = \frac{1-w}{2}a_F^2 k\epsilon_F q_1 \tag{4.124}$$

where the subscript 1 denotes a first-order small quantity, that is, a quantity small such that in the first approximation its square is negligible. The formal solution (4.113) gives us, to first order,

$$a_F^2 k\epsilon_F q_1 = -\left(1+w\right) S^{1+w} a_F^{-1-3w}\delta_1. \tag{4.125}$$

Meanwhile, we can always choose to let S take on its Friedmannian value [10], so $S^{1+w} = 3a_i^{1+3w}$. Therefore:

$$2\frac{a_F'}{a_F}\delta_1' + \left[3\left(1+w\right)\csc^2 x - 2\right]\delta_1 = 0 \tag{4.126}$$

$$\alpha_1'' + 2\frac{a_F'}{a_F}\alpha_1' + 8\alpha_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 = 0 \tag{4.127}$$

$$\beta_1'' + 2\frac{a_F'}{a_F}\beta_1' + 8\beta_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 = 0 \tag{4.128}$$

$$\gamma_1'' + 2\frac{a_F'}{a_F}\gamma_1' + 8\gamma_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 = 0 \tag{4.129}$$

which gives us the solution:

$$\delta_1 = c_1 \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}}. \tag{4.130}$$

The term governed by c_1 is a "removable" perturbation, that is, one not arising from a physical phenomenon but from small changes in our selection of the scale factor. Grishchuk, Doroshkevich & Iudin argue [10], and Grishchuk later proves in the case of high-frequency gravitational waves [103], that the the removable perturbation arises from the remaining freedom in having selected a synchronous reference system and represents a small change in the value of η. Therefore, the removable term represents the gauge freedom remaining in the Einstein equations. This coincides with the argument made by Bardeen [105] with regard to scalar and vector perturbations with wavelengths longer than the Hubble radius; Bardeen recommends a gauge choice minimizing shear. We always have the freedom to set c_1 to zero but do not do so yet. In a radiation-dominated universe, we have

$$\delta_1^{\text{radiation}} = c_1^{\text{radiation}} \cos\eta \csc^2 \eta \tag{4.131}$$

and in a matter-dominated universe

$$\delta_1^{\text{matter}} = c_1^{\text{matter}} \cos\frac{\eta}{2} \csc^3 \frac{\eta}{2}. \tag{4.132}$$

Therefore, the full first-order functions can be written:

$$\alpha_1'' + 2\cot x \alpha_1' + 8\alpha_1 = 3c_1 \left[\begin{array}{c} 1 + \frac{1+w}{2} (\csc x)^{1+3w} - \\ -\frac{1-w^2}{2} \csc^2 x \end{array} \right] (\csc x)^{\frac{3+3w}{1+3w}} \cos x \tag{4.133}$$

etc. Note that the right hand side contains *no* physical variables – no characteristic length or energy density. The Einstein equations at first order have the solutions

(denoted with a tilde for the $c_1 = 0$ case)

$$
\begin{aligned}
\tilde{\alpha}_1^{\text{radiation}} &= (C_{\alpha 1,1} \sin 3\eta + C_{\alpha 2,1} \cos 3\eta) \csc \eta \\
\tilde{\beta}_1^{\text{radiation}} &= (C_{\beta 1,1} \sin 3\eta + C_{\beta 2,1} \cos 3\eta) \csc \eta \\
\tilde{\gamma}_1^{\text{radiation}} &= (C_{\gamma 1,1} \sin 3\eta + C_{\gamma 2,1} \cos 3\eta) \csc \eta
\end{aligned}
\tag{4.134}
$$

and similarly for $\tilde{\beta}, \tilde{\gamma}$ in a radiation-dominated universe, and

$$
\tilde{\alpha}_1^{\text{matter}} = \frac{C_{\alpha 1,1}}{\sin \eta/2} \frac{d}{d\eta} \frac{\sin 3\eta}{\sin \eta/2} + \frac{C_{\alpha 2,1}}{\sin \eta/2} \frac{d}{d\eta} \frac{\cos 3\eta}{\sin \eta/2} \tag{4.135}
$$

etc. in a matter-dominated universe, in both cases constrained by the condition $C_{\alpha 1,1} + C_{\beta 1,1} + C_{\gamma 1,1} = C_{\alpha 2,1} + C_{\beta 2,1} + C_{\gamma 2,1} = 0$. A general solution for any constant equation of state, in terms of orthogonal polynomials in a, exists but is far too cumbersome to be of practical use in this work. We introduce the notation $C_{\alpha 1,1}$ *etc.* to be read in the following way: $C_{\alpha 2,1}$ is an arbitrary constant associated with the function α, the first index denoting the mode of the solution (1 for growing, 2 for decaying), the second index denoting the order of the constant in an expansion assuming $\alpha, \beta, \gamma \ll 1$. For convenience, we will sometimes write a generic solution to the differential equation (4.133) as

$$
\tilde{\alpha}_1 = C_{\alpha 1,1} y_1 + C_{\alpha 2,1} y_2. \tag{4.136}
$$

These solutions can be written in a less symmetric but easier-to-manipulate form:

$$\tilde{\alpha}_1^{\text{radiation}} = C_{\alpha 1,1} \left(2\cos 2\eta + 1\right) + C_{\alpha 2,1} \cos 3\eta \csc \eta \tag{4.137}$$

$$\tilde{\alpha}_1^{\text{matter}} = - \left[\begin{array}{c} C_{\alpha 1,1} \left(16\cos 2\eta + 10\cos\eta + 9\right) + \\ + \frac{1}{4} C_{\alpha 2,1} \csc^3 \frac{\eta}{2} \left(5\cos\frac{7}{2}\eta - 7\cos\frac{5}{2}\eta\right) \end{array} \right] \tag{4.138}$$

etc. When $\delta = 0$ we recognize the homogeneous first-order Einstein equations as describing weak gravitational waves with wavenumber $n = 3$ and a wave equation of the form

$$\nu'' + 2\cot(x)\,\nu' + \left(n^2 - 1\right)\nu = 0, \tag{4.139}$$

in line with [10]'s description.[9] In a radiation-dominated universe we have explicitly for the full first-order solution:

$$\alpha_1^{\text{radiation}} = C_{\alpha 1,1} \frac{\sin 3\eta}{\sin\eta} + C_{\alpha 2,1} \frac{\cos 3\eta}{\sin\eta} + \frac{c_1}{3} \cos\eta \csc^2\eta \tag{4.140}$$

etc. and in a matter-dominated universe we have

$$\alpha_1^{\text{matter}} = \frac{C_{\alpha 1,1}}{\sin\eta/2} \frac{d}{d\eta} \frac{\sin 3\eta}{\sin\eta/2} + \frac{C_{\alpha 2,1}}{\sin\eta/2} \frac{d}{d\eta} \frac{\cos 3\eta}{\sin\eta/2} + \frac{c_1}{3} \cos\frac{\eta}{2} \csc^3\frac{\eta}{2}. \tag{4.141}$$

It is common to refer to the decaying "cos" mode of these gravitational waves as "singularity-destroying" [10], in that they diverge as $\eta \to 0$, which could seem at first to imply $\lim\limits_{\eta\to 0} \gamma_{ab} \to \infty$. It is worth remembering that as the functions α, β, γ appear in the metric as exponents, that is, $\gamma_{11} = a_F^2 e^{2\alpha}$ *etc;* thus decaying functions

[9] Just as EQUATION (4.45) generalizes the Helmholtz differential equation to elliptical and hyperbolic spaces, the solutions y_1 and y_2 generalize the spherical Bessel functions $j_n(x)$ and $y_n(x)$; the radiation-dominated universe is solved by analogues of the $n = 0$ case and the matter-dominated universe by the $n = 1$ case. The wave functions in a matter-dominated universe always have longer trigonometric expansions than they do in the radiation-dominated universe and thus the equations in a matter-dominated universe are usually more difficult to solve.

are not necessarily "singularity-destroying" for the following reasons:

- their divergence must overcome the convergence of the Friedmannian term, which in the case of weak waves will occur when $w \leq 2/3$ but not generally;
- functions of the form $e^{-x^{-y}}$ for $x < 0, y < 0$ are non-analytic near $x = 0$, that is, they are not described by convergent Taylor series in that region.

As $C_{\alpha 2,1} + C_{\beta 2,1} + C_{\gamma 2,1} = 0$, either one or two decaying terms preserve the $t = 0$ singularity when the removable perturbation is removed, in a manner analogous to that found in the Kasner universe, in the case of weak gravitational waves (although the price of this is a divergence later).

When discussing high-frequency, localized waves, it is easy to define an amplitude of the waves by (for example) normalizing the root-mean-square (RMS) value over the wave's period. In the case of cosmological gravitational waves however this procedure is not possible in an absolute sense due to the diverging character of the decaying mode. Fortunately, mathematical conditions on the relation of linear-order terms to quadratic-order terms revealed at quadratic order (see SECTION 4.4.4) cause the term "weak" to give itself an objective meaning. If we wish to normalize the growing modes, they have the following RMS values:

$$y_1^{\text{RMS}} \equiv \left[\frac{2}{(1+3w)\,\pi}\int_0^{(1+3w)\pi/2} y_1^2 d\eta\right]^{1/2} \tag{4.142}$$

$$y_1^{\text{radiation,RMS}} = \sqrt{3} \tag{4.143}$$

$$y_1^{\text{matter,RMS}} = \sqrt{259} \approx 16.1. \tag{4.144}$$

It is interesting to note that in matter, the decaying "cos" mode of $\alpha_1, \beta_1, \gamma_1$ has the same η-dependence as the removable perturbation; a cosmologist attempting to remove what they assume, based on an incomplete picture of the sky, to be a removable perturbation may inadvertently be suppressing evidence of a non-removable gravitational wave!

Finally, the gravitational energy-momentum tensor's (entirely removable) components read, to linear order:

$$k\epsilon_{g(1)} = 3\,(1+w)\,\frac{c_1}{a_i^2}\cos x\,(\csc x)^{\frac{9+9w}{1+3w}} \tag{4.145}$$

$$kp^{(1)}_{g(1)} = kp^{(2)}_{g(1)} = kp^{(3)}_{g(1)} = 3w\,(1+w)\,\frac{c_1}{a_i^2}\cos x\,(\csc x)^{\frac{9+9w}{1+3w}} \tag{4.146}$$

while the back-reaction of the gravitational waves at linear order gives us matter EMT components which vary from background by:

$$q_1 = -3\,(1+w)\,c_1\cos x\,(\csc x)^{\frac{5+9w}{1+3w}}\,; \tag{4.147}$$

when removable perturbations have been removed, first-order weak gravitational waves have no effect on the distribution of matter, as is well-recognised in cosmological perturbation theory ([79, 4]).

4.4.4 Solutions at quadratic order

The Einstein equations to quadratic order read:

$$2\cot x\delta_2' + \left[3\left(1+w\right)\csc^2 x - 2\right]\delta_2 = \left\{\begin{array}{c} \left[3\csc^2 x\frac{(1+w)^2}{2} - 2\right]\delta_1^2 - \\ -\frac{1}{2}\left[\delta_1'^2 - \left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right)\right] + \\ +4\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) \end{array}\right\} \quad (4.148)$$

$$\alpha_2'' + \cot x\left(2\alpha_2' + \delta_2'\right) + 8\alpha_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(\frac{(1+w)^2}{2}\delta_1^2 - \left(1+w\right)\delta_2\right) - \\ -\alpha_1'\delta_1' + 8\left(\beta_1 - \gamma_1\right)^2 - \\ -16\alpha_1^2 + 16\alpha_1\delta_1 - 4\delta_1^2 \end{array}\right] \quad (4.149)$$

$$\beta_2'' + \cot x\left(2\alpha_2' + \delta_2'\right) + 8\beta_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(\frac{(1+w)^2}{2}\delta_1^2 - \left(1+w\right)\delta_2\right) - \\ -\beta_1'\delta_1' + 8\left(\gamma_1 - \alpha_1\right)^2 - \\ -16\beta_1^2 + 16\beta_1\delta_1 - 4\delta_1^2 \end{array}\right] \quad (4.150)$$

$$\gamma_2'' + \cot x\left(2\gamma_2' + \delta_2'\right) + 8\gamma_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(\frac{(1+w)^2}{2}\delta_1^2 - \left(1+w\right)\delta_2\right) - \\ -\gamma_1'\delta_1' + 8\left(\alpha_1 - \beta_1\right)^2 - \\ -16\gamma_1^2 + 16\gamma_1\delta_1 - 4\delta_1^2 \end{array}\right]. \quad (4.151)$$

Taking the ${}^{(2)}T^0_0$ equation (4.148) first,

$$2\cot x\delta_2' + \left[3\left(1+w\right)\csc^2 x - 2\right]\delta_2 = \left\{\begin{array}{c} \left[3\csc^2 x\frac{(1+w)^2}{2} - \frac{2}{3}\right]\delta_1^2 - \frac{1}{3}\delta_1'^2 + \\ +\frac{1}{2}\left(\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2\right) + \\ +4\left(\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2\right) \end{array}\right\}. \quad (4.152)$$

The homogeneous part $\tilde{\delta}_2$ of course has the same form as at first order, representing a removable perturbation, so

$$\tilde{\delta}_2 = c_2 \cos x \,(\csc x)^{\frac{3+3w}{1+3w}} \,. \tag{4.153}$$

The complete solution in integral form is

$$\begin{aligned}\delta_2 =& \frac{1}{2} \cos x \,(\csc x)^{\frac{3+3w}{1+3w}} \times \qquad (4.154)\\ &\times \left\{ \int \left[\begin{array}{c} \left[3 \csc^2 x \frac{(1+w)^2}{2} - \frac{2}{3}\right] \delta_1^2 - \\ -\frac{1}{3}\delta_1'^2 + \\ +\frac{1}{2}\left(\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2\right) + \\ +4\left(\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2\right) \end{array} \right] \sec^2 x \,(\sin x)^{\frac{4+6w}{1+3w}} \, d\eta + c_2 \right\} .\end{aligned}$$

We will continue to refer to the solutions $\alpha_2, \beta_2, \gamma_2$ as "gravitational waves" out of convention, as they solve the Laplacian equation (4.45), even though as will be seen these metric perturbations will at second order affect the distribution of matter. Define the following pseudo-vectors and their Euclidean dot products:

$$\left(C_{\alpha 1(1)}, C_{\beta 1(1)}, C_{\gamma 1(1)}\right) \equiv \boldsymbol{\sigma} \tag{4.155}$$

$$\left(C_{\alpha 2(1)}, C_{\beta 2(1)}, C_{\gamma 2(1)}\right) \equiv \boldsymbol{\tau} \tag{4.156}$$

$$\left(C^2_{\alpha 1(1)} + C^2_{\beta 1(1)} + C^2_{\gamma 1(1)}\right) = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \qquad \equiv \sigma^2 \tag{4.157}$$

$$\left(C^2_{\alpha 2,1} + C^2_{\beta 2,1} + C^2_{\gamma 2,1}\right) = \boldsymbol{\tau} \cdot \boldsymbol{\tau} \qquad \equiv \tau^2 \tag{4.158}$$

$$\left(C_{\alpha 1,1} C_{\alpha 2,1} + C_{\beta 1,1} C_{\beta 2,1} + C_{\gamma 1,1} C_{\gamma 2,1}\right) = \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \tag{4.159}$$

so

$$\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2 = \sigma^2 y_1^2 + \tau^2 y_2^2 + 2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} y_1 y_2 \tag{4.160}$$

$$\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2 = \sigma^2 y_1'^2 + \tau^2 y_2'^2 + 2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} y_1' y_2'. \tag{4.161}$$

Note that the solution $\delta_2 = 0$ is excluded except in the case of the background universe. When all removable perturbations are set to zero,

$$\delta_2^{\text{non-removable}} = \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}} \times \\ \times \left\{ \int^{\eta} \begin{bmatrix} \sigma^2 \left(2y_1^2 + \frac{1}{4} y_1'^2\right) + \\ +\tau^2 \left(2y_2^2 + \frac{1}{4} y_2'^2\right) + \\ +\boldsymbol{\sigma}\cdot\boldsymbol{\tau} \left(4y_1 y_2 + \frac{1}{2} y_1' y_2'\right) \end{bmatrix} \tan^2 x \left(\sin x\right)^{\frac{2}{1+3w}} d\bar{\eta} \right\}. \tag{4.162}$$

[10] We will discuss solutions to this equation term-by-term, noting that these terms can be solved entirely from information we obtained at first order.[11]

Contributions from the removable perturbations

Contributions from the removable perturbations at second order have the explicit forms:

[10]The Einstein equations for weak gravitational waves in a Bianchi IX universe have the elegant feature of being integrable in closed form, always reducible to functions form $\sin(n\eta)\csc^k(\eta)$ and $\cos(n\eta)\csc^k(\eta)$. Theoreticians working in regimes of higher-frequency gravitational waves on a slowly-moving background may find it felicitous to approximate a Euclidean universe as a closed one in order to avoid mathematical inconveniences associated with the function $\mathrm{sinc}(t)$!

[11]Li and Schwarz[107] obtain a similar result for a flat universe, but apply their results to a different domain. The averaging scheme they propose is not an applicable approach for cosmological gravitational waves. The result is generally stated in [2, ss. 96].

In a radiation-dominated universe:

$$\delta_2^{\text{removable}} = -\frac{c_1^2}{12}\left(4\sin^2\frac{\eta}{2} + \tan^2\frac{\eta}{2} + \cot^2\frac{\eta}{2} - 2\right)\csc^2\eta + c_2\cot\eta\csc\eta \quad (4.163)$$

Note that the terms deriving from the first-order removable perturbation diverge as $\mathcal{O}\left(\eta^{-4}\right)$, while those from the second-order removable perturbation diverge more slowly, as $\mathcal{O}\left(\eta^{-2}\right)$.

In a matter-dominated universe:

$$\delta_2^{\text{removable}} = -\frac{c_1^2}{12}\left(3\csc^4\frac{\eta}{2} + 8\csc^2\frac{\eta}{2} - 10\right)\csc^2\frac{\eta}{2} + c_2\cot\frac{\eta}{2}\csc^2\frac{\eta}{2}. \quad (4.164)$$

Similarly, terms deriving from the first-order perturbation diverge as $\mathcal{O}\left(\eta^{-6}\right)$ and so at small η will dominate terms deriving from the second-order removable perturbation which diverges as $\mathcal{O}\left(\eta^{-3}\right)$.

Contributions from the growing mode

Contributions from the growing mode have the following form:

$$\delta_2^{\text{growing}} = \sigma^2\cot x\,(\csc x)^{\frac{2}{1+3w}}\int^{\eta}\tan^2 x\left(\frac{1}{4}y_1'^2 + 2y_1^2\right)(\sin x)^{\frac{2}{1+3w}}\,d\bar{\eta}. \quad (4.165)$$

We can already discern that the sign on $\delta_2^{\text{growing}}$ must be positive in a young universe.

In a radiation-dominated universe:

$$\delta_2^{\text{growing,radiation}} = \sigma^2_{\text{radiation}} \cot\eta \csc\eta \left(-\frac{1}{3}\cos 3\eta + \frac{1}{5}\cos 5\eta + 2\sec\eta\right); \tag{4.166}$$

note the diverging contribution of $\mathcal{O}\left(\eta^{-2}\right)$ from growing modes.

