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Application of Åström's method to supply chain inventory problems: A reverse logistics scenario

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Abstract: A simple dynamic model of a hybrid manufacturing / remanufacturing system is investigated. In particular we study an infinite horizon, continuous time, APIOBPCS (Automatic Pipeline Inventory and Order Based Production Control System) model. We specifically highlight the effect of remanufacturing lead-time and the return rate on the inventory variance and bullwhip produced by the ordering policy. Our results clearly show that a larger return rate leads to less bullwhip and less inventory variance, thus returns can be used to absorb demand fluctuations to some extent. Longer remanufacturing lead-times have less impact at reducing inventory variance and bullwhip than shorter lead-times. We conclude, within our specified system, that inventory variance and bullwhip is always less in supply chain with returns than without returns.

Key words: manufacturing, remanufacturing, inventory variance, bullwhip, return rate, lead-time.

1. Introduction

Around the world, sustainability has become focal point of many economic development strategies. Leading companies may even use sustainability as a means of gaining competitive advantage, as the growing environmental awareness of customers is changing the marketplace, Mahadevan, Pyke, Fleischmann (2003). From a production viewpoint sustainability covers aspects of environmental production; green manufacturing, use of natural resources, recycling, material re-use and re-manufacturing. However, managing a reverse supply chain involves coping with many uncertainties, especially those concerned with the quantity, quality and timing of the returned products, Seitz, Disney and Naim (2003). In a number of recent papers, many issues have been raised, such as how to design a product so that it is easy to be disassembled and reused (Knodo *et al*, 2003), or how to make decisions on product recovery (van der Laan & Saloman 1999, Teunter & Vlachos 2002), for example reselling, recovery, or disposal. The recovery option may also include repair, refurbishing, remanufacturing, cannibalization and recycling, Thierry *et al.* (1995).

We focus on re-manufacturing here. In particular we look at the consequences of integrating the re-manufacturing process with the traditional supply chain concerned with the production of new products. We consider the scenario where "used" products are pushed through a remanufacturing process as soon as they are returned from the "customer" (or marketplace). There is a lead-time associated with the time to remanufacture a product and also a lead-time associated with the time that a product is "in use" by the customer. For convenience, we will join these two lead-times together and called it *Tr*, the Time to Remanufacture. Furthermore, we assume this remanufacturing lead-time is a stationary stochastic variable drawn from an exponential distribution. Only a fraction, $0<k<1$ of demand is returned from the marketplace, the rest we assume is either unusable or is lost to a landfill.

We also assume the remanufactured products are "as good as new" and thus form part of the serviceable stock. Serviceable stock is the finished goods from which customer demand is satisfied. In this study the terms net stock/inventory/serviceable stock are used interchangeably. We assume here that the customer demand is a stationary, independently and identically distributed (i.i.d) random process. Our analysis is independent of the actual distribution of the stochastic customer demand. That is, the demand distribution could be a normal, log normal, exponential or a gamma distribution for example. The manufacture of new products is controlled by a continuous time variant of the Order-Up-To policy, Dejonckheere, Disney, Lambrecht and Towill (2003). Figure 1 illustrates the material flow in this manufacturing / remanufacturing supply chain. We note that this system is different to the push/pull system of van der Laan & Saloman (1999).