In a matter-dominated universe:

$$\delta_2^{\text{growing,matter}} = \sigma^2_{\text{matter}} \cot\frac{\eta}{2} \csc^2\frac{\eta}{2} \begin{pmatrix} -\frac{6063}{4}\eta + \frac{13001}{8}\sin\eta - \frac{3237}{8}\sin 2\eta + \\ +\frac{933}{8}\sin 3\eta - 33\sin 4\eta + \\ +\frac{32}{5}\sin 5\eta + 900\tan\frac{\eta}{2} \end{pmatrix}. \tag{4.167}$$

In contrast to the radiation-dominated case, the growing mode's contribution does not diverge in a matter-dominated universe (the term in brackets equals $0+\mathcal{O}\left(\eta^5\right)$). Approximating to lowest orders in η,

$$\delta_2^{\text{growing,matter}} \approx \sigma^2_{\text{matter}} \left(245\eta^2 - \frac{21641}{84}\eta^4\right). \tag{4.168}$$

Contributions from the decaying mode

In a radiation-dominated universe In a radiation-dominated universe, the functions y_1 and y_2 have the property

$$y_1^2 + y_2^2 = \csc^2\eta \tag{4.169}$$

while the functions y_1' and y_2' are similarly related by

$$y_1'^2 + y_2'^2 = \left(8\sin^2\eta + 1\right)\csc^4\eta. \tag{4.170}$$

This simplifies calculations as we can readily say

$$\delta_2^{\text{decaying}} = \tau^2 \cot\eta \csc\eta \left(\frac{17}{4}\sec\eta + \frac{1}{4}\ln\tan\frac{\eta}{2}\right) - \frac{\tau^2}{\sigma^2}\delta_2^{\text{growing}}; \tag{4.171}$$

in a universe old enough that the diverging terms are negligible, the decaying mode intrinsically decreases the scale factor in the same way that the growing mode intrinsically increases it.

In a matter-dominated universe In a matter-dominated universe,

$$y_1^2 + y_2^2 = \csc^4\frac{\eta}{2}\left(9 + \frac{1}{4}\cot^2\frac{\eta}{2}\right) \tag{4.172}$$

and

$$y_1'^2 + y_2'^2 = \frac{1}{16}\csc^8\frac{\eta}{2}\left(-608\cos\eta + 140\cos 2\eta + 477\right) \tag{4.173}$$

so we can state

$$\delta_2^{\text{decaying,matter}} = \tau^2 \cos\frac{\eta}{2}\csc^3\frac{\eta}{2}\left(\begin{array}{c} 18\eta + 2450\tan\frac{\eta}{2} - \\ -\frac{10705}{48}\cot\frac{\eta}{2} - \\ -\frac{577}{96}\cot\frac{\eta}{2}\csc^2\frac{\eta}{2} \end{array}\right) - \frac{\tau^2}{\sigma^2}\delta_2^{\text{growing}}. \tag{4.174}$$

It is interesting to note that, due to the growing mode contribution's much slower contribution to change in the scale factor, the impact of the decaying mode on the dynamics of a young universe can be many orders of magnitude greater than the impact of the growing mode even when the decaying mode is several orders of magnitude weaker than the growing mode. The ratio

$$\left| \frac{\delta_2^{\text{decaying,matter}}}{\delta_2^{\text{growing,matter}}} \right| \approx \frac{\tau^2_{\text{matter}}}{\sigma^2_{\text{matter}}} \eta^{-8} \tag{4.175}$$

which means that in a matter-dominated universe with $\eta \approx 10^{-1}$ the decaying mode will have a greater impact on cosmic dynamics as long as $\tau^2_{\text{matter}} > 10^{-8}\sigma^2_{\text{matter}}$.

Contributions from the $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$ term

The contributions are described by the equation

$$\delta_2^{\text{mixed}} = \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}} \int^{\eta} \left(4y_1 y_2 + \frac{1}{2} y_1' y_2'\right) \tan^2 x \left(\sin x\right)^{\frac{2}{1+3w}} \, d\bar{\eta} \tag{4.176}$$

and have the following explicit forms:

Radiation-dominated universe In a radiation-dominated universe,

$$\delta_2^{\text{mixed,radiation}} = \frac{16}{15} \boldsymbol{\sigma} \cdot \boldsymbol{\tau}_{\text{radiation}} \sin\eta \cos\eta \left(3\cos 2\eta + 2\right). \tag{4.177}$$

Matter-dominated universe In a matter-dominated universe,

$$\delta_2^{\text{mixed}} = \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \cot\frac{\eta}{2} \csc^2\frac{\eta}{2} \begin{pmatrix} -4\cos\eta - 24\cos^2\eta - \cos 3\eta - \\ -\frac{15}{2}\cos 4\eta + 5\cos 5\eta \end{pmatrix}. \tag{4.178}$$

Gravitational waves at second order

Turning now to the R_a^b equations (4.149, 4.150, 4.151), to second order, the Einstein equations for $\epsilon - p^{(a)}$-terms read:

$$\left\{\begin{array}{c} \alpha_2'' + 2\cot x\alpha_2' + 8\alpha_2 \\ +\frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \\ -3\left[\frac{w}{2}(1+w)\csc^2 x + 1\right]\delta_2 \end{array}\right\} = \left\{\begin{array}{c} \left[\begin{array}{c} -3 + \frac{1}{16}(1+3w)^2\tan^2 x + \\ +\frac{3}{16}(1+w)(3w-1) + \\ +(1+w)^2\left(\frac{9}{16} - \frac{3}{4}w\right)\csc^2 x \end{array}\right]\delta_1^2 - \\ -\alpha_1'\delta_1' + \left(6\beta_1^2 - 16\beta_1\gamma_1 + 6\gamma_1^2\right) - \\ -18\alpha_1^2 + 16\alpha_1\delta_1 \end{array}\right\} \tag{4.179}$$

etc. If we suppress all removable terms, as we must for any practical observation of second-order terms, and taking into account (4.162), this further simplifies to

$$\alpha_2'' + 2\cot x\alpha_2' + 8\alpha_2 - 3\left[\frac{w}{2}(1+w)\csc^2 x + 1\right]\delta_2 = \left[\begin{array}{c} -26\alpha_1^2 + \\ +14\beta_1^2 + 14\gamma_1^2 - \\ -\frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \end{array}\right]. \tag{4.180}$$

Recalling the form of the gravitational waves including the removable perturbation at first order, we make the simple transformation $\alpha_2 \to \tilde{\alpha}_2 + \frac{1}{3}\delta_2$ to arrive at the equations:

$$\tilde{\alpha}_2'' + 2\cot x\tilde{\alpha}_2' + 8\tilde{\alpha}_2 = 40\left[\frac{1}{3}\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) - \alpha_1^2\right] \tag{4.181}$$

etc.; we recognize that linear-order gravitational waves act as a driving force on the waves at quadratic order. The solution of this equation is straightforward but tedious and we arrive at the following solutions:

In a radiation-dominated universe

$$\alpha_2^{\text{radiation}} = \left[\begin{array}{c} C_{\alpha 1,2}\frac{\sin 3\eta}{\sin\eta} + C_{\alpha 2,2}\frac{\cos 3\eta}{\sin\eta} + \\ +40\left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left(\frac{1}{36}\frac{\sin 3\eta}{\sin\eta} - \frac{1}{6}\eta\frac{\cos 3\eta}{\sin\eta}\right) + \\ +40\left(\frac{1}{3}\tau^2 - C_{\alpha 2,1}^2\right)\left(\begin{array}{c} \frac{1}{6}\eta\frac{\cos 3\eta}{\sin\eta} + \frac{1}{36}\frac{\sin 3\eta}{\sin\eta} + \frac{5}{24} + \\ +\frac{1}{16}\frac{\sin 5\eta}{\sin\eta} - \frac{1}{6}\frac{(2\eta-\pi)\cos 3\eta - 2\sin 3\eta \ln(2\sin\eta)}{\sin\eta} \end{array} \right) + \\ +40\left(\frac{2}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - 2C_{\alpha 1,1}C_{\alpha 2,1}\right)\left(\begin{array}{c} \frac{1}{6}\eta\frac{\sin 3\eta}{\sin\eta} + \frac{1}{8}\cot\eta + \\ +\frac{1}{36}\frac{\cos 3\eta}{\sin\eta} - \frac{1}{32}\frac{\cos 5\eta}{\sin\eta} \end{array} \right) \end{array} \right] + \frac{1}{3}\delta_2 \tag{4.182}$$

etc. with the second-order constants $C_{\alpha 1,2}$ *etc.* constrained such that

$$C_{\alpha 1,2} + C_{\beta 1,2} + C_{\gamma 1,2} = C_{\alpha 1,2} + C_{\beta 1,2} + C_{\gamma 1,2} = 0. \tag{4.183}$$

To lowest order in η the solution for α_2 reads

$$\alpha_2^{\text{radiation}} \approx \left[\begin{array}{c} C_{\alpha 1,2}\left(3 - 4\eta^2\right) + 20\left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left(-\frac{1}{6} + \frac{11}{9}\eta^2\right) \\ +C_{\alpha 2,2}\eta^{-1} + \frac{20\pi}{3}\left(\frac{1}{3}\tau^2 - C_{\alpha 2,1}\right)\eta^{-1} + \\ +\frac{175}{36}\left(\frac{2}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - 2C_{\alpha 1,1}C_{\alpha 2,1}\right)\eta^{-1} + \frac{1}{3}\delta_2^{\text{non-removable}} \end{array} \right] \tag{4.184}$$

etc. For the pure decaying mode, the contribution from δ_2 dominates, while for the pure growing mode and the mixed term the contributions from the homogeneous parts of α_2 dominate.

In a matter-dominated universe For a matter-dominated universe, the gravitational wave equation to second order has the following solution[12]:

$$\alpha_2 = \left(\begin{array}{c} C_{\alpha 1,2} \csc \frac{\eta}{2} \frac{d}{d\eta} \frac{\sin 3\eta}{\sin \eta/2} + C_{\alpha 2,2} \csc \frac{\eta}{2} \frac{d}{d\eta} \frac{\cos 3\eta}{\sin \eta/2} + \\ + \alpha_2^{\text{growing}} + \alpha_2^{\text{decaying}} + \alpha_2^{\text{mixed}} + \frac{1}{3}\delta_2^{\text{non-removable}} \end{array} \right) \tag{4.185}$$

$$\alpha_2^{\text{growing}} \equiv 5 \left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2 \right) \left\{ \begin{array}{c} \frac{1}{70} \sum_{n=0}^{10} g_n \cos n\eta + \\ + \frac{1}{56} \csc^3 \frac{\eta}{2} \left[\begin{array}{c} \eta \left(\begin{array}{c} -1128960 \cos \frac{5\eta}{2} + \\ +806400 \cos \frac{7\eta}{2} \end{array} \right) + \\ + \sum_{n=0}^{11} h_n \sin \left(\frac{2n+1}{2} \eta \right) \end{array} \right] \end{array} \right\} \tag{4.186}$$

$$g_0 = 32900, g_1 = 443310, g_2 = 90230, g_3 = 354221, g_4 = 20195, g_5 = 248918,$$

$$g_6 = -57025, g_7 = 68911, g_8 = -37880, g_9 = 15440, g_{10} = -22400$$

$$h_0 = 1166543, h_1 = -1664285, h_2 = 888216, h_3 = 990580, h_4 = -1262310, h_5 = 677390,$$

$$h_6 = -363895, h_7 = 197841, h_8 = -116900, h_9 = 66864, h_{10} = -34304, h_{11} = 8960$$

$$\alpha_2^{\text{growing}} \approx \left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2 \right) \left(82630 - \frac{4513087}{7} \eta^2 \right) \tag{4.187}$$

[12] There is no "royal road" to the explicit statement of this function, which was derived by substitution and variation of parameters with the assistance of a computer algebra system. With foreknowledge of the form of the solution, the equation (4.181) can be solved through the method of undetermined coefficients; this requires solving a 21-dimensional linear system. (4.181) may also admit a solution through the method of Fourier transforms, but only under torture.

$$\alpha_2^{\text{decaying}} \equiv \frac{1}{245}\left(\frac{1}{3}\tau^2 - C_{\alpha2,1}^2\right)\csc^4\frac{\eta}{2}\left(\begin{array}{c} -\frac{\eta}{2}\tan\frac{\eta}{2}\sum_{n=0}^{4} j_n\cos^n\eta + \\ +\sum_{n=0}^{6} k_n\cos^n\eta + \\ +\ln\left(-2\sin^2\frac{\eta}{2}\right)\sum_{n=0}^{4} l_n\cos^n\eta \end{array}\right) \tag{4.188}$$

$$j_0 = -34020, j_1 = -17010, j_2 = 153090, j_3 = 22680, j_4 = -113400$$

$$k_0 = 58329, k_1 = -514422, k_2 = 368937, k_3 = 675396,$$

$$k_4 = -678540, k_5 = 31500, k_6 = 61250$$

$$l_0 = -5670, l_1 = 102060, l_2 = -73710, l_3 = -136080, l_4 = 113400$$

$$\alpha_2^{\text{mixed}} \equiv \frac{4}{105}\left(\frac{1}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - C_{\alpha1,1}C_{\alpha2,1}\right)\csc^2\frac{\eta}{2}\left(\begin{array}{c} \frac{\eta}{2}\sum_{n=0}^{3} m_n\cos\eta - \\ -\cot\frac{\eta}{2}\sum_{n=0}^{5} n_n\cos^n\eta \end{array}\right) \tag{4.189}$$

$$m_0 = 2310, m_1 = -39270, m_2 = -9240, m_3 = 46200$$

$$n_0 = -936, n_1 = 15693, n_2 = 30204, n_3 = -58700, n_4 = -25200, n_5 = 42000$$

$$\alpha_2^{\text{mixed}} \approx -\frac{32}{105}\left(\frac{1}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - C_{\alpha1,1}C_{\alpha2,1}\right)\eta^{-3}\sum_{n=0}^{5} n_n$$

etc. The statement of the solutions to the gravitational wave equations to quadratic order in the matter-dominated universe are original to this work; the radiation-dominated quadratic order wave equations were presented in [10]. Note that $\sum_n l_n = \sum_n m_n = 0$.

Most interesting is the presence of ln-terms in (4.171) and (4.188), which on the one hand indicate the appearance of the power-law behavior of metric coefficients which typify the Kasner universe and the BKL universe in its quasi-isotropic phase; on the other hand, they show the breakdown of our approximation scheme and the limit of regular perturbation theory in solving the problem to hand. The Taylor expansion of the growing mode of α_2 indicates further that waves must be very weak ($\|\sigma\| = \mathcal{O}\left(10^{-4}\right)$) for the approximation scheme to be rigorously valid as the presence of $\csc^4 \frac{\eta}{2} \ln\left(-2\sin\eta\right)$-terms in (4.188) indicates a function which is both complex and pathological. In any case, indications are that the growing mode of hypothetical cosmological gravitational waves should be very much stronger than the decaying mode (see SECTION 6.3); we will not need to make use of the second-order solutions for the decaying mode and from here on will treat the decaying mode as being linear-order weak, that is, $C_{\alpha 2,1}^2 \approx C_{\beta 2,1}^2 \approx C_{\gamma 2,1}^2 \approx C_{\alpha 2,2} \approx C_{\beta 2,2} \approx C_{\gamma 2,2} \approx \tau^2 \approx 0$.

4.4.5 Strong growing waves in the quasi-isotropic regime

[10, part 3] begins the development of equations for a radiation-dominated universe describing strong gravitational waves in Bianchi IX. Similar equations in a matter-dominated universe are useful in considering observed acceleration, as $\Delta Q \approx -1$.

Consider the equations (4.109-4.112). Assume a solution of the form

$$\begin{aligned} \alpha &= \sum_{n=0}^{\infty} c_{2n}^{\alpha} \eta^{2n} \\ \beta &= \sum_{n=0}^{\infty} c_{2n}^{\beta} \eta^{2n} \\ \gamma &= \sum_{n=0}^{\infty} c_{2n}^{\gamma} \eta^{2n} \end{aligned} \tag{4.190}$$

with the terms c_n^{ξ} constants. It is convenient to define $e^{2c_0^{\alpha}} \equiv A, e^{2c_0^{\beta}} \equiv B, e^{2c_0^{\gamma}} \equiv G$. In a matter-dominated universe, to lowest two orders the solutions read

$$\begin{aligned} \alpha \approx & c_0^{\alpha} + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5A^2 - 3B^2 - 3G^2 + 6BG - 2AB - 2AG\right)\right]\eta^2 \\ \beta \approx & c_0^{\beta} + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5B^2 - 3G^2 - 3A^2 + 6AG - 2BG - 2AB\right)\right]\eta^2 \\ \gamma \approx & c_0^{\gamma} + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5G^2 - 3A^2 - 3B^2 + 6AB - 2AG - 2BG\right)\right]\eta^2 \end{aligned} \tag{4.191}$$

where $c_0^{\alpha}, c_0^{\beta}, c_0^{\gamma}$ are arbitrary; if we want to preserve the Friedmannian value of S then we need

$$c_0^{\alpha} + c_0^{\beta} + c_0^{\gamma} = 0 \tag{4.192}$$

[10]. We always have the freedom to set one of these to zero by a simple scaling of the metric; this preserves the two degrees of freedom for the gravitational wave.

If we apply the condition (4.192) and set the parameter $c_0^{\gamma} = 0$ by scaling, then the strong growing-mode waves are described by

$$c_0^\alpha \in \mathbb{R} \tag{4.193}$$

$$c_0^\beta = -c_0^\alpha \tag{4.194}$$

$$c_0^\gamma = 0 \tag{4.195}$$

$$c_2^\alpha = \frac{1}{20}\left(-5A^2 + 2A + 6 - 6A^{-1} + 3A^{-2}\right) \tag{4.196}$$

$$c_2^\beta = \frac{1}{20}\left(3A^2 - 6A + 6 + 2A^{-1} - 5A^{-2}\right) \tag{4.197}$$

$$c_2^\gamma = \frac{1}{20}\left(3A^2 + 2A - 10 + 2A^{-1} + 3A^{-2}\right) \tag{4.198}$$

with the single parameter c_0^α determining the whole system. Note that setting $c_0^\gamma = 0$ does not imply $\gamma' = 0$. We can also qualitatively say that for any value of A, two of functions α, β, γ will be positive, as will δ, unless $A = 1$ (the background case), in the regime that $A\eta$ is sufficiently small that $A^3\eta^3$ is negligible.

The functions (4.190) are linearly independent with y_2^{matter} to lowest order in η and therefore can be used together to describe a matter-dominated universe with arbitrarily strong growing gravitational waves and weak decaying gravitational waves up to order η^2, as long as the series (4.190) converge.

4.4.6 Dynamics

As in the Kasner universe (see SECTION 4.2.1), it is useful to generalize quantities pertaining to the expansion of space which are spherically symmetric in Friedmannian cosmology.

In terms of our statement of the metric (4.85), the generalized dynamical quantities for our space are

$$a_{ab} = a_F \begin{pmatrix} e^{\alpha} & 0 & 0 \\ 0 & e^{\beta} & 0 \\ 0 & 0 & e^{\gamma} \end{pmatrix} \tag{4.199}$$

$$\bar{a} = \frac{1}{3} a_F \left(e^{\alpha} + e^{\beta} + e^{\gamma} \right) \tag{4.200}$$

$$H_{ab} = \begin{pmatrix} \dot{a}_F/a_F + \dot{\alpha} & 0 & 0 \\ 0 & \dot{a}_F/a_F + \dot{\beta} & 0 \\ 0 & 0 & \dot{a}_F/a_F + \dot{\gamma} \end{pmatrix} \tag{4.201}$$

$$\bar{H} = \frac{\dot{a}_F}{a_F} + \frac{1}{3}\dot{\delta} \tag{4.202}$$

$$Q^1_1 \equiv \frac{d}{dt} H^{1c}\eta_{1c} - \delta^1_1 = -\frac{\begin{pmatrix} \ddot{a}_F/a_F + 2H_F\dot{\alpha} + \\ + \ddot{\alpha} + \dot{\alpha}^2 \end{pmatrix}}{\left(H_F + \dot{\alpha}\right)^2} \tag{4.203}$$

etc.

$$\bar{Q} = -\frac{1}{3} \begin{pmatrix} \frac{\ddot{a}_F/a_F + 2H_F\dot{\alpha} + \ddot{\alpha} + \dot{\alpha}^2}{\left(H_F + \dot{\alpha}\right)^2} + \\ + \frac{\ddot{a}_F/a_F + 2H_F\dot{\beta} + \ddot{\beta} + \dot{\beta}^2}{\left(H_F + \dot{\beta}\right)^2} + \\ + \frac{\ddot{a}_F/a_F + 2H_F\dot{\gamma} + \ddot{\gamma} + \dot{\gamma}^2}{\left(H_F + \dot{\gamma}\right)^2} \end{pmatrix}. \tag{4.204}$$

Our goal in undertaking the arduous task of solving the Einstein equations has been to derive the impact of long-wavelength gravitational waves on cosmic dynamics, particularly acceleration. We are now in a position to begin to discuss this impact.

Let each quantity in section (4.4.6) be expanded out into a background term plus corrections, such that for example

$$a_{ab} \approx a^{(0)}_{ab} + a^{(1)}_{ab} + a^{(2)}_{ab}. \tag{4.205}$$

Then the zero-order, background terms are simply

$$a^{(0)}_{ab} = a_F \eta_{ab} \tag{4.206}$$

$$H^{(0)}_{ab} = H_F \eta_{ab} \tag{4.207}$$

$${}^{(0)}Q^b_a = Q_F \delta^b_a. \tag{4.208}$$

While the gravitational energy-momentum tensor vanishes at first order with the removal of removable perturbations, the presence of weak gravitational waves can affect observed dynamic quantities. At first order:

$$a^{(1)}_{ab} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix} \tag{4.209}$$

$$\bar{a}_{(1)} = \frac{1}{3}\delta_1 \tag{4.210}$$

$$H^{(1)}_{ab} = \begin{pmatrix} \dot{\alpha}_1 & 0 & 0 \\ 0 & \dot{\beta}_1 & 0 \\ 0 & 0 & \dot{\gamma}_1 \end{pmatrix} \tag{4.211}$$

$$\bar{H}_{(1)} = \frac{1}{3}\dot{\delta}_1 \tag{4.212}$$

$$^{(1)}Q^1_1 = -H_F^{-1}\left[2\left(Q_F+1\right)\dot{\alpha}_1 + H_F^{-1}\ddot{\alpha}_1\right] \tag{4.213}$$

etc.,

$$\bar{Q}_{(1)} = -\frac{1}{3}H_F^{-1}\left[2\left(Q_F+1\right)\dot{\delta}_1 + H_F^{-1}\ddot{\delta}_1\right]. \tag{4.214}$$

Thus we illustrate the need for truly representative sky coverage in considering the problem of acceleration: gravitational waves can contribute to anisotropic acceleration even when they do not affect the distribution of matter. In domains when the first derivatives of a wave is small (that is, near peaks and troughs of the wave), the accelerative effect will not be accompanied by a large change in the Hubble flow. As before, a failure to completely suppress the removable perturbation may lead to incorrect evaluation of the strength of decaying modes. To first order, non-zero contribution to the average over the whole sky of the perturbations is removable; first-order weak gravitational waves in Bianchi IX do not produce isotropic acceleration.

To quadratic order, the dynamic quantities have the forms

$$a^{(2)}_{ab} = a_F \begin{pmatrix} \alpha_2 + \alpha_1^2/2 & 0 & 0 \\ 0 & \beta_2 + \beta_1^2/2 & 0 \\ 0 & 0 & \gamma_2 + \gamma_1^2/2 \end{pmatrix} \tag{4.215}$$

$$\bar{a}_2 = \frac{1}{3}a_F\left[\delta_2 + \frac{1}{2}\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right)\right] \tag{4.216}$$

$$H_{ab}^{(2)} = \begin{pmatrix} \dot{\alpha}_2 & 0 & 0 \\ 0 & \dot{\beta}_2 & 0 \\ 0 & 0 & \dot{\gamma} \end{pmatrix} \tag{4.217}$$

$$\bar{H}_{(2)} = \frac{1}{3}\dot{\delta}_2 \tag{4.218}$$

$$^{(2)}Q_1^1 = -H_F^{-1}\left[\begin{array}{c} 2\left(Q_F+1\right)\dot{\alpha}_2 + H_F^{-1}\ddot{\alpha}_2 - \\ -3H_F^{-1}\left(Q_F+1\right)\dot{\alpha}_1^2 - 2H_F^{-2}\dot{\alpha}_1\ddot{\alpha}_1 \end{array}\right] \tag{4.219}$$

etc.,

$$\bar{Q}_{(2)} = -\frac{1}{3}H_F^{-1}\left[\begin{array}{c} 2\left(Q_F+1\right)\dot{\delta}_2 + H_F^{-1}\ddot{\delta}_2 - \\ -3H_F^{-1}\left(Q_F+1\right)\left(\dot{\alpha}_1^2+\dot{\beta}_1^2+\dot{\gamma}_1^2\right) - \\ -2H_F^{-2}\left(\dot{\alpha}_1\ddot{\alpha}_1+\dot{\beta}_1\ddot{\beta}_1+\dot{\gamma}_1\ddot{\gamma}_1\right) \end{array}\right] \tag{4.220}$$

. At second order we begin to see a consequence of the non-linearity of the Bianchi IX Einstein equations which is potentially very important in the study of cosmic dynamics: isotropic changes to the Hubble parameter and to acceleration from anisotropic metric terms. With our knowledge of the Einstein equations at first and second order (4.133,4.152,4.180) we can show this explicitly:

$$
{}^{(2)}Q^1_1 = -\tan x \left\{ \begin{array}{c} \left(3w + (1+3w)\tan^2 x\right)\alpha_2' - 8\tan x\alpha_2 - \\ -40\tan x\alpha_1^2 + \frac{3}{2}\left(1 - 3w - (1+3w)\tan^2 x\right)\tan x\alpha_1'^2 + \\ +16\tan^2 x\alpha_1'\alpha_1 + \\ +\tan x \left[\begin{array}{c} 3\left[\frac{w}{2}(1+w)\csc^2 x + 1\right]\delta_2 + \\ +14\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) - \frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \end{array} \right] \end{array} \right\} \tag{4.221}
$$

etc. and

$$
\bar{Q}_{(2)} = \frac{1}{3}\tan^2 x \left\{ \begin{array}{c} \frac{1}{2}(1+3w)^2\sec^2 x\delta_2 - \\ -2(1+3w)\sec^2 x\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) + \\ +\frac{1}{4}\left[1 + 15w + 5(1+3w)\tan^2 x\right]\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) - \\ -16\tan x\left(\alpha_1'\alpha_1 + \beta_1'\beta_1 + \gamma_1'\gamma_1\right) \end{array} \right\}. \tag{4.222}
$$

Isotropic acceleration with quadratic-order strength arises from the non-linear interaction of linear-order gravitational waves, but in the regime of $|\alpha|, |\beta|, |\gamma| \ll 1$ the gravitational waves at linear order will dominate measurement of cosmological parameters.

In a matter-dominated universe with η small, the deceleration terms become, defining

$$
\Delta Q^a_b \equiv Q^a_b - Q_F\delta^a_b \tag{4.223}
$$

$$
\Delta\bar{Q} \equiv \bar{Q} - Q_F \tag{4.224}
$$

$$\Delta Q^1_{1,\text{matter}} \approx -\frac{\eta^2}{4}\left\{\begin{array}{c} \tan\frac{\eta}{2}\left(\alpha_1' + \alpha_2'\right) - 8\left(\alpha_1 + \alpha_2\right) - \\ -40\alpha_1^2 + \frac{3}{2}\alpha_1'^2 + 16\tan\frac{\eta}{2}\alpha_1'\alpha_1 + \\ +3\delta_2 + 14\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) - \frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \end{array}\right\} \quad (4.225)$$

$$\Delta\bar{Q}^{\text{matter}} \approx \frac{1}{48}\eta^2\left[\begin{array}{c} 2\delta_2 - 8\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) + \\ +\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) - \\ -16\tan\frac{\eta}{2}\left(\alpha_1'\alpha_1 + \beta_1'\beta_1 + \gamma_1'\gamma_1\right) \end{array}\right] \quad (4.226)$$

etc. Explicitly, these will have the lowest-order forms:

$$\Delta Q^1_{1,\text{matter}} \approx -\frac{\eta^2}{4}\left(\begin{array}{c} C_{\alpha 1,1}\left(280 - 259\eta^2\right) + \\ +C_{\alpha 2,1}\left(16\eta^{-3} - 72\eta^{-1} + \frac{2251}{5}\eta\right) + \\ +C^2_{\alpha 1,1}\left(710040 - 4687753\eta^2\right) + \\ +\sigma^2\left(-609590/3 + \frac{4402405}{3}\eta^2\right) \end{array}\right) \quad (4.227)$$

$$\Delta\bar{Q}^{\text{matter}} \approx \frac{\sigma^2}{24}\eta^2\left(-4900 + 2983\eta^2\right) \quad (4.228)$$

. These results are encouraging as, if we choose $\|\sigma\| \sim 10^{-4}$ (in order to make the gravitational waves weak) and $\eta \sim 10^{-2}$to match (6.1), we obtain $\Delta Q^1_{1,\text{matter}} \sim -10^{-6}$, which has the right sign as well as all the contributions at both first and second orders going in the "right" direction, toward acceleration. It is particularly encouraging that both growing and decaying modes contribute to acceleration to their lowest orders in η.

4.4.7 Back-reaction

Of interest in discussing the problem of acceleration is the effective equation of state of the gravitational waves' contribution to the energy density. Empirically, the equation of state of dark energy seems to be close to $w_X = -1$ (see SECTION 2.3), where the quantity w_x is related to the source of the energy such that the source evolves with regard to the scale factor at a rate of $a^{-3(1+w_X)}$. As noted in (SECTION 4.2.1) there is no unique way to define the scale factor, but a condition of quasi-isotropy is that expansion in every direction in the current epoch is proportional, that is to say, that they evolve as the same power of time. If the decaying mode of the cosmological gravitational wave is weak, then this evolution will be proportional to the Friedmannian scale factor.

To quadratic order, (4.99) reads

$$k\epsilon_g^{(2)} = 3\,(1+w)\,a_F^{-2}\csc^2 x\delta_2 \tag{4.229}$$

and so by (4.114)

$$q_{(2)} = -\,(1+w)\,a_F^{-2}\delta_2. \tag{4.230}$$

When the growing mode is dominant, δ_2 is always positive in a matter-dominated universe; therefore $q_{(2)}$ is negative. Thus the back-reaction appears to have negative energy density. A significant "mixed" $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$ term, however, can easily introduce intervals where $q_{(2)} > 0$.

In a matter-dominated universe and when the growing mode is dominant, $q_{(2)} \propto$

η^{-2} which, if the universe is evolving with a scale as $a_F \propto \eta^2$, implies an equation of state for the back-reaction of $w_X = -1/3$ (as compared to an equation of state for a cosmological constant of $w_X = -1$). While no investigation of the equation of state of dark energy includes this value within its highest confidence interval, measurements of w_X remain tentative, with large errors and high sensitivity both to single data points and to the algorithm for curve-fitting models to the data (see SECTION 2.3). In any case, a fluid with an equation of state of $w_X \approx -1/3$ can be responsible for acceleration only if it dominates the universe and if $w_X < -1/3$, in accordance with (1.16).

The dominant term in (4.99) is the a'_F/a_F-term. This stands in stark contrast to the commonly-considered case of gravitational waves in a background so slowly moving compared to the period of the waves that $\dot{a}_F \approx 0$, in which instance the quadratic combination of first-derivative terms dominates.

In regimes of stronger growing-mode gravitational waves, though, the scale factor as defined in (4.19) will be more dominated by terms of higher, even order and so $a_{ab} \propto \eta^4$ or higher. As the growing mode increases in strength, the equation of state decreases asymptotically toward a limit of $w_X = -1$; if the scale factor grows as η^{2s}, the equation of state for the back-reaction is given by $w_X = (1/3s) - 1$. As acceleration is empirically $Q_0 = -0.6$, this implies that in real life the gravitational wave strength is of order unity and therefore the effective equation of state is close to -1. Thus, the quasi-isotropic Bianchi IX model with strong growing-mode gravitational waves and weak or zero decaying-mode waves is compatible with the observed data on the equation of state of dark energy, without the invocation of a cosmological constant; the theory would be invalidated by definitive measurements of $w_X < -1$.

In any case, the fact of $w_X < 0$ allows us to draw a conclusion regarding cosmic evolution. [76] notes Kasner-like cosmologies go through two stages of evolution:

1. A "vacuum" stage, where matter's influence is, due to its evolution as a^{-4}, weak compared to the influence of the anisotropic expansion and contraction, influence which, in light of (SECTION 4.3), we now understand to be the result of gravitational waves in the BKL universe;

2. a "matter" stage, where expansion isotropizes [106] and is driven by, first relativistic ($w = 1/3$), then cold, non-relativistic ($w = 0$) matter. Formally, the contribution of curvature to cosmic evolution becomes important in this era ($w_K = -1/3$), but as the influence of curvature will be isotropic in Bianchi IX and the radius of curvature is very large compared to the Hubble radius (see SECTION 6.1), curvature will not have a practical influence on observations in and of itself.[13] To this second stage we can add a third stage:

3. A "dark energy" stage, in which growing modes of the cosmological gravitational waves which drove the initial isotropy return as the dominant influence on cosmic evolution.

[13] Formally we can also say that, due to the action of proton decay and positron annihilation, after sufficient time the $w = 0$ phase will return to a $w = 1/3$ phase where the universe is filled with neutrinos and photons. Following this period there will be another return to $w = 0$ as these free particles are absorbed by black holes. As these black holes evaporate by the process of Hawking radiation, there will then be a final return to $w = 1/3$. [80] gives a popular-science presentation of the universe in these phases, but as it was written only shortly after the discovery of acceleration its treatment of dark energy is highly speculative.

4.4.8 Amplification of gravitational waves

Grishchuk observed [8] that when the background of a cosmology containing gravitational waves varies rapidly, weak gravitational waves can be amplified where they would otherwise, in a slowly-moving background, decay rapidly [79]. With regard to the Bianchi IX cosmology, this is significant as when the growing mode of a cosmological gravitational wave dominates, the leading term in the gravitational energy density is of the form $(a'_F/a_F)\,\delta'_2 = \mathcal{O}\,(\text{constant})$. Cosmological observations (see SECTION 6.1) indicate the universe has $\eta < \mathcal{O}\,(10^{-1})$. In this regime, the term $a'_F/a_F = \cot(\eta/2) \approx 2/\eta$, which is dependent on the rate of change of the background, is arbitrarily large; therefore, weak waves may have an effect orders of magnitude greater than their amplitude. Similarly, the decaying mode of gravitational waves can have prominent or even dominant power in a sufficiently young universe even when the amplitude of the decaying mode is smaller than that of the growing mode.

4.5 Conclusions

Solutions have been presented for the gravitational wave equation for a Bianchi IX universe perturbed to quadratic order from the closed Friedmann case. Quadratic order is the limit of perturbation theory's applicability to explore nearly-Friedmannian Bianchi IX when decaying modes are sufficiently strong that they are not negligible.

At quadratic order, the non-linear interaction of the gravitational waves produces isotropic changes to dynamic quantities. While this isotropic change is likely to be dominated in any particular direction by linear-order contributions from the gravitational waves, in the regime of strong gravitational waves they will become more important and potentially even dominant. Where [98] discussed the possibility of acceleration in a non-vacuum Bianchi IX universe only qualitatively, we have shown it explicitly as well as illustrating a clear link between acceleration and the gravitational waves which are intrinsic to Bianchi IX in its full generality.

It is curious to note that the order-η^2 approximation we have made in (SECTION 4.4.5), α and δ in the normalization we have chosen take the form of Alexander polynomials [109, 110], although not Alexander polynomials for any knot of fewer than 11 crossings. Whether this mathematical observation is significant or coincidental is a subject for further debate, but as gravitational waves in Bianchi IX are moving equatorially around our background 3-sphere [10], and as a sub-class of knots (the "torus knots") are constructed by wrapping one 2-torus around another it is conceivable there could be a connection.

Back-reaction from growing modes of the gravitational waves appears to have negative energy density and an equation of state compatible with that observed for dark energy, especially in the regime of strong gravitational waves and quasi-isotropic expansion; when gravitational waves are strong, they become the dominant contributor to the evolution of the cosmos in an era following the era of matter domination.

Therefore, from the perspective of cosmic dynamics, cosmological gravitational waves in a quasi-isotropic Bianchi IX universe are a viable candidate for dark energy, without the invocation of a cosmological constant and without requiring any modification of the theory of relativity. An analysis of the impact of these gravitational waves on the cosmic microwave background is necessary in order to determine whether constraints from the CMB are compatible with the observed data on acceleration.

Chapter 5

The Cosmic Microwave Background of a Bianchi IX universe

While long-wavelength gravitational waves can cause both isotropic and anisotropic changes to the deceleration parameter in a Bianchi IX universe, the effect of such waves must be compatible with the observed cosmic microwave background in order to represent a practical model for explaining observed acceleration.

Sachs & Wolfe initiated [23] the systematic study of the effect of perturbations on the CMB, following a formalism developed by Kristian & Sachs [25]. Sachs & Wolfe's work developed the theory of scalar, vector and tensor perturbations on the CMB in a flat almost-isotropic universe to first order.

Sachs & Wolfe's work was generalized by Anile & Motta [26] to the almost-isotropic closed and open Friedmann cosmologies, again at first order. While Anile & Motta begin to consider the impact of long-wavelength gravitational waves on the CMB, they choose to explore the impact of waves with scales much smaller than the Hubble radius. Anile & Motta subsequently [27] ruled out the existence of these waves at significant strengths in the observable universe.

Doroshkevitch, Lukash & Novikov considered the impact of an anisotropic universe on the CMB in the case of the Bianchi VII, VIII and IX models [19], and concluded that a Bianchi IX model was potentially "compatible with observations, only if there was some secondary heating of the intergalactic gas". Doroshkevitch *et al*'s most important calculations are carried out on the assumption, then widespread, of $\Omega_M \approx 1$ and as such are of limited applicability; interestingly, in their conclusions they note that if $\Omega_M < 1$, "$\Delta T/T$ will be close to the maximum value only in a small 'spot' with an angular size $\theta \approx 4\Omega$" (where by "small" they give the example of $\Omega_M \approx 0.1 \implies \theta \approx 23°$).

Sung & Coles analytically and computationally explore the impact of various unperturbed Bianchi models, including Bianchi IX, on the CMB [21]. They report the useful theorem that "a gravitational field alone is not able to generate polarization", but do not consider the general case of Bianchi IX, only the isotropic case equivalent to the closed Friedmann universe.