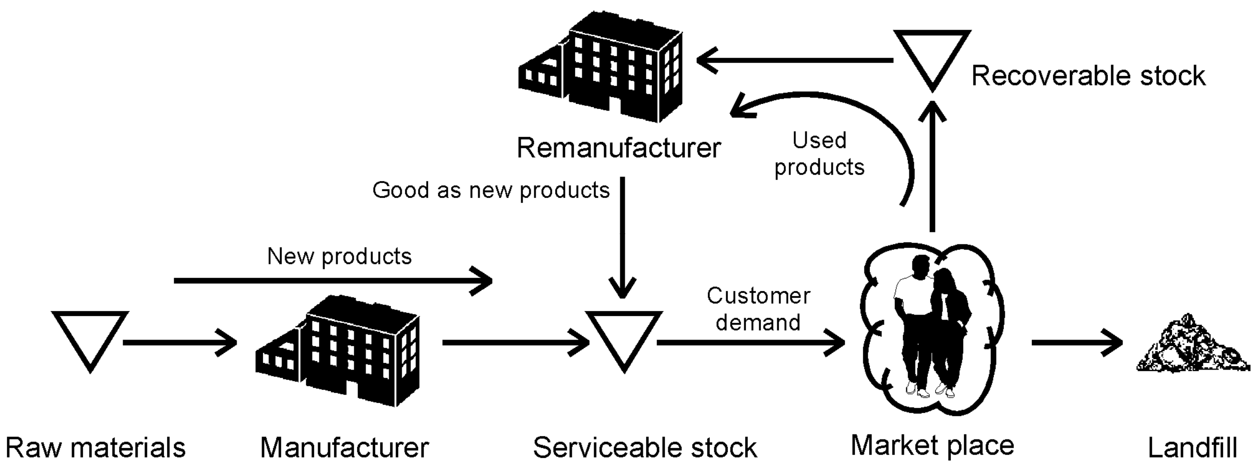

Figure 1. Material flow in a simple manufacturing / remanufacturing system

Much of the literature on reverse logistics has addressed inventory management. Fleischmann *et al.* (1997) provide a review of quantitative models in reverse logistics. Guide (2000) identifies and describes seven complicating characteristics of production planning and control activities for remanufacturing firms; uncertainty in the timing and the quantity of returns, balancing returns with demands, disassembly, uncertainty in materials recovered, reverse logistics, materials matching requirements, routing uncertainty and processing time uncertainty. Guide (2000) claims these special features require significant changes in production planning and control activities. It is this research opportunity we explore here.

Inderfurth and van der Laan (2001) study a simple four-parameter control rule for an inventory model with remanufacturing. There, the remanufacturing lead-time is treated as a decision variable to improve the performance of the policy. Kiesmüller and van der Laan (2001) investigate an inventory model with dependent product demands and returns. In this specific case, they conclude that neglecting the dependency between demands and returns of products may cause bad performance from the total average relevant costs viewpoint. Fleischmann and Kuik (2003) investigate an independent stochastic item returns scenario. Kleber *et al.* (2002) provide a continuous time inventory model to decide when returns should be used and when returns should be kept in inventory and not be remanufactured or disposed of immediately. Van der Laan (2003) analyze the economic consequences in a stochastic inventory system joint with manufacturing and remanufacturing from average cost and net present value point of view. Kiesmüller (2003) offers a new approach for a control problem with different lead-times for production and remanufacturing to decide the quantities and time of manufacturing and remanufacturing respectively. However, almost all quantitative literature is based upon a specified cost function; few papers study dynamic performance. Our aim here is to contribute to this field by using control theory as a medium for analysis by highlighting how the inventory variance and bullwhip phenomenon in a reverse logistics scenario.

The inventory variance determines stock levels required to meet a given target customer service levels. The higher the variance of inventory variance, the more stock will be needed to maintain customer service at the target level, Dejonckheere *et al.* (2002). The bullwhip effect relates to the order we place to maintain inventory level after the production rates. Both of inventory variance and bullwhip directly affect the economics of the situation, Disney and Grubbström (2003). Normally, inventory variance and bullwhip are conflicting phenomenon, that is, when inventory variance increases, the bullwhip often decreases. This leads to a trade-off analysis.

Our paper is organized as follows. First, we give a formal definition of our model and derive the corresponding continuous time, Laplace domain transfer function of the remanufacturing supply chain. Section 3 presents a general process for deriving variance ratio analytical expressions. This method is also powerful as it can be used on complex systems. Section 4 analyzes the variance of the serviceable stock levels. We then compare the inventory variance performance of the remanufacturing supply chain with the performance of traditional supply chain and draw out some managerial implications. Section 5 addresses bullwhip phenomena in the system. Again we also compare it with traditional supply chain. Here, we find that there is a trade-off to be made between the variance of serviceable stock and bullwhip. In section 6, we study the optimal conditions in our model that minimizes the sum of serviceable stock and order variance in the production of new products. Section 7 concludes.

2. Model description

In our analysis we assume that time passes continuously and consequently we exploit the Laplace transform in our analysis. The manufacturers' replenishment order is placed to produce new product based on a forecast of future demand, the serviceable inventory and the current work in progress in the original equipment factory. The ordering policy that we study here is based upon the APIOBPCS model. APIOBPCS is an acronym for Automatic Pipeline Inventory and Order Based Production Control System (John *et al* 1994). The policy can be expressed as; "let the production targets be equal to the sum of mean demand, plus a fraction ($1/Ti$) of the inventory error, plus a fraction ($1/Tw$) of the WIP error".

However, the remanufacturing scenario requires a slight modification to the classical APIOBPCS policy as shown in Figure 2. Here a fraction (k) of the demand is returned, brought to a good as new condition and added to the net stock of serviceable inventory after a random delay. This random delay is drawn from an exponential distribution with an average of Tr time units. Recall that we are assuming the demand is stationary i.i.d. This means the minimum mean squared error forecast of all future demands is given by the long-term expected value of demand, the mean. Furthermore, as we are considering a linear system, we may assume the mean demand is zero without loss of generality.

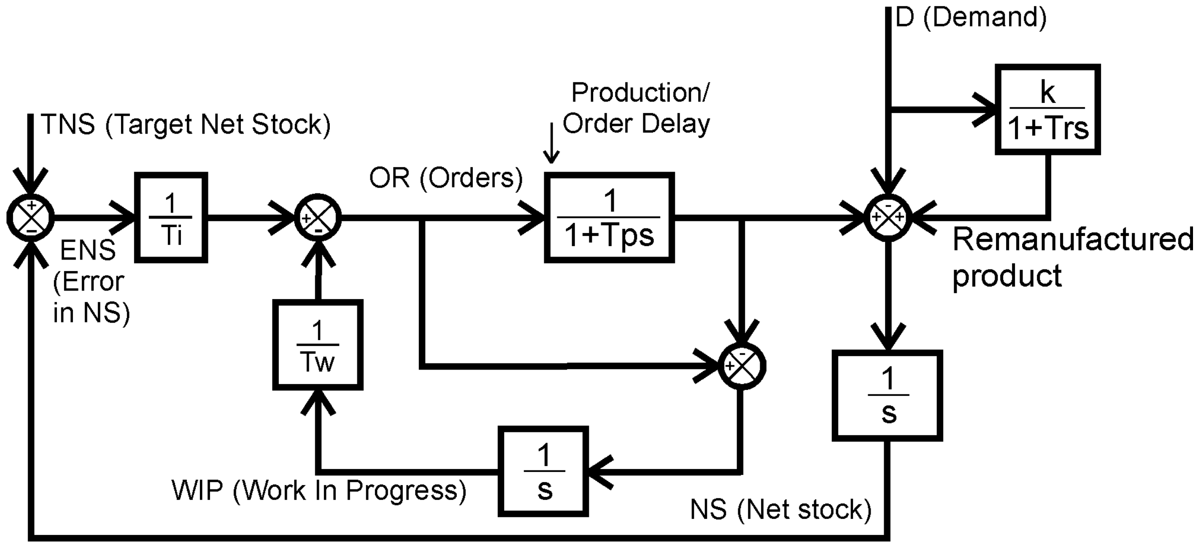

Figure 2. Block diagram of our re-manufacturing system

We may rearrange Figure 2 using standard techniques (see Nise 1994 for an introduction) to obtain the net stock/inventory and order rate transfer function,

$$\frac{NS}{D}=\frac{Ti\left(k-1-s\,Tr\right)\left(Tp+Tw+s\,Tp\,Tw\right)}{\left(1+s\,Tr\right)\left(Tw+s\,Ti\left(Tp+Tw+s\,Tp\,Tw\right)\right)} \tag{1}$$

$$\frac{OR}{D}=\frac{\left(1+sTp\right)\left(1-k+sTr\right)Tw}{\left(1+sTr\right)\left(Tw+sTi\left(Tp+Tw+sTpTw\right)\right)} \tag{2}$$

All of the constants in the system may be assumed to be larger than zero, so we have $Ti \geq 0$, $Tp \geq 0$, $Tw \geq 0$ $Tr \geq 0$ and $0 \leq k \leq 1$.

3. The method of deriving variance ratio

Disney and Towill (2003) have explored the relationship between "long-run" variance ratio measures and the system's transfer function. We adopt their definition for variance amplification as

$$VarAmp=\frac{\sigma^2_{output}}{\sigma^2_{input}}. \tag{3}$$

Furthermore, we assume without loss of generality that $\sigma^2_{input}=1$ throughout the paper. It can be shown that the integral of the square of the systems unit impulse response is equal to the variance of the systems output divided by the variance of the systems input, Newton, Gould and Kaiser (1957), see (4). We have sketched this mathematical proof in Appendix A for the reader. Here it is sufficient to present a realisation of the methodology to calculate the variance ratio.

$$VarAmp = \frac{\sigma_{output}^2}{\sigma_{input}^2} = I = \int_0^{\infty} \left[\mathrm{L}^{-1}(G(s)) \right]^2 dt \tag{4}$$

where $L^{-1}(G(s))$ is the inverse Laplace transform of $G(s)$, denoted by *g(t)*.