5.1 Geodesic equations

The effect of the metric on the CMB is determined by examining the change in geodesics of light rays relative to an isotropic, background case. Let the subscript E denote a function evaluated at the time of the emission of a photon, and the subscript R denote that function evaluated at the time of the photon's reception. Then the change in the temperature of the background radiation T is given by

$$T_R/T_E = \frac{1}{z+1}. \tag{5.1}$$

Consider the path of a light ray; let this be a four-vector denoted by k^μ such that $k^\mu k_\mu = 0$, with the light ray received in the direction $k^i_R = e^i$. The geodesic equation for the time part of k^μ in a Bianchi cosmology reads

$$\frac{dk^0}{d\lambda} + \Gamma^0_{ij} k^i k^j = 0 \tag{5.2}$$

and the equations for the space part of the vector read

$$\frac{dk^a}{d\lambda} + \Gamma^a_{00} + \Gamma^a_{0i} k^i + \Gamma^a_{i0} k^i + \Gamma^a_{bc} k^b k^c = 0. \tag{5.3}$$

Recalling (4.62) and (4.85) the Christoffel symbols

$$\Gamma^0_{ij} = \frac{1}{2}\gamma_{ab,0} e^a_i e^b_j, \tag{5.4}$$

, $\Gamma^a_{00} = \Gamma^a_{0i} = \Gamma^a_{i0} = 0$ and the Ricci rotation coefficients read[1]:

$$
\begin{aligned}
\Gamma^a_{bc} =& \frac{1}{2}\left(\delta^a_f \epsilon_{bcd} + \gamma^{ag}\gamma_{cd}\epsilon_{gbf} - \gamma^{ag}\gamma_{db}\epsilon_{cgf}\right)\eta^{df} \\
\Gamma^1_{23} =& \frac{1}{2}\left(\gamma^{11}\left(\gamma_{33}-\gamma_{22}\right)+1\right) = && \frac{1}{2}\left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}+1\right) \\
\Gamma^1_{32} =& \frac{1}{2}\left(\gamma^{11}\left(\gamma_{33}-\gamma_{22}\right)-1\right) = && \frac{1}{2}\left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}-1\right) \\
\Gamma^2_{31} =& \frac{1}{2}\left(\gamma^{22}\left(\gamma_{11}-\gamma_{33}\right)+1\right) = && \frac{1}{2}\left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}+1\right) \\
\Gamma^2_{13} =& \frac{1}{2}\left(\gamma^{22}\left(\gamma_{11}-\gamma_{33}\right)-1\right) = && \frac{1}{2}\left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}-1\right) \\
\Gamma^3_{12} =& \frac{1}{2}\left(\gamma^{33}\left(\gamma_{22}-\gamma_{11}\right)+1\right) = && \frac{1}{2}\left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}+1\right) \\
\Gamma^3_{21} =& \frac{1}{2}\left(\gamma^{33}\left(\gamma_{22}-\gamma_{11}\right)-1\right) = && \frac{1}{2}\left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}-1\right)
\end{aligned}
\tag{5.5}
$$

with all others zero; note that the form of the rotation coefficients guarantees that only anisotropic parts of the metric tensor will have an effect on k^i (and therefore δ-terms, whether removable or non-removable always vanish in the geodesic equations; recall SECTION 4.2.1). Using the same method of conformally-related objects as described in [23, part IIe], define the vector $\bar{k}^\mu : a_F^2\bar{k}^\mu = k^\mu$ and the tensor $\bar{\gamma}_{ab} : a_F^2\bar{\gamma}_{ab} = \gamma_{ab}$; recall that $k^0_R = -k^i_R k^R_i = 1$. This gives us geodesic equations:

$$\frac{d\bar{k}^0}{d\lambda} + \frac{1}{2}\bar{\gamma}_{ab,0}\bar{k}^a\bar{k}^b = 0 \tag{5.6}$$

$$\frac{d\bar{k}^1}{d\lambda} + \left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}\right)\bar{k}^2\bar{k}^3 = 0 \tag{5.7}$$

$$\frac{d\bar{k}^2}{d\lambda} + \left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}\right)\bar{k}^1\bar{k}^3 = 0 \tag{5.8}$$

$$\frac{d\bar{k}^3}{d\lambda} + \left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}\right)\bar{k}^1\bar{k}^2 = 0. \tag{5.9}$$

Despite the symmetry of these equations, their nonlinearity has inhibited the discovery of exact solutions and research into their properties is ongoing; see for

[1] The symbol ε_{abc} represents the Levi-Civita symbol defined such that $\varepsilon_{123} = 1$

example [24]. However, with solutions up to quadratic order for the metric in hand (4.140, 4.141, 4.184, 4.185), we can explicitly solve the equations in the case of weak waves. Let $\bar{k}^a = \bar{k}^a_R + \Delta\bar{k}^a(\lambda)$. Expanding out the geodesic equations to second order in the metric:

$$\frac{d\Delta\bar{k}^0_1}{d\lambda} + \frac{1}{2}\left[\alpha'_1\left(\bar{k}^1_R\right)^2 + \beta'_1\left(\bar{k}^2_R\right)^2 + \gamma'_1\left(\bar{k}^3_R\right)^2\right] = 0 \tag{5.10}$$

$$\frac{d\Delta\bar{k}^1_1}{d\lambda} + 2\left(\gamma_1 - \beta_1\right)\bar{k}^2_R\bar{k}^3_R = 0 \tag{5.11}$$

$$\frac{d\Delta\bar{k}^1_2}{d\lambda} + 2\left(\alpha_1 - \gamma_1\right)\bar{k}^1_R\bar{k}^3_R = 0 \tag{5.12}$$

$$\frac{d\Delta\bar{k}^1_3}{d\lambda} + 2\left(\beta_1 - \alpha_1\right)\bar{k}^1_R\bar{k}^2_R = 0 \tag{5.13}$$

$$\frac{d\Delta\bar{k}^0_2}{d\lambda} + \frac{1}{2}\begin{bmatrix} \left(\alpha'_2 + 2\alpha'_1\alpha_1\right)\left(\bar{k}^1_R\right)^2 + 2\bar{k}^1_R\alpha'_1\Delta\bar{k}^1_1 + \\ + \left(\beta'_2 + 2\beta'_1\beta_1\right)\left(\bar{k}^2_R\right)^2 + 2\bar{k}^2_R\beta'_1\Delta\bar{k}^2_1 + \\ + \left(\gamma'_2 + 2\gamma'_1\gamma_1\right)\left(\bar{k}^3_R\right)^2 + 2\bar{k}^3_R\gamma'_1\Delta\bar{k}^3_1 \end{bmatrix} = 0 \tag{5.14}$$

$$\frac{d\Delta\bar{k}^1_2}{d\lambda} + 2\begin{bmatrix} \left(\gamma_1 - \beta_1\right)\left(\bar{k}^2_R\Delta\bar{k}^3_1 + \bar{k}^3_R\Delta\bar{k}^2_1\right) + \\ + \left(\gamma_2 - \beta_2 + 3\gamma_1^2 - 3\beta_1^2\right)\bar{k}^2_R\bar{k}^3_R \end{bmatrix} = 0 \tag{5.15}$$

$$\frac{d\Delta\bar{k}^2_2}{d\lambda} + 2\begin{bmatrix} \left(\alpha_1 - \gamma_1\right)\left(\bar{k}^3_R\Delta\bar{k}^1_1 + \bar{k}^1_R\Delta\bar{k}^3_1\right) + \\ + \left(\alpha_2 - \gamma_2 + 3\alpha_1^2 - 3\gamma_1^2\right)\bar{k}^1_R\bar{k}^3_R \end{bmatrix} = 0 \tag{5.16}$$

$$\frac{d\Delta\bar{k}^3_2}{d\lambda} + 2\begin{bmatrix} \left(\beta_1 - \alpha_1\right)\left(\bar{k}^1_R\Delta\bar{k}^2_1 + \bar{k}^2_R\Delta\bar{k}^1_1\right) + \\ + \left(\beta_2 - \alpha_2 + 3\beta_1^2 - 3\alpha_1^2\right)\bar{k}^1_R\bar{k}^2_R \end{bmatrix} = 0. \tag{5.17}$$

To first order, the equations are trivially solved by choosing $\lambda = \eta$ as the affine parameter; the problem of determining $d\lambda/d\eta$ is overcome by our choice of reference system, the lack of vector perturbations and the homogeneity of space:

$$
\begin{aligned}
\Delta\bar{k}_1^0 =& -\frac{1}{2}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]_{\eta=\eta_E}^{\eta=\eta_R} \qquad (5.18)\\
=& -\frac{1}{2}\left[\tilde{\alpha}_1\left(\bar{k}_R^1\right)^2+\tilde{\beta}_1\left(\bar{k}_R^2\right)^2+\tilde{\gamma}_1\left(\bar{k}_R^3\right)^2+\frac{1}{3}\delta_1\right]_{\eta=\eta_E}^{\eta=\eta_R}\\
\Delta\bar{k}_1^1 =& 2\bar{k}_R^2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left(\beta_1-\gamma_1\right)d\eta \qquad (5.19)\\
\Delta\bar{k}_1^2 =& 2\bar{k}_R^3\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left(\gamma_1-\alpha_1\right)d\eta \qquad (5.20)\\
\Delta\bar{k}_1^3 =& 2\bar{k}_R^1\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left(\alpha_1-\beta_1\right)d\eta. \qquad (5.21)
\end{aligned}
$$

The relationship (5.18) explicitly shows the quadrupolar nature of changes to the CMB alluded to in [19]. An unremoved removable perturbation, that is, a gauge term which is not accounted for, changes the temperature of the whole sky isotropically; this confirms the effect noted by Hwang & Noh [42].

The equations for quadratic-order corrections read

$$
\frac{d\Delta\bar{k}_2^0}{d\lambda}+\frac{1}{2}\left[\begin{array}{c}
\left(\alpha_2'+2\alpha_1'\alpha_1\right)\left(\bar{k}_R^1\right)^2+2\bar{k}_R^1\alpha_1'\Delta\bar{k}_1^1+\\
+\left(\beta_2'+2\beta_1'\beta_1\right)\left(\bar{k}_R^2\right)^2+2\bar{k}_R^2\beta_1'\Delta\bar{k}_1^2+\\
+\left(\gamma_2'+2\gamma_1'\gamma_1\right)\left(\bar{k}_R^3\right)^2+2\bar{k}_R^3\gamma_1'\Delta\bar{k}_1^3
\end{array}\right]=0 \qquad (5.22)
$$

which due to the cancellation of the terms in the right column integrates trivially to

$$
\Delta\bar{k}_2^0=-\frac{1}{2}\left[\left(\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+\left(\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\right]_{\eta=\eta_E}^{\eta=\eta_R} \qquad (5.23)
$$

(reiterating the quadrupolar character of the change to the CMB, but generalizing it to anisotropic expansion); meanwhile for the space part of the vector

$$\frac{d\Delta\bar{k}_2^1}{d\lambda}+2\left[\begin{array}{c}(\gamma_1-\beta_1)\left(\bar{k}_R^2\Delta\bar{k}_1^3+\bar{k}_R^3\Delta\bar{k}_1^2\right)+\\ +\left(\gamma_2-\beta_2+3\gamma_1^2-3\beta_1^2+2\delta_1\left(\beta_1-\gamma_1\right)\right)\bar{k}_R^2\bar{k}_R^3\end{array}\right]=0 \tag{5.24}$$

$$\frac{d\Delta\bar{k}_2^2}{d\lambda}+2\left[\begin{array}{c}(\alpha_1-\gamma_1)\left(\bar{k}_R^3\Delta\bar{k}_1^1+\bar{k}_R^1\Delta\bar{k}_1^3\right)+\\ +\left(\alpha_2-\gamma_2+3\alpha_1^2-3\gamma_1^2+2\delta_1\left(\gamma_1-\alpha_1\right)\right)\bar{k}_E^3\bar{k}_E^1\end{array}\right]=0 \tag{5.25}$$

$$\frac{d\Delta\bar{k}_2^3}{d\lambda}+2\left[\begin{array}{c}(\beta_1-\alpha_1)\left(\bar{k}_R^1\Delta\bar{k}_1^2+\bar{k}_R^2\Delta\bar{k}_1^1\right)+\\ +\left(\beta_2-\alpha_2+3\beta_1^2-3\alpha_1^2+2\delta_1\left(\alpha_1-\beta_1\right)\right)\bar{k}_R^1\bar{k}_R^2\end{array}\right]=0 \tag{5.26}$$

which has solutions

$$\Delta\bar{k}_2^1=-2\left\{\begin{array}{c}2\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\gamma}_1-\tilde{\beta}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^2\right)^2\int^{\eta}\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)d\bar{\eta}+\\ +\left(\bar{k}_R^3\right)^2\int^{\eta}\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\ +\bar{k}_R^2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left(\gamma_2-\beta_2+\tilde{\gamma}_1^2-\tilde{\beta}_1^2+2\tilde{\alpha}_1\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)\right)d\eta\end{array}\right\} \tag{5.27}$$

$$\Delta\bar{k}_2^2=-2\left\{\begin{array}{c}2\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\alpha}_1-\tilde{\gamma}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^3\right)^2\int^{\eta}\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)d\bar{\eta}+\\ +\left(\bar{k}_R^1\right)^2\int^{\eta}\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\ +\bar{k}_R^3\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left(\alpha_2-\gamma_2+\tilde{\alpha}_1^2-\tilde{\gamma}_1^2+2\tilde{\beta}_1\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)\right)d\eta\end{array}\right\} \tag{5.28}$$

$$\Delta\bar{k}_2^3=-2\left\{\begin{array}{c}2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\beta}_1-\tilde{\alpha}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^1\right)^2\int^{\eta}\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)d\bar{\eta}+\\ +\left(\bar{k}_R^2\right)^2\int^{\eta}\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\ +\bar{k}_R^1\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left(\beta_2-\alpha_2+\tilde{\beta}_1^2-\tilde{\alpha}_1^2+2\tilde{\gamma}_1\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)\right)d\eta\end{array}\right\}. \tag{5.29}$$

5.2 Redshift and CMB variations

The geodesic of a light ray is related to its observed redshift by

$$z+1=\frac{\left(k^{\mu}u_{\mu}\right)_R}{\left(k^{\mu}u_{\mu}\right)_E} \tag{5.30}$$

[23]. Having determined $u_0 = 1$ and $u_i = 0$ this simplifies to

$$z+1=\frac{a_F\left(\eta_R\right)}{a_F\left(\eta_E\right)}\bar{k}_R^0 \tag{5.31}$$

so, to quadratic order,

$$z+1\approx\frac{a_F\left(\eta_R\right)}{a_F\left(\eta_E\right)}\left\{1-\frac{1}{2}\left[\begin{array}{c}\left(\alpha_1+\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+\\+\left(\beta_1+\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+\\+\left(\gamma_1+\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\end{array}\right]_{\eta=\eta_E}^{\eta=\eta_R}\right\} \tag{5.32}$$

. Meanwhile, the temperature field

$$\frac{T_R}{T_E}=\frac{1}{z+1}\approx\frac{a_F\left(\eta_E\right)}{a_F\left(\eta_R\right)}\left\{\begin{array}{c}1+\frac{1}{2}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]+\\+\frac{1}{4}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]^2+\\+\frac{1}{2}\left[\begin{array}{c}\left(\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+\\+\left(\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+\\+\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\end{array}\right]\end{array}\right\}_{\eta=\eta_E}^{\eta=\eta_R} \tag{5.33}$$

so

$$\frac{\Delta T}{T_R} \approx \frac{a_F(\eta_E)}{a_F(\eta_R)} \left\{ \begin{array}{c} \frac{1}{2}\left[\alpha_1\left(\bar{k}^1_R\right)^2 + \beta_1\left(\bar{k}^2_R\right)^2 + \gamma_1\left(\bar{k}^3_R\right)^2\right] + \\ +\frac{1}{4}\left[\alpha_1\left(\bar{k}^1_R\right)^2 + \beta_1\left(\bar{k}^2_R\right)^2 + \gamma_1\left(\bar{k}^3_R\right)^2\right]^2 + \\ +\frac{1}{2}\left[\begin{array}{c} \left(\alpha_2+\alpha_1^2\right)\left(\bar{k}^1_R\right)^2 + \\ +\left(\beta_2+\beta_1^2\right)\left(\bar{k}^2_R\right)^2 + \\ +\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}^3_R\right)^2 \end{array}\right] \end{array} \right\}_{\eta=\eta_E}^{\eta=\eta_R} . \tag{5.34}$$

5.3 Comparison with the observed CMB

Five-year and seven-year results [16, 18] from WMAP [30] give the best picture to date of the CMB. The WMAP observations reconfirm the constraint of the quantity $\Delta T/T < 10^{-4}$ [20]; any change to the CMB from acceleration must be equal to or smaller than this value in order to be compatible with observations, placing an additional constraint on cosmological models. This implies that in the current epoch, and in the absensce of further special alignment, $|\alpha|, |\beta|, |\gamma| \lesssim 10^{-5}$. In a matter dominated universe, under ordinary circumstances, this implies (since $\eta \lesssim 10^{-1}$; see SECTION 6.1)

$$\left|C_{\alpha 1,1} y_1^{\text{matter}}\right| \lesssim 10^{-5} \implies |C_{\alpha 1,1}| \lesssim 10^{-6} \tag{5.35}$$

$$\left|C_{\alpha 2,1} y_2^{\text{matter}}\right| \lesssim 10^{-5} \implies |C_{\alpha 2,1}| \lesssim 10^{-8}; \tag{5.36}$$

meanwhile in a radiation-dominated universe,

$$\left|C_{\alpha 1,1} y_1^{\text{radiation}}\right| \lesssim 10^{-5} \implies |C_{\alpha 1,1}| \lesssim 10^{-5} \tag{5.37}$$

$$\left|C_{\alpha 2,1} y_2^{\text{radiation}}\right| \lesssim 10^{-5} \implies |C_{\alpha 2,1}| \lesssim 10^{-6}. \tag{5.38}$$

The coefficients associated with the decaying mode are constrained to be smaller than those associated with the growing mode without further theoretical considerations.

5.3.1 CMB anomalies

Since the publication of the latest generation of CMB maps [28], numerous claims have been made (for example, [28, 32, 34, 39]) of anomalous structure in the CMB. While the WMAP team argue [17] that these phenomena are not of statistical significance, if a quasi-isotropic Bianchi IX universe could produce any of the perceived patterns it would point the way toward further observational studies of the CMB to determine cosmological parameters, and establish the quasi-isotropic Bianchi IX universe as a viable model for cosmology.

In all cases, we emphasize that the most likely explanation for any perceived pattern in the CMB which is not shown to be statistically significant is the null hypothesis: that is, the human perceptive phenomenon of pareidolia, the same phenomenon responsible for observing familiar shapes in clouds or the “Man in the Moon”.

Cold spots, “fingers” and the “Axis of Evil”

Two compact, supposedly anomalous areas of low temperature have been noted in the CMB, the so called “cold spots”.

The first of these (called Cold Spot I in [17]) is a region [31] covering approximately 15000 square degrees in the direction of the galactic center, much of which is 194 microkelvin [28] colder than the CMB mean temperature ($\Delta T/T_R = -7.12 \times 10^{-5}$).

Particularly noteworthy regarding Cold Spot I is its membership in one of four "fingers" spaced at roughly 90-degree angles around the galactic equator, intersticed by four areas of higher ($\Delta T/T_R = 7.12 \times 10^{-5}$) temperature[2]. Qualitatively, such a pattern is roughly consistent with the expected pattern if two of the functions $\alpha, \beta, \gamma > 0$ and if two of the the principle axes of the metric tensor lie on the axes of the cold and hot zones (implying the third axis points along the "Axis of Evil", see below). The so-called "Cold Spot II" reported by Vielva *et al.* [34, 37] also forms part of these "finger" structures [17].

Cold Spot I also has the angular size [19] predicts for the observed value of $\Omega_M \approx .3$.

Due to the coincidence of the cold spot with the direction of the galactic center, there are no optical observations in its direction (see FIGURE 2.2), and therefore there is no data on cosmic acceleration in the direction of Cold Spot I.

(EQUATION 5.34) implies that any cold spot resulting from anisotropy in the metric should be accompanied by an identical cold spot at a point antipodal to the original spot. Tegmark's examination [28] of the one-year WMAP data on the CMB low-order multipoles revealed an alignment between the CMB quadrupole and octupole in the direction of $(l, b) \approx (-110°, 60°)$ along which the quadrupole is nearly zero, an axis which Land & Maguiejo found [32] extended to the 16-pole and 32-pole

[2]The CMB dipole is defined as such a way as to be traceless, so $\int \Delta T_{\text{quadrupole}}/T_R dS = 0$.

as well; the alignment has been dubbed the "Axis of Evil". While examination of the three-year WMAP data [33] found the Axis of Evil to be of lower significance than initially thought (94%-98%), it still persists; the WMAP team's discussion of the alignment [17, pt. 7] admits the "remarkability" of this alignment and, while assigning its existence to chance, does not attempt to explain the "Axis of Evil" in full.

The Axis of Evil, which in equatorial coordinates [35, p. 43] lies close to RA 10:44 Dec $+7.6°$, falls within the zone in which redshift data has been collected for measurement of the cosmic deceleration parameter. To simplest linear approximation with a pure growing mode, (that is, that the functions α and α' are both small such that $\alpha^2 \approx 0$) this alignment rules out a CMB arising from cosmological gravitational waves as a source of cosmic acceleration. However, the fact of the alignment of the quadrupole, octopole, 16-pole and 32-pole indicates that non-linear contributions of gravitational waves to acceleration are not ruled out.

The question of the overall magnitude of the quadrupole, which is only 14% of the expected value [28, 38], has also been raised. The WMAP team [17, pt. 4] agree with Tegmark that the depressed quadrupole falls within the 95% confidence interval for simulations of the CMB, but do not attempt an explanation for the unusually strong octopole term. Long-wavelength gravitational waves can easily explain both through judicious choice of the arbitrary constants $C_{\alpha 1,1}$ etc. in a manner compatible with the CMB. Efstathiou [29] supposes that the depressed quadrupole could be an indication of a closed universe; however, the relationships he proposes generate zero contributions to the CMB power spectrum from the genuinely cosmological, intrinsic $n = 3$ waves found in Bianchi IX, and any observational test using his framework must rely on correct evaluation of gauge terms whose effective wavelengths must be far longer than the cosmic horizon. Furthermore, Efstathiou's conclusion that a closed universe would automatically require a scrapping of current inflationary models is contradicted by others; for example, Guth argues that a universe that is closed but with a very large radius of curvature is not ruled out [40].

The quasi-isotropic Bianchi IX model cannot provide an explanation for hemispherical dipole asymmetry claimed by Ericksen *et al.* [39].

5.4 Conclusions

The long-wavelength gravitational waves intrinsic to a quasi-isotropic Bianchi IX will cause a change in the cosmic microwave background with a distinctive quadrupolar signature. A radially-symmetric pattern of light deflections in the CMB resulting from shear may also be observed.

The almost-isotropic Bianchi IX model can be compatible with the CMB as observed, and can provide an explanation for perceived anomalies observed in the CMB by COBE and WMAP. However, the existence of these anomalies beyond the level of statistical noise is not certain; a possible route of cross-disciplinary research is open in the form of examination of the phenomenon of pareidolia as applied to the CMB.

Models of quasi-isotropic Bianchi IX relying on pure growing modes or pure decaying modes of the gravitational waves cannot simultaneously explain observed cosmic acceleration and the observed cosmic microwave background. Research into the non-linear regime of the Bianchi IX cosmology will elucidate the existence of a model of an accelerating Bianchi IX universe preserving an almost-isotropic CMB.

Chapter 6

An accelerating Bianchi IX universe preserving an almost-isotropic CMB

In order for a Bianchi IX universe to both appear nearly isotropic in the cosmic microwave background and to accelerate through the existence of long-wavelength gravitational waves, it must fulfill two conditions. The first is that the function $k^0(\eta_R)$ must have absolute value less than the limit imposed by observations of the cosmic microwave background, $\Delta T/T_R$. The second is that at least one of the functions $Q_a^b < 0$. It is possible for both these conditions to be simultaneously filled while remaining compatible with other observational constraints on cosmological parameters.

The idea of long-wavelength gravitational waves causing anisotropy in the CMB has been proposed, but not applied to the Bianchi IX universe. Grishchuk &

Zel'dovich consider the possibility of long-wavelength gravitational waves existing in a Friedmann universe without violating the limits imposed by the CMB [41], but do not apply their work to the gravitational waves of cosmological character which appear in some homogeneous cosmologies. Campanelli *et al.* suggest that such a universe could exist and propose a Taub-type Bianchi I universe which also includes anisotropic dark energy as an initial explanation for the observed CMB, complementing Rodrigues [113]. Critically, they do not consider gravitational waves as a generator of the anisotropy and treat the parameters of the Taub universe as if dark energy were simply established by fiat. Similarly, Kovisto and Mota [115] do not look beyond the Bianchi I model and instead fall back on exotic theories to explain dark energy.

6.1 Cosmological parameters

WMAP [18, 16] has produced an all-sky survey of the CMB which, if the universe is almost Friedmannian, can be used to constrain cosmological parameters.

Let the radius of curvature a_0 and conformal time η of the background Friedmann cosmology be treated as a free parameters; assume a closed universe. The WMAP seven-year data gives

$$H_0 = 70.4^{+1.3}_{-1.4} \text{km/s/Mpc} \tag{6.1}$$

$$\Omega_K = -.0025 \pm 0.0109 \tag{6.2}$$

(WMAP's analysis includes the value of Ω_K measured by baryon acoustic oscillations reported in [101]). The radius of curvature, Hubble parameter and curvature energy density are related by

$$a_0 = H_0^{-1}\sqrt{-\Omega_K^{-1}} \tag{6.3}$$

while the Hubble parameter, radius of curvature and η-time are related by

$$H_0 a_0 = \cot\left(\eta_0/2\right). \tag{6.4}$$

Therefore we have limiting values (as defined by the 95% confidence boundary of the WMAP observations)

$$a_0 \geq 1.12 \times 10^{29}\mathrm{cm} \tag{6.5}$$

$$\eta_0 \leq 0.0266 \tag{6.6}$$

and highest-confidence values

$$a_0 = 2.68 \times 10^{29}\mathrm{cm} \tag{6.7}$$

$$\eta_0 = 0.00499. \tag{6.8}$$

Meanwhile, the ratio of Hubble radius to radius of curvature is at least

$$H_0 a_0 \geq 8.67 \tag{6.9}$$

with a best-fit value of

$$H_0 a_0 = 20.0. \tag{6.10}$$

In other words, if the universe is closed, then the cosmological gravitational waves of the Bianchi IX cosmology are of much, much longer wavelength than the observable universe.

Finally, from the value of the redshift of decoupling, $z_{\text{last scattering}} = 1090$, we can say by (2.2) that

$$\eta_R/\eta_E \approx 33.0. \tag{6.11}$$

As the available data, including that from supernovae (see TABLE 2.1), do not exclude a flat universe, we are always free, in developing the theory of Bianchi IX and acceleration, to set the parameter η as close to zero as necessary. Doing so will not, in and of itself, violate observations, but will instead be constrained by the impact of the decaying mode of the gravitational waves on the CMB.

6.2 Compatibility with the redshift

Of all the observed cosmological parameters observed by WMAP and other probes of the CMB, the ones that are directly observed are $\Delta T/T_R$ and $z_{\text{last scattering}}$. From these we can say that in the current epoch the universe appears isotropic and that its expansion since last scattering has, on average to the present time, been isotropic. Neither of these facts necessarily imply that the overall expansion was isotropic at any time before the present. Instead, the condition of quasi-isotropy simply implies that

$$\frac{dk^0}{d\eta} + \frac{1}{2}\gamma_{ab,0}k^a k^b \approx 0. \tag{6.12}$$

This implies that shear is small, so

$$k^a \approx k_0^a \tag{6.13}$$

$$z+1 \approx a_F\left(\eta_R\right)/a_F\left(\eta_E\right) \tag{6.14}$$

as in the background Friedmann case.

We can obtain a near-zero value to the wave functions in the present epoch by admitting the presence of both growing and decaying modes in the gravitational waves. We want the condition (assuming $\Delta T/T_R$ is positive; in the case that it is negative the inequalities must be reversed)

$$0 \leq \frac{a_F\left(\eta_E\right)}{a_F\left(\eta_R\right)} e^{\alpha(\eta_R)-\alpha(\eta_R)} \leq \left|\Delta T/T_R\right| \tag{6.15}$$

and similarly for β, γ. In its full form this equation is transcendental even when discussing weak waves, but expanding (4.135) to lowest surviving order in η, we obtain

$$\left|37C_{\alpha 1,1}\left(\eta_R^2-\eta_E^2\right)+4C_{\alpha 2,1}\left(\eta_R^{-3}-\eta_E^{-3}\right)\right| \leq \left|\Delta T/T_R\right|. \tag{6.16}$$

In a young universe, the times of emission and reception of a light ray are related by $\eta_E \approx \eta_R\left(z+1\right)^{-1/2}$ so

$$\left|37C_{\alpha 1,1}\left(1-\left(z+1\right)^{-1}\right)\eta_R^2+4C_{\alpha 2,1}\left(1-\left(z+1\right)^{3/2}\right)\eta_R^{-3}\right| \leq \left|\Delta T/T_R\right|. \tag{6.17}$$

Let:

- 10^{-g} be the amplitude of the growing mode $C_{\alpha 1,1}$, so $C_{\alpha 1,1} = \mathrm{sgn}\left(C_{\alpha 1,1}\right)10^{-g}$;
- 10^{-d} be the amplitude of the decaying mode $C_{\alpha 2,1}$, so $C_{\alpha 2,1} = \mathrm{sgn}\left(C_{\alpha 2,1}\right)10^{-d}$;

- 10^{-b} be the value of η_R;
- 10^{-T} be the value of $|\Delta T/T_R|$

so noting that $z \sim 1000 = 10^3$, our condition becomes approximately

$$\left|\mathrm{sgn}\left(C_{\alpha 1,1}\right) 10^{-2b-g+3/2} - \mathrm{sgn}\left(C_{\alpha 2,1}\right) 10^{3b-d+5}\right| \lesssim \left|10^{-T}\right|. \tag{6.18}$$

When the amplitude of the growing mode term dominates, this approximate inequality is satisfied by

$$-2b - g + 3/2 \lesssim -T; \tag{6.19}$$

when the decaying mode dominates, the inequality is satisfied by

$$3b - d + 5 \lesssim -T. \tag{6.20}$$

WMAP constrains $T \approx 4$ (the difference between lowest and highest temperatures is $2\Delta T_R/T = 1.4 \times 10^{-4}$) and $b \gtrsim 1$. This constrains the growing and decaying modes, when they act on their own, to:

$$g \gtrsim 7/2 \tag{6.21}$$

$$d \gtrsim 12. \tag{6.22}$$

There exists a third possibility, in which the growing and decaying contributions are, in the current epoch, of equal size and opposite sign. For this to be the case, we need

$$-2b - g + 3/2 \approx 3b - d + 5; \tag{6.23}$$

this approach relies on the observation of amplification of weak gravitational waves in rapidly-changing backgrounds (see SECTION 4.4.8). Since b is a free parameter, this approximate equation can always be satisfied, but we still need to satisfy the constraints of the CMB.

6.3 Acceleration in the Bianchi IX universe

6.3.1 Order of magnitude estimates for gravitational wave amplitudes

We could naively attempt to relate an assumed isotropic acceleration to the constraints of the CMB by using (5.34) to constrain the amplitude of the gravitational wave functions and determining the value of (4.228) that results. This gives us, to lowest orders in η, assuming a pure growing mode of the gravitational waves and choosing $\bar{k}^a = (1, 0, 0)$ for simplicity,

$$-\frac{35}{2}\frac{\eta_E^2}{\eta_R^2}C_{\alpha 1,1}\eta_R^2 \sim \Delta T/T_R = 10^{-5} \implies C_{\alpha 1,1} \sim -2\times 10^{-7}\eta_R^{-2} \tag{6.24}$$

and therefore, if we assume that $C_{\alpha 1,1}, C_{\beta 1,1}, C_{\gamma 1,1}$ are all of the same order of magnitude,

$$\Delta \bar{Q}_2 \approx -\frac{4900}{24}\sigma^2\eta_R^2 \sim -\left(1\times 10^{-11}\right)\eta_R^{-2} \tag{6.25}$$

and thus we could conclude that the observed difference between the deceleration parameter and its Friedmann value

$$\Delta\bar{Q}_2 \approx -1 \implies \eta_R \sim 3 \times 10^{-6} \tag{6.26}$$

which is certainly on its own allowable under the observed cosmological parameters. To do this, however, would require $C_{\alpha 1,1} \sim 2 \times 10^5$, well beyond the limit of applicability of what could be called "weak" waves. Nonetheless, we can confidently say we have shown that weak gravitational waves can contribute to cosmic acceleration. This statement is the main result of this work.

Meanwhile, consider the anisotropic deceleration parameter:

$$Q_1^1 \equiv -\frac{\ddot{a}_{11} a_{11}}{(\dot{a}_{11})^2} = \left[Q_0 - 2\frac{a_F}{\dot{a}_F}\dot{\alpha} - \frac{a_F^2}{\dot{a}_F^2}\left(\ddot{\alpha} + \dot{\alpha}^2\right)\right]\left(1 + 2\frac{a_F}{\dot{a}_F}\dot{\alpha} + \frac{a_F^2}{\dot{a}_F^2}\dot{\alpha}^2\right)^{-1} \tag{6.27}$$

& similarly for Q_2^2, Q_3^3; this relationship is exact. Evaluating (4.203) gives to lowest surviving order in η

$$\Delta Q_{11,\text{growing}}^{(1)} \approx -70 C_{\alpha 1,1}\eta^2 \tag{6.28}$$

$$\Delta Q_{11,\text{decaying}}^{(1)} \approx \frac{19}{2} C_{\alpha 2,1}\eta^{-1}. \tag{6.29}$$

With an observed $\Delta Q_{11} \approx -1$ we can write:

$$10^0 \approx \operatorname{sgn}\left(C_{\alpha 1,1}\right) 10^{9/5-2b-g} - \operatorname{sgn}\left(C_{\alpha 2,1}\right) 10^{1-d+b}. \tag{6.30}$$

In the case of the growing mode dominating we need $\operatorname{sgn}\left(C_{\alpha 1,1}\right) = +1$ and $9/5 + 2b - g \approx 0$. This forms a system of equations with (6.19) so we have, at the limit of the allowed CMB perturbation,

$$\begin{cases} 9/5 - 2b - g \approx 0 \\ -2b - g + 11/2 \approx 0 \end{cases} \implies \text{no solution}; \tag{6.31}$$

the growing mode cannot, on its own, cause the observed acceleration and be compatible with the CMB. For the decaying mode, we need $\text{sgn}\,(C_{\alpha 2,1}) = -1$ and have

$$\begin{cases} 1 - d + b \approx 0 \\ 3b - d + 5 \approx -4 \end{cases} \implies \begin{cases} b \approx & -5 \\ d \approx & -6 \end{cases} \implies \begin{cases} \eta \sim 1 \times 10^5 \\ C_{\alpha 2,1} \sim 1 \times 10^6 \end{cases} \tag{6.32}$$

which is a nonsense result. Therefore neither the growing or decaying modes, on their own, can both cause observed acceleration and preserve the CMB. In the cases of the two modes having comparable effect on the metric and opposite sign, though, we can solve (6.30) with $\text{sgn}\,(C_{\alpha 1,1}) = +1$, $\text{sgn}\,(C_{\alpha 2,1}) = -1$ and

$$10^0 \approx -\left(C_{\alpha 1,1}\right) 10^{9/5-2b-g} + \left(C_{\alpha 2,1}\right) 10^{1-d+b} \tag{6.33}$$

$$\begin{cases} g - d \approx -5b - 7/2 \\ 9/5 - 2b - g \approx 0 \end{cases} \implies d \approx \frac{17}{10} + 7b, g \approx 2b - \frac{52}{10} \tag{6.34}$$

when the growing mode dominates the change in acceleration; this sets estimated limits on the parameters (since $b \gtrsim 2$):

$$C_{\alpha 1,1} \gtrsim 2 \times 10^1 \tag{6.35}$$

$$C_{\alpha 2,1} \lesssim 2 \times 10^{-16}. \tag{6.36}$$

When the decaying mode dominates the change in acceleration,

$$\begin{cases} g - d \approx -5b - 7/2 \\ 1 - d + b \approx 0 \end{cases} \implies g \approx -4b - 5/2, d \approx b + 1 \tag{6.37}$$

which constrains the parameters

$$C_{\alpha 1,1} \gtrsim 3 \times 10^{10} \tag{6.38}$$

$$C_{\alpha 2,1} \lesssim 1 \times 10^{-3}. \tag{6.39}$$

While the values for the growing mode are far greater than those for what could be called "weak" waves (recalling the constraints of SECTION 4.4.4), our educated estimate for $C_{\alpha 1,1}$ in the growing-mode dominated regime aligns nicely with the necessary strong-wave growing-mode value for ΔQ^1_1 disregarding the CMB; we could not have expected a change in acceleration at order unity in a universe where η is small to be driven by anything less than a gravitational wave so strong as to dominate the Friedmannian expansion. Therefore we can turn to an analysis in the quasi-isotropic regime.

6.3.2 Quasi-isotropic, strong growing mode acceleration

We apply the same reasoning as in the previous section, but we are aware of constraints (from [28]) not just on the CMB in the direction of the observed acceleration (which we continue to assign as the "α" or e^1_i direction) but on the CMB in the other two (the "beta" and "gamma" directions):

$$\Delta T_\alpha/T_R + \frac{1}{2}\left(\Delta T_\beta/T_R + \Delta T_\gamma/T_R\right) = 7.1\times 10^{-5} \tag{6.40}$$

$$2\Delta T_\beta/T_R = 1.4\times 10^{-4} \tag{6.41}$$

$$2\Delta T_\gamma/T_R = 1.4\times 10^{-4} \tag{6.42}$$

$$Q_1^1 = -0.6 \tag{6.43}$$

$$\eta_R \lesssim 3\times 10^{-2}. \tag{6.44}$$

In this and all regimes to follow we can also approximate $Q_F \approx Q_F^{\text{flat}} = 1/2$ to the limit of precision given the constraints on η; Q_F will be 1% stronger than Q_F^{flat} only when $\eta \approx 0.51$. Between the constraints (6.40-6.43) and the average over the sky of $\Delta T/T_R = 0$, we have four equations with seven unknowns (η, $c_0^\alpha, c_0^\beta, c_0^\gamma$, $C_{\alpha 2}, C_{\beta 2}, C_{\gamma 2}$). These equations are, explicitly (see equations 4.141, 4.203, 5.34):

$$7.1\times 10^{-5} \gtrsim (\eta_E/\eta_R)^2\left(e^{\alpha(\eta_R)-\alpha(\eta_E)} - 1\right) \tag{6.45}$$

$$1.4\times 10^{-4} \gtrsim (\eta_E/\eta_R)^2\left(e^{\beta(\eta_R)-\beta(\eta_E)} - 1\right) \tag{6.46}$$

$$1.4\times 10^{-4} \gtrsim (\eta_E/\eta_R)^2\left(e^{\gamma(\eta_R)-\gamma(\eta_E)} - 1\right) \tag{6.47}$$

$$Q_1^1 = \frac{Q_F - \tan(\eta_R/2)\,\alpha_R' - \tan^2(\eta_R/2)\,\alpha_R'' - \tan^2(\eta_R/2)\,\alpha_R'^2}{1 + 2\tan(\eta_R/2)\,\alpha_R' + \tan^2(\eta_R/2)\,\alpha_R'^2}. \tag{6.48}$$

Trivially, we can see that in the limit of $\alpha,\beta,\gamma \to \infty$, we must have $Q_1^1 \approx Q_2^2 \approx Q_3^3 \to -1$; if acceleration is driven by growing modes of long-wavelength gravitational waves then in the long run, the universe asymptotically approaches de Sitter expansion as if driven by a cosmological constant, indicating a solution in the regime of quasi-isotropy.