As our system is linear time invariant, its Laplace transform is a rational expression. However the integral of *I* is often difficult to obtain, thus we may exploit Parseval's Relation. What we wish to know is the area, denoted by *I*, beneath the product of two time functions $g_1(t)$ and $g_2(t)$ over the infinite interval of time from $-\infty$ to $+\infty$. Obviously when $g_1 = g_2$, I will be directly equivalent to the variance ratio.

$$I = \int_{-\infty}^{+\infty} g_1(t)\, g_2(t)\, dt \tag{5}$$

Now suppose that the time functions are Fourier transformable and have transforms $G_1(s)$ and $G_2(s)$ respectively. We wish to get I directly from these transforms to avoid inverse transforming $G_1(s)$ and $G_2(s)$. To do this we observe that,

$$g_2(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{st} G_2(s)\, ds\,. \tag{6}$$

Substitution of this value of $g_2(t)$ into (5) yields

$$I = \int_{-\infty}^{+\infty} g_1(t)\; dt \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{st}\, G_2(s)\, ds\,. \tag{7}$$

Interchanging the order of (7) in order to integrate with respect to time first, we have,

$$I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G_2(s)\, ds \int_{-\infty}^{+\infty} e^{st} g_1(t)\, dt\,. \tag{8}$$

By the direct Fourier transform we know that

$$G_1(s) = \int_{-\infty}^{\infty} e^{-st} g_1(t)\, dt\,. \tag{9}$$

Thus the integral with respect to time on the right side of (8) may be evaluated as

$$\int_{-\infty}^{\infty} e^{st} g_1(t)\, dt = G_1(-s)\,.$$

So (5) can be expressed as

$$I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G_1(-s) G_2(s)\, ds\,. \tag{10}$$

This is the desired expression for the integral. Furthermore, when $g_1(t)$ is equal to $g_2(t)$, the integral, I, becomes

$$I = \int_{-\infty}^{\infty} [g_2(t)]^2 \, dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G_1(-s)G_2(s)ds .$$

This result is known as Parseval's relation (Newton, Gould and Kaiser (1957)). In summary we have

$$\int_{-\infty}^{\infty}\left[\mathrm{L}^{-1}[(G(s))]\right]^2 dt = \int_{-\infty}^{0_-}\left[\mathrm{L}^{-1}[(G(s))]\right]^2 dt + \int_{0_+}^{\infty}\left[\mathrm{L}^{-1}[(G(s))]\right]^2 dt = 0 + \int_{0}^{\infty}\left[\mathrm{L}^{-1}[(G(s))]\right]^2 dt$$
$$= \int_{0}^{\infty}\left[\mathrm{L}^{-1}[(G(s))]\right]^2 dt = \int_{0}^{\infty} g^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(-s)G(s)ds = Variance\ \ Ratio. \tag{11}$$

Note that the integral in time domain $(-\infty, 0_-]$ is zero. So, we only have to integrate between $[0_+, +\infty)$. As $G(s)$ is a rational function, the integral-square transform function will appear in the form

$$\frac{\sigma^2_{output}}{\sigma^2_{input}} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{B(s)B(-s)}{A(s)A(-s)}ds , \tag{12}$$

where A and B are polynomials with real coefficients

$$A(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = \sum_{i=0}^{n} a_i s^{n-1} ,$$
$$B(s) = b_1 s^{n-1} + \cdots + b_{n-1}s + b_n = \sum_{i=1}^{n} b_i s^{n-1} . \tag{13}$$

The integral (11) will always exist (but may not always be easy to obtain) if the zeros of the polynomial, A*(s)*, do not have positive real components. Note that the polynomial B*(s)* must be at least one degree less than the polynomial A*(s)* in order to guarantee that the Laplace transform *G(s)* is rational. On the basis of Parseval's theorem, Åström (1970) evaluated the integrals I in terms of the coefficients appearing in the polynomials.

We will now exploit Åström's approach to solving the integral square of signals. We will focus on the application of the method and refer interested readers to Åström (1970) for the rigorous mathematical proof. Expressing the transfer function as two polynomials is *A(s)* and *B(s)*, with real coefficients we have,

$$\frac{B(s)}{A(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} . \tag{14}$$

The first step of Åström's method is to build two new polynomials, $A_k(s)$ and $B_k(s)$ with a lower order than *n* of *A(s)* in (13). This is done with

$$A_k(s) = a_0^k s^k + a_1^k s^{k-1} + \cdots + a_k^k \qquad \text{and}$$

$$B_k(s) = b_1^k s^{k-1} + b_2^k s^{k-2} + \cdots + b_k^k \ , \tag{15}$$

which are defined recursively from the equations,

$$a_i^{k-1} = \begin{cases} a_{i+1}^k & i\ even \\ a_{i+1}^k - \alpha_k a_{i+2}^k & i\ odd, \quad and \quad \alpha_k = a_0^k / a_1^k \end{cases} \qquad \forall\ i = 0, \cdots, k-1 \tag{16}$$

$$b_i^{k-1} = \begin{cases} b_{i+1}^k & i\ odd \\ b_{i+1}^k - \beta_k a_{i+1}^k & i\ even, \quad and \quad \beta_k = b_1^k / a_1^k \end{cases} \qquad \forall\ i = 1, \cdots, k-1 . \tag{17}$$