Consider the quasi-isotropic solution to the growing mode of the Einstein equations, normalized as in EQUATIONS (4.193-4.198). In the regime where c_0^α is sufficiently large that $A \gg 1$, we can approximate

$$c_2^\alpha \approx -\frac{1}{4}A^2 \tag{6.49}$$

$$c_2^\beta \approx c_2^\gamma \approx \frac{3}{20}A^2 \tag{6.50}$$

(an identical argument, with the functions α and β transposing their roles, applies for the case where $c_0^\alpha < 0$). From these terms we can also approximate the next order terms in the series:

$$c_4^\alpha \approx \frac{521}{5600}A^4 \tag{6.51}$$

$$c_4^\beta \approx c_4^\gamma \approx -\frac{15}{224}A^4. \tag{6.52}$$

Thus we see that the three functions α, β, γ are related in a Taub-like but not exactly-Taub fashion (this corresponds to case C_1 as described by [146]). Approximating equation (6.48) to order $A^4\eta^4$ we obtain the relationships

$$Q_1^1(\eta_R) = \frac{Q_F + \frac{3}{8}A^2\eta_R^2 - \left(\frac{521}{1120} + \frac{1}{16}\right)A^4\eta_R^4}{1 - \frac{1}{2}A^2\eta_R^2 + \left(\frac{1}{16} + \frac{521}{1400}\right)A^4\eta_R^4} + \mathcal{O}\left(\left(\frac{1}{2}A\eta_R\right)^6\right)$$

$$Q_2^2(\eta_R) \approx Q_3^3(\eta_R) = \frac{Q_F - \frac{9}{40}A^2\eta_R^2 + \left(\frac{75}{224} - \frac{9}{400}\right)A^4\eta_R^4}{1 + \frac{3}{10}A^2\eta_R^2 + \left(\frac{9}{400} - \frac{15}{56}\right)A^4\eta_R^4} + \mathcal{O}\left(\left(\frac{1}{2}A\eta_R\right)^6\right).$$

When $Q_1^1(\eta_R) = -0.6$ then $A\eta_R \approx 1.5 \pm 0.2$ ($c_0^\alpha \gtrsim 1.9$), within the limit of applicability of the expansion and also in the regime where the infinite series (4.190) converge. Thus, we have shown analytically that long-wavelength gravitational waves can explain cosmic acceleration if that acceleration is anisotropic.

We can also make the following qualitative assessments about acceleration. Firstly, its time-evolution is non-monotonic. In the α direction, the universe will at first exhibit slightly increased deceleration before starting to accelerate. In the β and γ directions, deceleration will asymptotically increase toward infinity but then acceleration will decrease from infinity, quickly converging on the strong-field value of $Q^2_2 = Q^3_3 = -1$. Acceleration in the α direction begins at $A\eta \approx 1.2$ and the universe accelerates in every direction after $A\eta \approx 1.6$; thus the supposition that acceleration is a recent phenomenon is supported.

A universe that is accelerating in every direction is within the region allowed by the model. FIGURE (6.1) illustrates the evolution of the deceleration parameters as a function of time. The constraints placed on the decaying mode in (SECTION 6.1) and the upper limit on η_R show that the decaying mode of long-wavelength gravitational waves has not played a significant role in cosmic acceleration; in the epoch of last scattering, the deceleration parameter was almost isotropic and had a close to Friedmannian value.

We now turn our attention to the preservation of the CMB. We have three equations in three unknowns, taking the lowest term in the decaying mode and the lowest

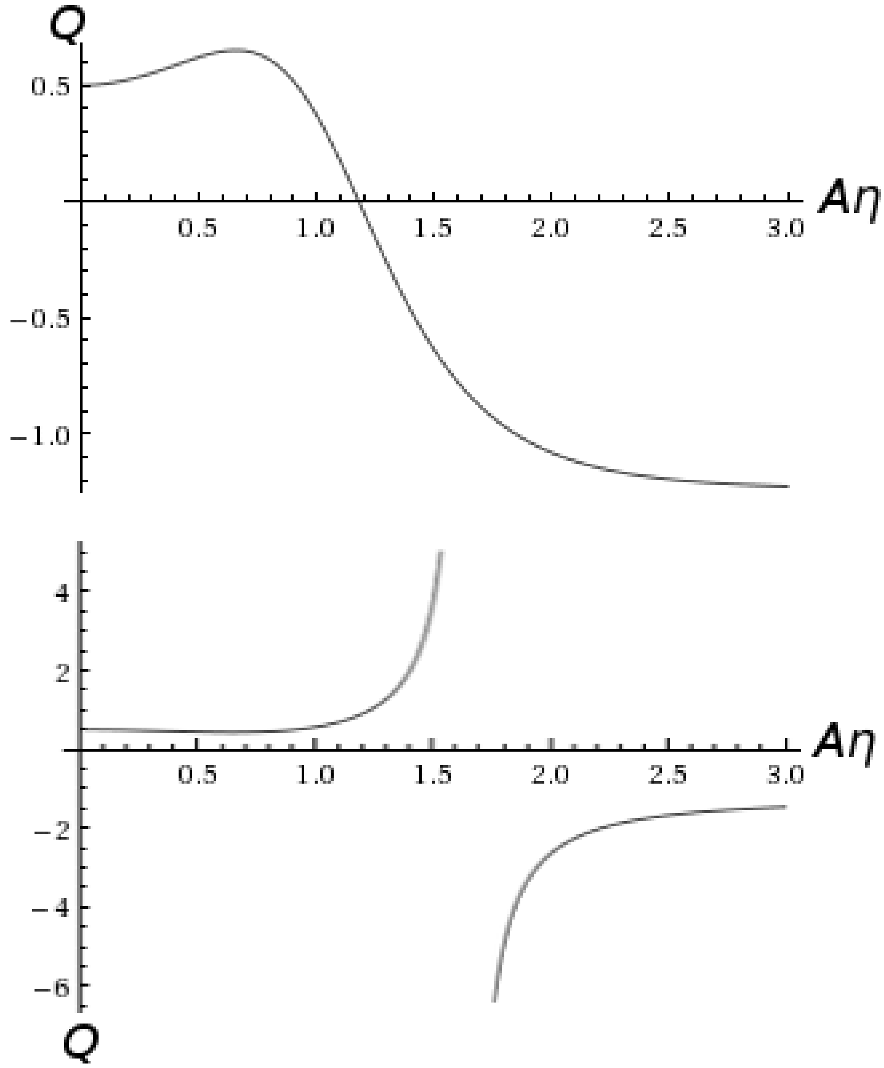

Figure 6.1: Deceleration parameter versus time

Along a preferred axis, the universe at first decelerates, then quickly begins accelerating, with the deceleration parameter asymptotically approaching −1. Along the other two axes, the deceleration parameter goes to infinity before converging asymptotically from negative infinity to the value −1. The vertical axis of each graph gives Q_a^b; the horizontal axis is in units of $A\eta$.

two terms in the growing mode:

$$7.1\times10^{-5} \gtrsim 4\left(\eta_E/\eta_R\right)^2 \left[\begin{array}{c} c_2^{\alpha}\left(\eta_R^2-\eta_E^2\right)+\frac{1}{2}\left(c_2^{\alpha}\right)^2\left(\eta_R^2-\eta_E^2\right)^2+ \\ +c_4^{\alpha}\left(\eta_R^4-\eta_E^4\right)+C_{\alpha 2,1}\left(\eta_R^{-3}-\eta_E^{-3}\right) \end{array}\right] \tag{6.53}$$

$$1.4\times10^{-4} \gtrsim 4\left(\eta_E/\eta_R\right)^2 \left[\begin{array}{c} c_2^{\beta}\left(\eta_R^2-\eta_E^2\right)+\frac{1}{2}\left(c_2^{\beta}\right)^2\left(\eta_R^2-\eta_E^2\right)^2+ \\ +c_4^{\beta}\left(\eta_R^4-\eta_E^4\right)+C_{\beta 2,1}\left(\eta_R^{-3}-\eta_E^{-3}\right) \end{array}\right] \tag{6.54}$$

$$1.4\times10^{-4} \gtrsim 4\left(\eta_E/\eta_R\right)^2 \left[\begin{array}{c} c_2^{\gamma}\left(\eta_R^2-\eta_E^2\right)+\frac{1}{2}\left(c_2^{\gamma}\right)^2\left(\eta_R^2-\eta_E^2\right)^2+ \\ +c_4^{\gamma}\left(\eta_R^4-\eta_E^4\right)+C_{\gamma 2,1}\left(\eta_R^{-3}-\eta_E^{-3}\right) \end{array}\right]. \tag{6.55}$$

As $30\eta_E \approx \eta_R$ we can further approximate

$$7.1\times10^{-5} \gtrsim \frac{1}{225}\left[\frac{1}{4}A^2\eta_R^2+\left(\frac{1}{32}-\frac{521}{5600}\right)A^4\eta_R^4+27000C_{\alpha 2,1}\eta_R^{-3}\right] \tag{6.56}$$

$$1.4\times10^{-4} \gtrsim \frac{1}{225}\left[-\frac{3}{20}A^2\eta_R^2+\left(\frac{9}{800}+\frac{15}{224}\right)A^4\eta_R^4+27000C_{\beta 2,1}\eta_R^{-3}\right] \tag{6.57}$$

$$1.4\times10^{-4} \gtrsim \frac{1}{225}\left[-\frac{3}{20}A^2\eta_R^2+\left(\frac{9}{800}+\frac{15}{224}\right)A^4\eta_R^4+27000C_{\gamma 2,1}\eta_R^{-3}\right]. \tag{6.58}$$

If we take the inequalities as approximate equivalences and use $A\eta_R \approx 1.5$ then this system has solutions

$$C_{\beta 2,1} \approx C_{\gamma 2,1} \approx \qquad 3\times10^{-7}\eta_R^3$$

$$C_{\alpha 2,1} \approx \qquad 9\times10^{-6}\eta_R^3$$

which is compatible with the estimates of (SECTION 6.3.1). That $C_{\alpha 2,1}+C_{\beta 2,1}+C_{\gamma 2,1} \neq 0$ is a consequence of the impossibility of *a priori* choosing an "unperturbed" temperature against which to compare anisotropic CMB fluctuations; the significance of non-linear terms means we cannot at the same time have the average over the whole sky of $\Delta T/T_R = 0$ and have $\delta_1 = 0$, recalling (EQUATION 5.18).

We exhaust almost all the freedom in the system (6.45-6.48) in choosing to explain the "Axis of Evil" at the same time as acceleration; if this requirement is dropped and we treat CMB variations as insignificant then a broad range of solutions opens up. In particular, if the ratio of growing mode to decaying mode is approximately equal for all three of α, β, γ we always have sufficient freedom to choose a η that reduces CMB variation to below the level of detectability, at the expense of "tuning" the universe to place us as observers in the era when the CMB is nearly isotropic.

Compatibility with an almost-isotropic Hubble flow

The objection could be raised that the necessity of the universe contracting along two axes demands that a large region of the sky be blue-shifted, which would surely have been observed. This problem can be made to vanish into statistical noise by the choice of a sufficiently small η as (4.201) implies $a_F H_{11} = a'_F/a_F + \alpha' \approx 2\left(\eta^{-1} + c_2^\alpha \eta\right)$ *etc.* It is notable that the very limited indications [149, 150] of anisotropic Hubble flow roughly align with the Axis of Evil and show angular scales on the order of 40°.

6.4 Conclusions

It is possible for a Bianchi IX universe with initial conditions $c_0^\alpha, c_0^\beta, c_0^\gamma \sim 1$ to display the acceleration observed in our universe while not only remaining compatible with the observed CMB but providing an explanation for potentially meaningful patterns in the CMB, specifically the so-called "Axis of Evil" and its associated phenomena such as cold spots. These conditions can be attained without additional constraints on the cosmological parameter of Ω_K, a parameter which is subject to further scrutiny and potentially tightening toward the flat universe case of $\Omega_K = 0$.

The method of combining strong growing modes with linear-order weak decaying modes of cosmological gravitational waves is borne out by observational data, which imply a difference of at least 17 orders of magnitude in amplitude between the growing and decaying modes. In the current epoch, decaying modes of cosmological gravitational waves can be neglected entirely. However, in the time close to last scattering, these modes may have participated at a strength comparable to the growing modes. Furthermore, the action of growing or decaying modes on their own is ruled out as an explanation for acceleration as neither on its own can preserve the CMB.

The question of how the ratio of growing mode to decaying mode is equal along all three principle axes of the metric tensor is answered easily if we postulate that cosmological gravitational waves present at the earliest moment in time were all in phase (the easiest way to do this is to postulate that they consisted of pure growing modes). As the functions α, β and γ would have all crossed the boundary from a $w = 1/3$ medium to a $w = 0$ medium at the same time, they would thus have remained in phase after last scattering, implying equal growing-to-decaying ratios for all three functions. As this transition happened in the very young universe ($\eta_E \lesssim 2 \times 10^{-3}$), the decaying mode that exists after last scattering would be very small.

The nonlinearity of Bianchi IX causes growing modes with initial values of order unity to develop exponentially and cause very powerful effects. The structure of the equations also indicates that multiple sets of initial conditions can lead to the same set of cosmological parameters. In light of the requirement of this model that both strong growing modes and weak but non-zero decaying modes of the gravitational waves exist, the possibility that these long-wavelength gravitational waves constituted the "pump field" of inflation [103] in the early universe should be explored.

The model proposed can be tested and is falsifiable by observation of acceleration in areas of the sky 90° from the highly-observed field; in areas of the sky away from the currently-observed acceleration, we will see either a very large deceleration parameter or a negative one. From the analysis of acceleration data in (Chapter 2) it is easy to see that, in the current state of observations, there are several possible areas of the sky where evidence of a gravitational-wave nature of cosmic acceleration could be sitting undetected.

Chapter 7

Conclusions

We have examined the current dominant hypotheses explaining cosmic acceleration and we have identified shortfallings in them, notably the overapplication of the assumption of isotropic acceleration in the absence of data covering large areas of the sky. We have completed an analysis of the almost-Friedmannian Bianchi IX cosmology perturbed to quadratic order in small corrections to the background metric and we have shown that both weak and strong cosmological gravitational waves could contribute to cosmic acceleration under some circumstances. We have completed an analysis of the effects quadratic-order weak gravitational waves would produce on the cosmic microwave background and we have shown that not only could cosmological gravitational waves be compatible with the CMB but their presence could account for many suggested anisotropic anomalies in the CMB. We have presented a set of cosmological parameters including strong growing cosmological gravitational waves and weak decaying cosmological gravitational waves which is compatible with observations of cosmic acceleration without the invocation of scalar fields of exotic equation of state, at the expense of strong anisotropy in the Hubble flow. We have proposed observational tests which would provide evidence for or against this model.

7.1 Directions for future research

The possibility of explaining cosmic acceleration through a Bianchi IX cosmological model opens up numerous possibilities for future research, both theoretical and observational.

While the difficulties with carrying out a full-sky optical survey of supernovae are understandable, experimental verification or falsification of a Bianchi IX model for acceleration requires nearly full sky coverage at high z to discover or rule out regions of anisotropy in the acceleration field. Infrared astronomy with wide sky coverage, for example WFIRST [65], presents the best possibility for these new observations through traditional astronomy. The Einstein telescope provides the tantalizing possibility of independent verification of the properties of dark energy through the examination of gravitational radiation [111].

Meanwhile, the available supernova data can be re-examined for signs of acceleration, although given the comparatively small datasets in any particular area other than the highly-observed field and the equatorial bias in the distribution of the data this re-examination is less likely to produce definitive results. Célérier is justified in her criticisms [91] of the assumptions being made in proposed models of cosmological acceleration; it is curious that the authors if [89] reasoned, with

44 low-z sources, that “poor coverage at low and moderate Galactic latitudes [...] makes it practically impossible to distinguish between a peculiar monopole and a quadrupole” but that [1], which shares two authors with [89], does not even mention the possibility of cosmographic bias in its smaller sample of high-z sources.

Consideration should be given to the question of why cosmographic bias exists, and whether it points to an unexpected privileging of the observer: namely, the fact that modern observatories are hosted only in regions of the Earth that can afford to host them.

Perturbative methods for solving the Einstein equations for weak gravitational waves in Bianchi IX can be considered exhausted, having reached the limit of practical utility at quadratic order. Further analytic explorations should concentrate on the quasi-isotropic approach. The fact of Bianchi IX’s easy reduction to a system of non-linear second-order ordinary differential equations combined with the divergence of Taylor series describing strong gravitational waves point toward either a Fourier-series approach or numerical methods for further analysis; the likelihood of chaotic behavior [73] in Bianchi IX, though, merits caution in the selection of initial conditions for any simulation.

Numerical examination of the quasi-isotropic regime should also be pursued for a fuller exploration of the space allowing for anisotropic acceleration while preserving an almost-isotropic cosmic microwave background. The next generation of microwave anisitropy probe should settle the question of whether the “Axis of Evil” and similar phenomena are genuine artifacts or statistical noise; in the meantime, the question of pareidolia in relation to the CMB has not been explored and deserves formal examination in order to raise awareness within the scientific community of the issue.

Overall, any theory is only as good as its ability to predict future results. Cosmic acceleration needs to be more closely examined, not only for time dependence, but for spatial dependence, before any theory can emerge as preferred.

7.2 Implications of the Bianchi IX cosmological model

Since the discovery of cosmic acceleration, a wide range of scalar theories, ranging from the mundane to the exotic, have been put forward to explain the phenomenon. While the fact of acceleration, the discovery of which was the logical culmination of the hunt for the "missing mass" of the universe above and beyond that provided by dark matter, necessarily implies the slaughter of at least one sacred cow, the community of physicists has no consensus over which should be sacrificed the most readily.

Attempts to surrender homogeneity are physically the best-grounded but philosophically the most rash. Certainly the idea of a purely homogeneous cosmology is an approximation, but a universe which is not on average homogeneous, that is, where the homogeneous regions are rare exceptions, is one in which cosmology as a science ceases to be possible. The "Swiss cheese" universe has the advantage of making use of a known, exact solution to the Einstein equations and at least avoids the exceptionalism of the "Hubble bubble" proposal, but defeats itself on the grounds of testability.

Meanwhile, postulation of exotic states of matter has been done too enthusiastically for the evidence available. The simple fact of noting that the available data on acceleration was anisotropic exposes as irrational exuberance the rush to explain the phenomenon through the medium of a substance which has never been seen or even indicated in the laboratory, and whose theoretical justification is far beyond testability. The willingness of many to see acceleration as a falsification of the theory of general relativity looks all the more bizarre when counterposed with the unwillingness to explore gravitational-wave solutions to the problem.

The objection could be raised that asserting acceleration to potentially be anisotropic, in the weak sense of the word "isotropy", violates the cosmological principle by saying that our telescopes are privileged observers, in that our observational field happens to align with an axis of acceleration. This is no more so true than the "privilege" hypothesized by, for example, Riess *et al.* when they assert, from a few dozen data points, that acceleration is a recent phenomenon, and that implicitly we are privileged observers in time for taking up cosmology just as the universe has begun to exhibit this behavior. While a cosmological constant is the simplest explanation for $w_X = -1$ on mathematical grounds, the lack of physical justification for a non-zero cosmological constant puts it in the same class as scalar-field theories. The simple fact is, $w_X = -1$ is, in the long run, the natural equation of state for any function which grows faster than the matter-driven terms in the background cosmology. The idea of the "Big Rip" [99], while intellectually (and emotionally) intriguing, makes the same mistake in the other direction, privileging observers to be alive just as the universe is beginning to tear itself apart. In this sense, a $w_X = -1$ field is the best preserver of the cosmological principle, and when the cosmological constant has been excluded the simplest explanation for acceleration comes from a tensorial field.

Similarly, when cosmic flatness is called into question – and it has never been and can never be definitively proven, it can only be disproven – the next-simplest model is the closed model. Recall that the Bianchi models are distinguished by their symmetries, and of all the Bianchi models with Friedmann universes as special cases, Bianchi IX has the most symmetric symmetries, obeying a "handedness" rule students learn before their first year of university. The fact of this "handedness" – parity – may even provide a neat explanation of the CP violation in particle physics [100], as Grishchuk alluded to [10].

The least speculative fact revealed by the assessment of available acceleration data is that more data is needed, from broader areas of the sky. The anticipated launch of WFIRST is likely to prove more momentous for cosmology than the flight of WMAP; WMAP largely reconfirmed what we already believed we knew, but WFIRST and SNAP will clearly illustrate how much we do not know. We also need techniques to see deeper into the sky and measure the distance-redshift relationship further into the past; the standard ladder of baryon acoustic oscillations [101] combined with better redshift data from WiggleZ may provide the necessary window.

That Bianchi IX could in principle contain accelerating regimes was never really in doubt. Numerical and qualitative analysis has indicated this ever since [10] noted that the vacuum equations contained a regular minimum, implying a positive first derivative for the Hubble parameter. The character of the acceleration has now been more properly investigated, bringing with it the possibility of a purely gravitational explanation for inflation, especially in light of the divergence of δ constructed only from growing modes in the radiation-dominated universe. An exploration of the differences between Bianchi I and Bianchi IX in a universe filled with ultra-relativistic matter could make Bianchi IX into a panacea for all the major problems of large-scale cosmology.

The unwillingness of the perturbed Bianchi IX cosmology to support decaying-mode gravitational waves stronger than linear order is puzzling, especially as the BKL universe always has a divergent term. The BKL universe, though, never reaches a singularity, and so the divergence of the a decaying mode never has time to take effect. Furthermore, the power law contraction along one axis could always be explained by a "growing" (non-diverging) function with negative coefficients, due to the exponential term in the metric.

The impact of strong waves on the CMB, meanwhile, also requires deeper explanation. Preservation of the CMB's apparent anisotropy at first glance appears to require some "tuning", a particular growing-decaying ratio which merits deeper questioning; there is also the outstanding matter of why we happen to live in one of the few periods of time when the CMB appears nearly isotropic. Clever examination of the symmetries of Bianchi IX may reveal a more satisfying answer, although the ability of Bianchi IX to explain CMB anomalies is one of its most satisfying features.

Most fundamentally, the biggest impact of the Bianchi IX theory of cosmic acceleration is the expansion of the cosmologist's parameter space. While in scalar models the only parameter truly open for discussion is the function describing the equation of state of dark energy, the gravitational waves of the Bianchi IX universe have four degrees of freedom; while the strength a non-zero cosmological constant has some theoretical justification in fundamental physics independent of large-scale cosmology, there is no immediately apparent reason why the gravitational waves in Bianchi IX should have any particular amplitude. As always in cosmology, we need more information than we have.

Appendix

Table 7.1: Supernova observations used i

of acceleration

	Right ascension, J2000	Declin
Riess 1998 supernovae:[90]		
SN1994U	13:04:56	−6:3:3
SN1997bp	12:46:54	−10:21
SN1996V	11:21:31	2:48:40
SN1994C	07:56:40	44° 52
SN1995M	09:38:42	−11:39
SN1995ae	23:16:56	−1:55:
SN1994B	08:20:41	15:43:

SN1995ao	02:57:31	−0:18:40
SN1995ap	03:12:28	0:41:43
SN1996R	11:16:10	0:11:39
SN1996T	10:05:28	−6:32:36
SN1997I	04:59:37	−2:50:58
SN1997ap	13:47:10	2:23:57
SDSS-II SNIa observations:[3]		
(Corner 1)	20:00:00	1:15:00
(Corner 2)	20:00:00	−1:15:00
(Corner 3)	04:00:00	1:15:00
(Corner 4)	04:00:00	−1:15:00
ESSENCE windows:[11]		

waa1	**23:29:52.92**	**—08:38**
waa2	**23:27:27.02**	**—08:38**
waa3	**23:25:01.12**	**—08:38**
waa5	**23:27:27.02**	**—09:14**
waa6	**23:25:01.12**	**—09:14**
waa7	**23:30:01.20**	**—09:44**
waa8	**23:27:27.02**	**—09:50**
waa9	**23:25:01.12**	**—09:50**
wbb1	**01:14:24.46**	**00:51:4**
wbb3	**01:09:36.40**	**00:46:4**
wbb4	**01:14:24.46**	**00:15:4**
wbb5	**01:12:00.46**	**00:15:4**

wbb6	**01:09:00.16**	**00:10:43.3**
wbb7	**01:14:24.46**	**−00:20:17**
wbb8	**01:12:00.46**	**−00:20:17**
wbb9	**01:09:36.40**	**−00:25:16**
wcc1	**02:10:00.90**	**−03:45:00**
wcc2	**02:07:40.60**	**−03:45:00**
wcc3	**02:05:20.30**	**−03:45:00**
wcc4	**02:10:01.20**	**−04:20:00**
wcc5	**02:07:40.80**	**−04:20:00**
wcc7	**02:10:01.55**	**−04:55:00**
wcc8	**02:07:41.03**	**−04:55:00**
wcc9	**02:05:20.52**	**−04:55:00**

wdd2	**02:31:00.25**	**—07:48**
wdd3	**02:28:36.25**	**—07:48**
wdd4	**02:34:30.35**	**—08:19**
wdd5	**02:31:00.25**	**—08:24**
wdd6	**02:28:36.25**	**—08:24**
wdd7	**02:33:24.25**	**—08:55**
wdd8	**02:31:00.25**	**—09:00**
wdd9	**02:28:36.25**	**—09:00**
HST supernovae:[14]		
SCP05D0	**02:21:42.066**	**—03:21**
SCP06H5	**14:34:30.140**	**34:26:5**
SCP06K0	**14:38:08.366**	**34:14:1**

SCP06K18	14:38:10.665	34:12:47.1
SCP06R12	02:23:00.083	−04:36:03
SCP06U4	23:45:29.430	−36:32:45
SCP06C1	12:29:33.013	01:51:36.6
SCP06F12	14:32:28.749	33:32:10.0
SCP05D6	02:21:46.484	−03:22:56
SCP06G4	14:29:18.744	34:38:37.3
SCP06A4	22:16:01.078	−17:37:22
SCP06C0	12:29:25.655	01:50:56.5
SCP06G3	14:29:28.430	34:37:23.1
SCP06H3	14:34:28.879	34:27:26.6
SCP06N33	02:20:57.699	−03:33:23

SCP05P1	03:37:50.352	−28:43
SCP05P9	03:37:44.513	−28:43
SCP06X26	09:10:37.888	54:22:2
SCP06Z5	22:35:24.967	−25:57
Riess "gold" dataset:[56, 54]		
Window 1	03:32:30	−27:46
Window 2	12:37:00	62:10:0

Bibliography

[1] Riess, A *et al.* "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant". The Astronomical Journal, Volume 116, Issue 3, pp. 1009-1038. (09/1998) arXiv:astro-ph/9805201

[2] Landau, L; Lifshitz, E. *The Classical Theory of Fields, Fourth Revised English Edition.* Oxford: Pergamon Press (1975)

[3] Sako, M *et al.* "The Sloan Digital Sky Survey-II Supernova Survey: Search Algorithm and Follow-Up Observations". The Astronomical Journal, Volume 135, Issue 1, pp. 348-373 (2008). (01/2008) arXiv:0708.2750

[4] Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Wiley & Sons, Chicago. (1975)

[5] Bondi, H. *Cosmology.* Cambridge Monographs on Physics, Cambridge: University Press, 1961, 2nd ed. (1961)

[6] Huterer, D; Turner, M. "Prospects for probing the dark energy via supernova distance measurements". Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 60, Issue 8, 15 October 1999, id. 081301 (10/1999) arXiv:astro-ph/9808133

[7] Grishchuk, L. "On Spatially Uniform Gravitational Fields" Soviet Physics Doklady, Vol. 15, p.130 (08/1970)

[8] Grishchuk, L. "Amplification of gravitational waves in an isotropic universe." Zh. Ehksp. Teor. Fiz., Tom 67, p. 825-838 (1974)

[9] Sollerman, J *et al.* "First-Year Sloan Digital Sky Survey-II (SDSS-II) Supernova Results: Constraints On Non-Standard Cosmological Models". The Astrophysical Journal, Volume 703, Issue 2, pp. 1374-1385 (2009). (10/2009) arXiv:0908.4276

[10] Grishchuk, L *et al.,* "Long Gravitational Waves in a Closed Universe". Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, vol. 69, Dec. 1975, p. 1857-1871. In Russian. [Sov. Phys. JETP, 42, 6 (1975)] (12/1975)

[11] Mikinatis, G *et al.* "The ESSENCE Supernova Survey: Survey Optimization, Observations, and Supernova Photometry". The Astrophysical Journal, Volume 666, Issue 2, pp. 674-693. (09/2007) arXiv:astro-ph/0701043

[12] Sullivan, M *et al.* "The dependence of Type Ia Supernovae luminosities on their host galaxies". Monthly Notices of the Royal Astronomical Society, Volume 406, Issue 2, pp. 782-802. (08/2010) arXiv:1003.5119

[13] Sullivan, M *et al.* "SNLS3: Constraints on Dark Energy Combining the Supernova Legacy Survey Three Year Data with Other Probes". The Astrophysical Journal, Volume 737, Issue 2, article id. 102 (2011). (08/2011) arXiv:1104.1444v2

[14] Suzuki, N *et al.* "The *Hubble Space Telescope* Cluster Supernova Survey: V. Improving The Dark Energy Constraints Above Z>1 And Building An Early-Type-Hosted Supernova Sample". eprint arXiv:1105.3470 (05/2011) arXiv:1105.3470v1

[15] Blake, C *et al.* "The WiggleZ Dark Energy Survey: the growth rate of cosmic structure since redshift z = 0.9". Monthly Notices of the Royal Astronomical Society, Volume 415, Issue 3, pp. 2876-2891. (08/2011) arXiv:1104.2948v1

[16] Jarosik, N *et al.* "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Sky Maps, Systematic Errors, and Basic Results". The Astrophysical Journal Supplement, Volume 192, Issue 2, article id. 14 (2011). (02/2011) arXiv:1001.4744

[17] Bennett, C *et al.* "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Are There Cosmic Microwave Background Anomalies?" The Astrophysical Journal Supplement, Volume 192, Issue 2, article id. 17 (2011). (02/2011) arXiv:1001.4758

[18] Hinshaw, G *et al.* "Five-Year Wilkinson Microwave Anisotropy Probe Observations: Data Processing, Sky Maps, And Basic Results". The Astrophysical Journal Supplement, Volume 180, Issue 2, pp. 225-245 (2009). (02/2009) arXiv:0803.0732

[19] Doroshkevitch, A; Lukash V; Novikov I. "Primordial radiation in a homogeneous but isotropic universe". (Astronomicheskii Zhurnal, vol. 51, Sept.-Oct. 1974, p. 940-950.) Soviet Astronomy, vol. 18, Mar.-Apr. 1975, p. 554-560. Translation. (04/1975)

[20] Wright, E. "Theoretical Overview of Cosmic Microwave Background Anisotropy". In *Measuring and Modeling the Universe,* from the Carnegie Observatories Centennial Symposia. Published by Cambridge University Press, as part of the Carnegie Observatories Astrophysics Series. Edited by W. L. Freedman, 2004, p. 291. (2004) arXiv:astro-ph/0305591

[21] Sung, R; Coles, P. "Temperature and Polarization Patterns in Anisotropic Cosmologies". Journal of Cosmology and Astroparticle Physics, Issue 06, pp. 036 (2011). (06/2011) arXiv:1004.0957

[22] Barrow, J. "Cosmological Limits on Slightly Skew Stresses". Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 55, Issue 12, 15 June 1997, pp.7451-7460. (06/1997) arXiv:gr-qc/9701038

[23] Sachs, R; Wolfe A. "Perturbations of a Cosmological Model". Astrophysical Journal, vol. 147, p.73 (01/1967)

[24] Nilsson, U *et al.* "A dynamical systems approach to geodesics in Bianchi cosmologies". General Relativity and Gravitation, Volume 32, Issue 7, p.1319-1343 (07/2000) arXiv:gr-qc/9908062

[25] Kristian, J; Sachs, R. "Observations in Cosmology". Astrophysical Journal, vol. 143, p.379 (02/1966)

[26] Anile, A; Motta, S. "Perturbations of the general Robertson-Walker universes and angular variations of the cosmic blackbody radiation". Astrophysical Journal, vol. 207, Aug. 1, 1976, pt. 1, p. 685-692. Research supported by the Consiglio Nazionale delle Ricerche. (08/1976)

[27] Anile, A; Motta, S. "Long-wavelength gravitational waves and the fine-scale anisotropy of the cosmic microwave background". Royal Astronomical Society, Monthly Notices, vol. 184, Aug. 1978, p. 319-326. Research supported by the Consiglio Nazionale delle Ricerche. (08/1978)

[28] Tegmark, M *et al.,* "A high resolution foreground cleaned CMB map from WMAP". Physical Review D, vol. 68, Issue 12, id. 123523 (12/2003) arXiv:astro-ph/0302496

[29] Efstathiou, G. “Is the Low CMB Quadrupole a Signature of Spatial Curvature?” Monthly Notice of the Royal Astronomical Society, Volume 343, Issue 4, pp. L95-L98. (08/2003) arXiv:astro-ph/0303127

[30] Bennett, C *et al.* “The Microwave Anisotropy Probe Mission”. The Astrophysical Journal, Volume 583, Issue 1, pp. 1-23. (01/2003) arXiv:astro-ph/0301158

[31] C. L. Bennett *et al.* “First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results”. The Astrophysical Journal Supplement Series, Volume 148, Issue 1, pp. 1-27. (09/2003) arXiv:astro-ph/0302207

[32] Land, K; Maguiejo, J. “The Axis of Evil”. Phys.Rev.Lett. 95 (2005) 071301. (08/2005) arXiv:astro-ph/0502237v2

[33] Land, K; Maguiejo, J. “The Axis of Evil Revisited”. Monthly Notices of the Royal Astronomical Society, Volume 378, Issue 1, pp. 153-158. (06/2007) arXiv:astro-ph/0611518

[34] Cruz, M.; Martínez-González, E.; Vielva, P.; Cayón, L. “Detection of a non-Gaussian spot in WMAP”. Monthly Notices of the Royal Astronomical Society, Volume 356, Issue 1, pp. 29-40. (01/2005) arXiv:astro-ph/0405341

[35] Duffett-Smith, P. *Practical astronomy with your calculator,* 3rd edition. Cambridge University Press, 1988

[36] Zhang, R; Huterer, D. “Disks in the sky: A reassessment of the WMAP ’cold spot”’ Astroparticle Physics, 33, 69 (2010). (03/2010) arXiv:0908.3988

[37] Vielva, P *et al.* “Detection of Non-Gaussianity in the Wilkinson Microwave Anisotropy Probe First-Year Data Using Spherical Wavelets”. The Astro-

physical Journal, Volume 609, Issue 1, pp. 22-34. (07/2004) arXiv:astro-ph/0310273

[38] Smoot, G *et al.* "Structure in the COBE differential microwave radiometer first-year maps". Astrophysical Journal, Part 2 - Letters (ISSN 0004-637X), vol. 396, no. 1, Sept. 1, 1992, p. L1-L5. Research supported by NASA. (09/1992)

[39] Eriksen, H; Hansen, F; Banday, A; Górski, K; Lilje, P. "Asymmetries in the Cosmic Microwave Background Anisotropy Field". The Astrophysical Journal, Volume 605, Issue 1, pp. 14-20. (04/2004) arXiv:astro-ph/0307507

[40] Guth, A; Pi, S.-Y. "Fluctuations in the new inflationary universe" Physical Review Letters, vol. 49, Oct. 11, 1982, p. 1110-1113. (10/1982)

[41] Grishchuk, L; Zeldovich, Ia. "Long-wavelength perturbations of a Friedmann universe, and anisotropy of the microwave background radiation". (Astronomicheskii Zhurnal, vol. 55, Mar.-Apr. 1978, p. 209-215.) Soviet Astronomy, vol. 22, Mar.-Apr. 1978, p. 125-129. Translation. (04/1978)

[42] Hwang, J; Noh, H. "Sachs-Wolfe effect: Gauge independence and a general expression". Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 59, Issue 6, 15 March 1999, id. 067302. (03/1999) arXiv:astro-ph/9812007

[43] Drinkwater, M *et al.* "The WiggleZ Dark Energy Survey: Survey Design and First Data Release". Monthly Notices of the Royal Astronomical Society, Volume 401, Issue 3, pp. 1429-1452. (01/2010) arXiv:0911.4246

[44] Colgate, S. "Supernovae as a standard candle for cosmology". Astrophysical Journal, Part 1, vol. 232, Sept. 1, 1979, p. 404-408. Research supported by the U.S. Department of Energy and NSF. (09/1979)

[45] Wagoner, R. "Determining q_0 from Supernovae". Astrophysical Journal Letters v.214, p.L5. (05/1977)

[46] Roddier, F (ed.). *Adaptive Optics in Astronomy.* pp. 419. ISBN 0521612144. Cambridge, UK: Cambridge University Press, November 2004. (11/2004)

[47] Spitzer, L. "The Beginnings and Future of Space Astronomy". American Scientist, Vol. 50, No. 3, September 1962. (09/1962)

[48] Perlmutter, S *et al.* "Cosmology from Type Ia Supernovae" American Astronomical Society, 191st AAS Meeting, #85.04; Bulletin of the American Astronomical Society, Vol. 29, p.1351 (12/1997) arXiv:astro-ph/9812473

[49] Perlmutter, S *et al.* "Measurements of Omega and Lambda from 42 High-Redshift Supernovae" Astrophysical Journal 517:565-586, 1999. (12/1998) arXiv:astro-ph/9812133v1

[50] Astier, P *et al.* "The Supernova Legacy Survey: measurement of Ω_M, Ω_Λ and w from the first year data set". Astronomy and Astrophysics, Volume 447, Issue 1, February III 2006, pp.31-48. (02/2006) arXiv:astro-ph/0510447

[51] Kowalski, M *et al.* "Improved Cosmological Constraints from New, Old, and Combined Supernova Data Sets". The Astrophysical Journal, Volume 686, Issue 2, pp. 749-778. (10/2008) arXiv:0804.4142v1

[52] Wood-Vasey, W *et al.* "Observational Constraints on the Nature of Dark Energy: First Cosmological Results from the ESSENCE Supernova Survey". The Astrophysical Journal, Volume 666, Issue 2, pp. 694-715. (09/2007) arXiv:astro-ph/0701041

[53] Davis, T *et al.* "Scrutinizing Exotic Cosmological Models Using ESSENCE Supernova Data Combined with Other Cosmological Probes". The Astro-

physical Journal, Volume 666, Issue 2, pp. 716-725. (09/2007) arXiv:astro-ph/0701510

[54] Riess, A *et al.* "New *Hubble Space Telescope* Discoveries of Type Ia Supernovae at $z \geq 1$: Narrowing Constraints on the Early Behavior of Dark Energy". The Astrophysical Journal, Volume 659, Issue 1, pp. 98-121. (04/2007) arXiv:astro-ph/0611572

[55] Perlmutter, S *et al.* "Measurements of the Cosmological Parameters Omega and Lambda from the First Seven Supernovae at $z \geq 0.35$". Astrophysical Journal v.483, p.565. (07/1997) arXiv:astro-ph/0510447

[56] Riess, A *et al.* "Type Ia Supernova Discoveries at $z > 1$ from the Hubble Space Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution". The Astrophysical Journal, Volume 607, Issue 2, pp. 665-687. (06/2004) arXiv:astro-ph/0402512

[57] Drinkwater, M *et al.* "The WiggleZ Dark Energy Survey: survey design and first data release" Monthly Notices of the Royal Astronomical Society, Volume 401, Issue 3, pp. 1429-1452. (01/2010) arXiv:0911.4246v2

[58] Blake, C *et al.* "The WiggleZ Dark Energy Survey: measuring the cosmic expansion history using the Alcock-Paczynski test and distant supernovae" (08/2011) arXiv:1108.2637v1

[59] Jha, S; Riess, A; Kirshner, R. "Improved Distances to Type Ia Supernovae with Multicolor Light-Curve Shapes: MLCS2k2". The Astrophysical Journal, Volume 659, Issue 1, pp. 122-148. (04/2007) arXiv:astro-ph/0612666

[60] Kessler, R *et al.* "First-Year Sloan Digital Sky Survey-II Supernova Results: Hubble Diagram and Cosmological Parameters". The Astrophysi-

cal Journal Supplement, Volume 185, Issue 1, pp. 32-84 (2009). (11/2009) arXiv:0908.4274

[61] Mörtsell, E; Clarkson, C. “Model independent constraints on the cosmological expansion rate”. Journal of Cosmology and Astroparticle Physics, Issue 01, pp. 044 (2009). (01/2009) arXiv:0811.0981v2

[62] Komatsu, E *et al.* “Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation”. The Astrophysical Journal Supplement, Volume 192, Issue 2, article id. 18 (2011). (02/2011) arXiv:1001.4538v3

[63] Kraan-Korteweg, R; Lahav, O. “The Universe behind the Milky Way”. The Astronomy and Astrophysics Review, Volume 10, Issue 3, pp. 211-261 (2000). arXiv:astro-ph/0005501

[64] Kocevski, D; Mullis, C; Ebeling, H. “The Dipole Anisotropy of the First All-Sky X-Ray Cluster Sample”. The Astrophysical Journal, Volume 608, Issue 2, pp. 721-730. (06/2004) arXiv:astro-ph/0403275

[65] Myers, J (curator). “WFIRST: Wide-Field Infrared Survey Telescope”. Goddard Space Flight Center, 04 August 2011. (08/2011) http://wfirst.gsfc.nasa.gov/

[66] Yamato, M (curator). “SNAP, Super Nova Acceleration Probe”. Lawrence Berkeley National Laboratory, _____. (_____) http://snap.lbl.gov/

[67] SNAP Collaboration: G. Aldering *et al.* “Supernova / Acceleration Probe: A Satellite Experiment to Study the Nature of the Dark Energy”. eprint arXiv:astro-ph/0405232. (05/2004) arXiv:astro-ph/0405232

[68] ______. "JDEM, Joint Dark Energy Mission". Lawrence Berkeley National Laboratory, 2010. (05/2010) http://jdem.lbl.gov/

[69] Gehrels, N. "The Joint Dark Energy Mission (JDEM) Omega". eprint arXiv:1008.4936. (08/2010) arXiv:1008.4936

[70] Bianchi, L. "Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti" ["On the three-dimensional spaces which admit a continuous group of motions"] Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza, Tomo XI, pp. 267–352 (1898). Translated by Robert Jantzen.