The polynomials A_{k-1} and B_{k-1} can only be defined if $a_1^k \neq 0$. Åström (1970) shows these new coefficients may be obtained by the following table.

$$\begin{array}{cccccccccccc} a_0^k & a_1^k & a_2^k & a_3^k & a_4^k & \cdots & b_1^k & b_2^k & b_3^k & b_4^k & b_5^k \\ & a_0^{k-1} & a_1^{k-1} & a_2^{k-1} & a_3^{k-1} & & & b_1^{k-1} & b_2^{k-1} & b_3^{k-1} & b_4^{k-1} \\ & & & & \vdots & & & & & & \vdots \\ & & & a_0^1 & a_1^1 & & & & & & b_1^1 \end{array} \tag{18}$$

After having obtained the values α_k and β_k, the value of the integral *I* is then given by

$$I = \sum_{k=1}^{n} \beta_k^2 / (2\alpha_k) = \sum_{k=1}^{n} (b_1^k)^2 / (2 a_0^k a_1^k) \ . \tag{19}$$

Because of its recursive nature, Åström's method is very easy implement in computer software packages. Let us highlight Åström's method explicitly here. Rearranging the net stock transfer function (1), we have

$$\begin{aligned} A(s) &= TiTpTrTw\, s^3 + (TiTpTr + TiTpTw + TiTrTw)s^2 + (TiTp + TiTw + TrTw)s + Tw, \\ B(s) &= -TiTpTrTw\, s^2 - Ti(TpTr + TpTw - kTpTw + TrTw)s - Ti(Tp - kTp + Tw - kTw). \end{aligned} \tag{20}$$

Because the denominator is a third order polynomial, table (18) then has three rows,

$$\begin{array}{cccccccc} a_0^3 & a_1^3 & a_2^3 & a_3^3 & & b_1^3 & b_2^3 & b_3^3 \\ & a_0^2 & a_1^2 & a_2^2 & & & b_1^2 & b_2^2 \\ & & a_0^1 & a_1^1 & & & & b_1^1 \end{array} \tag{21}$$

where;

$a_0^3 = TiTpTrTw;$

$a_1^3 = TiTpTr + TiTpTw + TiTrTw;$

$a_2^3 = TiTp + TiTw + TrTw;$

$a_3^3 = Tw;$

$b_1^3 = -TiTpTrTw;$

$b_2^3 = -Ti\left(TpTr + TpTw - kTpTw + TrTw\right);$

$b_3^3 = -Ti\left(Tp - kTp + Tw - kTw\right).$

Step 1; Use $\alpha_k = a_0^k / a_1^k$ and $\beta_k = b_1^k / a_1^k$ to calculate α_3 and β_3

$$\begin{aligned} \alpha_3 &= a_0^3 / a_1^3 = \frac{TpTrTw}{TrTw + Tp\left(Tr + Tw\right)}; \\ \beta_3 &= b_1^3 / a_1^3 = -\frac{TpTrTw}{TrTw + Tp\left(Tr + Tw\right)}. \end{aligned} \tag{22}$$

Step 2; Using (16) and (17) translate $a_{1\,to\,3}^3$ into $a_{0\,to\,2}^2$ and $b_{2\,to\,3}^3$ into $b_{1\,to\,2}^2$

$$\begin{aligned} a_0^2 &= a_1^3 = TiTpTr + TiTpTw + TiTrTw; \\ a_1^2 &= a_2^3 - \alpha_3 a_3^3 = \left(Tp + Tw\right)\left(Ti + \frac{Tr^2 Tw}{TrTw + Tp\left(Tr + Tw\right)}\right); \\ a_2^2 &= a_3^3 = Tw; \\ b_1^2 &= b_2^3 = -Ti\left(TpTr + TpTw - kTpTw + TrTw\right); \\ b_2^2 &= b_3^3 - \beta_3\, a_3^3 = \frac{TpTrTw^2 + \left(-1 + k\right)Ti\,\left(Tp + Tw\right)\left(TrTw + Tp\left(Tr + Tw\right)\right)}{TrTw + Tp\left(Tr + Tw\right)}. \end{aligned} \tag{23}$$

Step 3; Repeat Step 1 to find α_2 and β_2,

$$\begin{aligned} \alpha_2 &= a_0^2 / a_1^2 = \frac{Ti\left(TrTw + Tp\left(Tr + Tw\right)\right)^2}{\left(Tp + Tw\right)\left(TiTpTr + Tr^2 Tw + Ti\left(Tp + Tr\right)Tw\right)}; \\ \beta_2 &= b_1^2 / a_1^2 = -\frac{Ti\left(TrTw + Tp\left(Tr + Tw\right)\right)\left(TrTw + Tp\left(Tr + Tw - kTw\right)\right)}{\left(Tp + Tw\right)\left(TiTpTr + Tr^2 Tw + Ti\left(Tp + Tr\right)Tw\right)}. \end{aligned} \tag{24}$$

Step 4; Using (16) and (17) translate $a^2_{1\,to\,2}$ into $a^1_{0\,to\,1}$ and b^2_2 into b^1_1 ,

$$a_0^1 = a_1^2 = (Tp+Tw)\left(Ti + \frac{Tr^2 Tw}{TrTw + Tp(Tr+Tw)}\right);$$
$$a_1^1 = a_2^2 = Tw; \tag{25}$$
$$b_1^1 = b_2^2 = \frac{TpTrTw^2 + (-1+k)Ti\,(Tp+Tw)(TrTw+Tp(Tr+Tw))}{TrTw+Tp(Tr+Tw)}.$$

Step 5; Repeat Step 1 to find α_1 and β_1

$$\alpha_1 = a_0^1 / a_1^1 = \frac{(Tp+Tw)(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw)}{Tw(TrTw+Tp(Tr+Tw))};$$
$$\beta_1 = b_1^1 / a_1^1 = \frac{TpTrTw^2 + (-1+k)Ti(Tp+Tw)(TrTw+Tp(Tr+Tw))}{Tw\,(TrTw+Tp(Tr+Tw))}. \tag{26}$$

Step 6; Finally we may use (19) to reveal the integral I as,

$$\sigma^2_{NS} = \frac{\sigma^2_{NS}}{\sigma^2_D} = \sum_{k=1}^{3}\beta_k^2/(2\alpha_k) = \frac{\left[\begin{array}{l} TpTr^2Tw^3 + (k-1)^2 Ti^2 (Tp+Tw)^2 (TrTw + Tp(Tr+Tw)) + \\ TiTw\left(\begin{array}{l} Tr^2Tw^2 + TpTrTw(2Tr+Tw) \\ + Tp^2\left(Tr^2 + TrTw + (k-1)^2 Tw^2\right)\end{array}\right)\end{array}\right]}{2Tw(Tp+Tw)\left(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw\right)}. \tag{27}$$

(27) is our desired closed form expression for the inventory variance. This has been further verified via simulation in the Matlab software package. As we assume the demand rate is drawn from an arbitrary i.i.d distribution we may select a number of typical i.i.d. inputs in this example. We have chosen the Normal distribution and the Exponential distribution. We have also compared our simulation results with the exact analytical expressions of (27). Our results are shown in Table 1, for various combinations of the parameters in the solution space.

Ti	*Tw*	*Tp*	*Tr*	*k*	Normal distribution		Exponential distribution		Analytical expression (19)
					σ^2_{NS}	Error (%)	σ^2_{NS}	Error (%)	σ^2_{NS}
4	4	8	4	0.3	4.55	0.22	4.37	4.17	4.56
4	8	8	4	0.3	4.05	2.27	3.88	2.02	3.96
8	8	8	8	0.3	6.81	1.93	6.53	2.39	6.69
8	16	16	8	0.6	5.65	7.01	5.39	2.08	5.28
16	16	16	32	0.6	12.25	0.74	11.73	3.54	12.16
16	32	16	32	0.6	12.19	3.92	11.52	1.79	11.73
32	32	4	16	0.9	6.85	5.71	6.50	0.31	6.48
32	4	4	16	0.9	7.14	2.15	6.86	1.86	6.99
Average error					2.99%		2.27%		0

Table 1. The inventory variance in continuous time when input is i.i.d.

4. Analysis of the inventory variance

The inventory variance had been derived in (27). As it is quite complex, let us first review some special cases. By setting k=0, we may determine the inventory variance in a traditional supply chain. This is shown in (28). Recall, Ti and Tw are control parameters that we may used to tune the dynamic response of the system and Tp is the lead-time associated with the manufacture of new product.

$$\sigma^2_{NS,k=0} = \frac{TpTw^2 + Ti\left(Tp+Tw\right)^2}{2Tw\left(Tp+Tw\right)} \tag{28}$$

We notice that Ti only occurs in the numerator. This means that reducing Ti will reduce inventory variance. An important subset of the control parameters occurs when Tw=Ti. It allows further simplification to (27) and (28) as shown in (29) and (30) respectively.