[71] Behr, Ch. "Eine Verallgemeinerung des Friedmannschen Weltmodells mit positiver Raumkrümmung." Zeitschrift für Astrophysik, Vol. 54, p.268 (1962).

[72] Carroll, S; Press, W; Turner, E. "The cosmological constant". In: Annual review of astronomy and astrophysics. Vol. 30 (A93-25826 09-90), p. 499-542. (1992)

[73] Misner, C. "Mixmaster universe". Physical Review Letters, vol. 22, Issue 20, pp. 1071-1074. (05/1969)

[74] Belinskij, V; Khalatnikov, I; Lifshits, E. "Oscillatory approach to a singular point in the relativistic cosmology". Adv. Phys., Vol. 19, p. 525 - 573 (1970)

[75] Lifshits, E; Khalatnikov, I. "Problems Of Relativistic Cosmology". Sov. Phys. Usp. 6 495 (1964) (translation)

[76] Misner, C; Thorne, K; Wheeler, J. *Gravitation.* San Francisco: W.H. Freeman and Co., 1973.

[77] Davis, M; Peebles, P. "Evidence for local anisotropy of the Hubble flow". IN: Annual review of astronomy and astrophysics. Volume 21 (A84-10851 01-90). Palo Alto, CA, Annual Reviews, Inc., 1983, p. 109-130. (1983)

[78] Wheeler, J. "Gravitation as geometry". in: Hong-yee Chiu, William F. Hoffmann (ed.), *Gravitation and Relativity: Physical investigations of the universe.* W.A. Benjamin, 1964

[79] Lifshitz, E. "On the gravitational stability of the expanding universe". Zh. Eksp. Teor. Fiz. Tom 16, 587. (1946)

[80] Adams, F; Laughlin, G. *The five ages of the universe: inside the physics of eternity.* Simon & Schuster, 2000.

[81] Chernin, A. "How Gamow calculated the temperature of the background radiation or a few words about the fine art of theoretical physics". Physics Uspekhi, Volume 37, Issue 8, pp. 813-820 (1994). (08/1994)

[82] de Sitter, W. "Einstein's theory of gravitation and its astronomical consequences. Third paper". Monthly Notices of the Royal Astronomical Society, Vol. 78, p.3-28. (11/1917).

[83] Bondi, H; Gold, T. "The Steady-State Theory of the Expanding Universe". Monthly Notices of the Royal Astronomical Society, Vol. 108, p.252 (1948).

[84] Alpher, R; Bethe, H; Gamow, G. "The Origin of Chemical Elements". Physical Review, vol. 73, Issue 7, pp. 803-804 (04/1948).

[85] Penzias, A; Wilson, R. "A Measurement of Excess Antenna Temperature at 4080 Mc/s." Astrophysical Journal, vol. 142, p.419-421 (07/1965)

[86] Friedmann, A. "Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes". Zeitschrift für Physik, Volume 21, Issue 1, pp.326-332 (12/1924)

[87] Krauss, L; Turner, M. "The cosmological constant is back". General Relativity and Gravitation, Volume 27, Issue 11, pp.1137-1144. (11/1995) arXiv:astro-ph/9504003

[88] Kamenshchik, A; Moschella, U; Pasquier, V. "An alternative to quintessence". Physics Letters B, Volume 511, Issue 2-4, p. 265-268. (07/2001) arXiv:gr-qc/0103004

[89] Zehavi, I; Riess, A; Kirshner, R; Dekel, A. "A Local Hubble Bubble from Type IA Supernovae?" Astrophysical Journal v.503, p.483. (08/1998) arXiv:astro-ph/9802252

[90] Riess, A *et al.* "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant". The Astronomical Journal, Volume 116, Issue 3, pp. 1009-1038. (09/1998) arXiv:astro-ph/9805201

[91] Célérier, M. "Do we really see a cosmological constant in the supernovae data?" Astronomy and Astrophysics, v.353, p.63-71 (2000) (01/2000) arXiv:astro-ph/9907206

[92] Brown, I; Schrempp, L; Ananda, K. "Accelerating the universe with gravitational waves". Journal of Cosmology and Astroparticle Physics, Issue 02, pp. 011 (2010). (02/2010) arXiv:0909.1922

[93] Einstein, A; Straus, E. "The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Individual Stars". Reviews of Modern Physics, vol. 17, Issue 2-3, pp. 120-124 (04/1945)

[94] Dyer, C; Roeder, R. "Observations in Locally Inhomogeneous Cosmological Models". Astrophysical Journal, Vol. 189, pp. 167-176 (1974) (04/1974)

[95] Freese, K; Lewis, M. "Cardassian expansion: a model in which the universe is flat, matter dominated, and accelerating". Physics Letters B, Volume 540, Issue 1-2, p. 1-8. (07/2002) arXiv:astro-ph/0201229

[96] Dvali, G; Gabadadze, G; Porrati, M. "Metastable gravitons and infinite volume extra dimensions". Physics Letters B, Volume 484, Issue 1-2, p. 112-118. (06/2000) arXiv:hep-th/0002190

[97] Vilenkin, A; Shellard, E. *Cosmic Strings and Other Topological Defects.* ISBN 0521654769. Cambridge, UK: Cambridge University Press, July 2000. (07/2000)

[98] Hobill, D; Guo, W. "Cosmology without dark energy: Weyl curvature solutions". Canadian Journal of Physics, vol. 86, issue 4, pp. 571-577 (2008). (09/2011)

[99] Caldwell, R; Kamionkowski, M; Weinberg, N. "Phantom Energy and Cosmic Doomsday". Physical Review Letters, vol. 91, Issue 7, id. 071301 (08/2003) arXiv:astro-ph/0302506v1

[100] Gibbons, G. "Cosmological fermion-number non-conservation". Physics Letters B Volume 84, Issue 4, 16 July 1979, Pages 431-434. (1979)

[101] Eisenstein, D *et al.* "Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies". The Astrophysical Journal, Volume 633, Issue 2, pp. 560-574. (11/2005) arXiv:astro-ph/0501171v1

[102] Cooray, A; Caldwell, R. "Large-scale bulk motions complicate the Hubble diagram". Physical Review D, vol. 73, Issue 10, id. 103002 (05/2006) arXiv:astro-ph/0601377v1

[103] Grishchuk, L. "Density perturbations of quantum-mechanical origin and anisotropy of the microwave background". Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 50, Issue 12, 15 December 1994, pp.7154-7172. (12/1994) arXiv:gr-qc/9405059

[104] Ellis, G.; MacCallum, M. "A class of homogeneous cosmological models". Communications in Mathematical Physics, Volume 12, Issue 2, pp.108-141. (06/1969)

[105] Bardeen, J. "Gauge-invariant cosmological perturbations" Physical Review D (Particles and Fields), Volume 22, Issue 8, 15 October 1980, pp.1882-1905 (10/1980)

[106] Tsagas, C; Challinor, A; Maartens, R. "Relativistic cosmology and large-scale structure". Physics Reports, Volume 465, Issue 2-3, p. 61-147. (08/2008) arXiv:0705.4397

[107] Li, N; Schwarz, D. "On the onset of cosmological backreaction". Physical Review D, vol. 76, Issue 8, id. 083011 (10/2007) arXiv:gr-qc/0702043

[108] Taub, AH. "Empty space-times admitting a three parameter group of motions". Annals of Mathematics, second series, 53, #3 (May 1951), 472-490. (05/1951)

[109] Alexander, J. "Topological invariants of knots and links". Transactions of the American Mathematical Society 30 (1928), 275-306. (1928)

[110] Kawauchi, A. *A survey of knot theory.* Birkhäuser Verlag, 1996. (1996)

[111] Zhao, W; van den Broeck, C; Baskaran, D; Li, T. "Determination of dark energy by the Einstein Telescope: Comparing with CMB, BAO, and SNIa observations". Physical Review D, vol. 83, Issue 2, id. 023005 (01/2011) arXiv:1009.0206

[112] Landau, L; Lifshitz, E. *Teoria Polya [Theory of Fields], Seventh Corrected Edition*. Moskva "Nauka" Glavnaya Redakchiya Fiziko-Matematicheskoye Literatur (1988)

[113] Rodrigues, Davi C. "Anisotropic cosmological constant and the CMB quadrupole anomaly". Physical Review D, vol. 77, Issue 2, id. 023534. (01/2008) arXiv:0708.1168

[114] Campanelli, L.; Cea, P.; Fogli, G. L.; Tedesco, L. "Anisotropic Dark Energy and Ellipsoidal Universe". International Journal of Modern Physics D, Volume 20, Issue 06, pp. 1153-1166 (2011). (2011) arXiv:1103.2658

[115] Koivisto, Tomi; Mota, David F. "Anisotropic dark energy: dynamics of the background and perturbations". Journal of Cosmology and Astroparticle Physics, Issue 06, pp. 018 (2008). (06/2008) arXiv:0801.3676

[116] Dyson, F; Eddington, A; Davidson, C. "A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919". Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, Volume 220, pp. 291-333 (1920)

[117] Jenet, F; Lommen, A; Larson, L; Wen, L. "Constraining the Properties of Supermassive Black Hole Systems Using Pulsar Timing: Application to 3C 66B". The Astrophysical Journal, Volume 606, Issue 2, pp. 799-803. arXiv:astro-ph/0310276

[118] Birkhoff, G; Langer, R. *Relativity and Modern Physics.* Harvard University Press, Cambridge, Mass., USA (1927)

[119] Kopeikin, S; Schäfer, G; Gwinn, C; Eubanks, TM. "Astrometric and Timing Effects of Gravitational Waves from Localized Sources". Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 59, Issue 8, 15 April 1999, id. 084023 (04/1999) arXiv:gr-qc/9811003

[120] Damour, T; Esposito-Farese, G. "Light Deflection by Gravitational Waves from Localized Sources".Physical Review D (Particles, Fields, Gravitation, and Cosmology), Volume 58, Issue 4, 15 August 1998, id. 044003 (08/1998) arXiv:gr-qc/9802019

[121] Adler, R; Bazin, M; Schiffer M. *Introduction to General Relativity,* second edition. New York: McGraw-Hill Inc, 1975.

[122] Schwarzschild, K. "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie" Sitzungsberichte der Koniglichen Preussischen Akademie der Wissenschaften zu Berlin, Klasse fur Mathematik, Physik, und Technik, pages 189-196. (1916)

[123] Northover, K. "The Radio Galaxy 3C 66*". Monthly Notices of the Royal Astronomical Society, Vol. 165, p. 369 (1973)

[124] Matthews, T; Morgan, W; Schmidt, M. "A Discussion of Galaxies Identified with Radio Sources". Astrophysical Journal, vol. 140, p.35 (07/1964)

[125] Spergel, D *et al.* "First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters". The Astrophysical Journal Supplement Series, Volume 148, Issue 1, pp. 175-194. (09/2003) arXiv:astro-ph/0302209

[126] Ekejiuba, I; Okeke, P; Okoye, S. "Origin of the Central Radio Gaps in Extragalactic Radio Sources". Astrophysics and Space Science (ISSN 0004-640X), vol. 187, no. 2, Jan. 1992, p. 209-214 (01/1992)

[127] Sudou, H; Iguchi, S; Murata, Y; Taniguchi, Y. "Orbital Motion in 3C 66B: Evidence for a Supermassive Black Hole Binary". Science, Volume 300, Issue 5623, pp. 1263-1265 (05/2003) arXiv:astro-ph/0306103

[128] De Paolis, F; Ingrosso, G; Nucita, A. "A super massive black hole binary in 3C66B: future observational perspectives". Astronomy and Astrophysics, v.426, p.379-385 (11/2004) arXiv:astro-ph/0407498

[129] Croston, J; Hardcastle, M; Birkinshaw, M; Worrall, D. "*XMM-Newton* observations of the hot-gas atmospheres of 3C 66B and 3C 449" Monthly Notices of the Royal Astronomical Society, Volume 346, Issue 4, pp. 1041-1054. (12/2003) arXiv:astro-ph/0309150

[130] Darwin, C. "The Gravity Field of a Particle". Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Volume 249, Issue 1257, pp. 180-194 (01/1959)

[131] Giovannini, G; Cotton, W; Feretti, L; Lara, L; Venturi, T. "VLBI Obervations of a Complete Sample of Radio Galaxies: 10 Years Later". The Astrophysical Journal, Volume 552, Issue 2, pp. 508-526. (05/2001) arXiv:astro-ph/0101096v1

[132] Fraix-Burnet, D. "An Optical Counterjet in 3C66B?". *Mon. Not. R. astr. Soc.* (1996) 226, 1-7. (09/1996) arXiv:astro-ph/9610259v1

[133] ________________. "Welcome to the Very Long Baseline Array!". http://www.vlba.nrao.edu/ (08/2009)

[134] Romney, J. "Angular resolution & *u-v* coverage". http://www.vlba.nrao.edu/astro/obstatus/current/node24.html (06/2011)

[135] Hirabayashi, H *et al.* "On the Near-term Space VLBI Mission VSOP-2". (01/2005) arXiv:astro-ph/0501020

[136] ______. "MSC - Space Interferometry Mission". NASA Exoplanet Science Institute. http://msc.caltech.edu/missions/SIMPQ/

[137] Edberg, S; Shao, M; Beichman, C ed. "SIM PlanetQuest: A Mission for Astrophysics and Planet Finding". Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, USA, 2 May 2005. (05/2005)

[138] Gendreau, K C, *et al.* "The MAXIM X-ray Interferometry Mission Concept Study". In: *From optical to millimetric interferometry: scientific and technological challenges. Proceedings of the 36th Liège International Astrophysics Colloquium*, Liège, Belgium, July 2-5, 2001. Edited by J. Surdej, J. P. Swings, D. Caro, and A. Detal. Liège, Belgium: Université de Liège, Institut d'Astrophysique et de Géophysique, 2001, p. 11-16. (07/2001)

[139] Kopeikin, S; Masshoon, B. "Gravitomagnetic Effects in the Propagation of Electromagnetic Waves in Variable Gravitational Fields of Arbitrary-Moving and Spinning Bodies". Physical Review D, vol. 65, Issue 6, id. 064025 (03/2002). arXiv:gr-qc/0110101v2

[140] Einstein, A. "Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes". Annalen der Physik, 35, 898 (1911). Translated in English in H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity*, Dover (1952)

[141] Bozza, V; Scarpetta, G. "Strong deflection limit of black hole gravitational lensing with arbitrary source distances". Physical Review D, vol. 76, Issue 8, id. 083008 (10/2007). arXiv:gr-qc/0705.0246

[142] Kopeikin, S; Makarov, V. "Gravitational bending of light by planetary multipoles and its measurement with microarcsecond astronomical interferometers". Physical Review D, vol. 75, Issue 6, id. 062002 (03/2007). arXiv:astro-ph/0611358v2

[143] Abramowitz, M; Stegun, I, ed. *Handbook of Mathematical Functions*, ninth printing, National Bureau of Standards Press (1970)

[144] Burke-Spolaor, S. "A radio Census of binary supermassive black holes". Monthly Notices of the Royal Astronomical Society, Volume 410, Issue 4, pp. 2113-2122. (02/2011) arXiv:1008.4382

[145] Iguchi, S; Okuda, T; Sudou, H. "A Very Close Binary Black Hole in a Giant Elliptical Galaxy 3C 66B and its Black Hole Merger". The Astrophysical Journal Letters, Volume 724, Issue 2, pp. L166-L170 (2010). (12/2010) arXiv:1011.2647

[146] Fay, Stéphane; Lehner, T. "Bianchi type IX asymptotical behaviours with a massive scalar field: chaos strikes back". General Relativity and Gravitation, Volume 37, Issue 6, pp.1097-1117. (06/2005) arXiv:gr-qc/0509071

[147] Schwarz, D; Weinhorst, B. "(An)isotropy of the Hubble diagram: comparing hemispheres". Astronomy and Astrophysics, Volume 474, Issue 3, November II 2007, pp.717-729 (11/2007) arXiv:0706.0165

[148] Blomqvist, M; Enander, J; Mörtsell, E. "Constraining dark energy fluctuations with supernova correlations". Journal of Cosmology and Astroparticle Physics, Issue 10, pp. 018 (2010). (10/2010) arXiv:1006.4638

[149] Blomqvist, M; Mörtsell, E; Nobili, S. "Probing dark energy inhomogeneities with supernovae". Journal of Cosmology and Astroparticle Physics, Issue 06, pp. 027 (2008). (06/2008) arXiv:0806.0496

[150] Bochner, B. "Looking for Signs of Anisotropic Cosmological Expansion in the High-z Supernova Data". In Proceedings of the 22nd Texas Symposium on Relativistic Astrophysics at Stanford University, Dec. 13-17, 2004; paper #1301, pp. 1-6 (02/2007) arXiv:astro-ph/0702730

[151] Grishchuk, L. "Cosmological models and spatial homogeneity criteria". Soviet Astronomy – AJ, vol. 18, no. 5 (March-April 1968). (03/1968)

[152] Schluessel, E. "Constraining the parameters of binary systems through time-dependent light deflection". General Relativity and Gravitation, Volume 40, Issue 5, pp.1013-1027. (05/2008) arXiv:0710.1682