$$\sigma^2_{NS,Tw=Ti} = \frac{\left[\begin{array}{l}(k-1)^2 Ti^3 (Tp+Tr) + Tp^2 Tr\left((k-1)^2 Tp + Tr\right) + Ti^2 (Tp+Tr)\cdot \\ \left(3(k-1)^2 Tp + Tr\right) + TiTp\left((k-1)^2 Tp + Tr\right)(Tp+3Tr)\end{array}\right]}{2(Ti+Tp)(Ti+Tr)(Tp+Tr)} \tag{29}$$

$$\sigma^2_{NS,k=0,Ti=Tw} = \frac{Ti^2 + 3TiTp + Tp^2}{2(Ti+Tp)} = \frac{1}{2}(Ti+Tp) + \frac{TiTp}{2(Ti+Tp)} \tag{30}$$

In (29), we find that the order of Ti and Tp is numerator higher than in denominator, so both Tp and Ti should be reduced in order to dampen inventory variance. This result is more obvious in (30), a supply chain with no returns, as subtracting (29) from (30) results in (31).

$$(29)-(30) = \frac{(-2+k)k\left(TiTp\left(Ti^2+3TiTp+Tp^2\right)+(Ti+Tp)^3 Tr\right)}{2(Ti+Tp)(Ti+Tr)(Tp+Tr)} < 0 \tag{31}$$

Here we can see that when Tw=Ti, that is, when the two feedback gains are equal, then inventory variance in a remanufacturing supply chain will always be less than in a traditional supply chain. This is because we assume $0 \le k \le 1$, thus (31) is always negative.

When all of the products are returned from the marketplace (after the stochastic exponential delay), k=1. Here (27) reduces to,

$$Var_{k=1,Tw=Ti} = \frac{Tr\left(TiTp(Ti+Tp)+Tr\left(Ti^2+3TiTp+Tp^2\right)\right)}{2(Ti+Tp)(Ti+Tr)(Tp+Tr)}. \tag{32}$$

Again, compared to inventory variance in a traditional supply chain, we have

$$(32)-(30) = -\frac{TiTp\left(Ti^2+3TiTp+Tp^2\right)+(Ti+Tp)^3 Tr}{2(Ti+Tp)(Ti+Tr)(Tp+Tr)} < 0, \tag{33}$$

where we can clearly see the smoothing effect of the returns is reduced. Returning now to the general case of (27). We may factor this into the following,

$$\sigma_{NS}^2 = \frac{TpTw^2 + Ti(Tp+Tw)^2}{2Tw(Tp+Tw)} + \frac{(k-2)kTiTp^2Tw^2}{2(Tp+Tw)(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw)} + \frac{(k-2)kTi^2(Tp+Tw)(TrTw + Tp(Tr+Tw))}{2Tw(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw)} \quad (34)$$

In (34) the first term on the RHS is the inventory variance generated by a traditional supply chain. The second and third terms are always negative as $0 \le k \le 1$. This result reveals that the inventory variance with returns, in our specified model, is always less than without returns.

Returning again to (27). Differentiating (27) with respect to *k* and *Tr* yields

$$\frac{\partial \sigma_{NS}^2}{\partial k} = \frac{(k-1)Ti(Tp^2Tw^3 + Ti(Tp+Tw)^2(TpTw + Tr(Tp+Tw)))}{Tw(Tp+Tw)(Tr^2Tw + TiTpTw + TrTi(Tp+Tw))} \quad (35)$$

$$\frac{\partial \sigma_{NS}^2}{\partial Tr} = \frac{(2-k)kTi(TiTp^2Tw^2(Tp+Tw) + Tr^2Ti(Tp+Tw)^3 + 2TrTpTw(TpTw^2 + Ti(Tp+Tw)^2))}{2(Tp+Tw)(Tr^2Tw + TiTpTw + TrTi(Tp+Tw))^2} \quad (36)$$

(35) is monotone and always negative (or zero when k=1) in the return rate, *k*. (36) is monotone and positive in the remanufacturing lead-time, *Tr*.

The relationship of inventory variance between return rate *k* and manufacturing lead-times can be illustrated in Figure 3 where we have set Tp=3, Ti = 4, and Tw = 8.

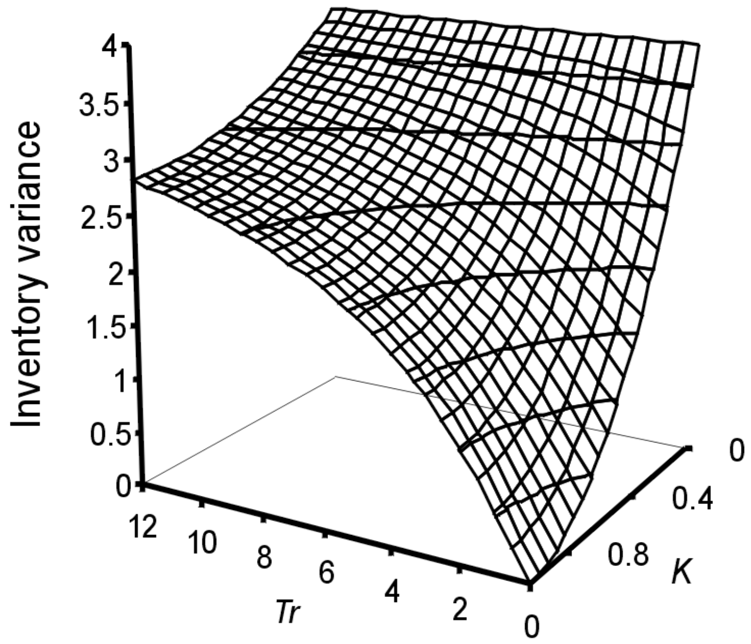

Figure 3. The effect of return rate and lead-time on inventory variance when *Tp=Tr=3* and *Tw=Ti*

5. Bullwhip in the remanufacturing supply chain

"Bullwhip" is the phenomenon where the orders at the supplier level tend to have a larger variance than sales to the buyer (that is the demand gets distorted), and the distortion propagates upstream in an amplified form (i.e. variance amplification), Dejonckheere *et al* (2002). Carlsson and Fullér (2000) have summarized the negative impact of bullwhip problem as follows; excessive inventory investments throughout out the supply chain to cope with the increased demand variability, reduced customer service due to the inertia of the production/distribution system, lost revenues due to shortages, reduced productivity of capital investment, increased investment in capacity, inefficient use of transport capacity and increased missed production schedules. Thus, avoiding or reducing bullwhip has a real and important impact on the performance of a commercial company.

A mathematical definition of bullwhip, that we will adopt here, has been proposed by Chen, Drezner, Ryan and Simchi-Levi (2000) as,

$$Bullwhip = \frac{\sigma_{OR}^2}{\sigma_D^2}. \tag{37}$$

From (37), we can see that bullwhip is also a variance ratio problem. We therefore can employ the same methods as used in Section 3 to acquire the bullwhip analytical expression as:

$$Bullwhip = \frac{Tw\left(Tr^2(Ti+Tp)Tw + (k-1)^2 TiTp(Ti+Tp)Tw + TrTi\left((k-1)^2 Ti + Tp\right)(Tp+Tw)\right)}{2Ti^2(Tp+Tw)\left(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw\right)} \tag{38}$$

Zhou, Disney, Lalwani, Wu (2004) have also derived (38) using Parsevel's Relation. Again we start by setting k=0 to determine the bullwhip in a traditional supply chain as shown in (39).

$$Bullwhip_{k=0} = \frac{Tw(Ti+Tp)}{2Ti^2(Tp+Tw)} \tag{39}$$

(39) shows that in a supply chain with no returns, the bullwhip decreases as Ti increases and the lead-time, Tp, should be reduced in order to smooth production and reduce the associated capacity on-costs. Again, in the subset of $Tw=Ti$, then (38) and (39) are further simplified as (40) and (41) respectively.

$$Bullwhip_{Tw=Ti} = \frac{(k-1)^2 Ti + Tr}{2Ti(Ti+Tr)} = \frac{1}{2Ti} + \frac{k^2-2k}{2(Ti+Tr)} \tag{40}$$

$$Bullwhip_{k=0,Tw=Ti} = \frac{1}{2Ti} \tag{41}$$

Here we can see that when $Tw=Ti$, then bullwhip in a remanufacturing supply chain will always be less than in a traditional supply chain due to the return rate $0 \le k \le 1$, thus the last term of (40) is always negative. (40) and (41) show that when $Tw=Ti$, the Tp drops out of the bullwhip expression.

When all of the products are returned from the marketplace (after the stochastic exponential delay), k=1. Here (38) reduces to,

$$Bullwhip_{k=1,Tw=Ti} = \frac{1}{2Ti} - \frac{1}{2(Ti+Tr)}, \tag{42}$$

where we can clearly see that; bullwhip in a remanufacturing supply chain is always less than in a traditional supply chain, but this smoothing effect is reduced with longer remanufacturing lead-times, Tr. Returning now to the general case of (38). We may factor this into the following,

$$Bullwhip = \frac{(Ti+Tp)Tw}{2Ti^2(Tp+Tw)} + \frac{(k-2)kTw}{2Ti(Tp+Tw)} + \frac{(k-2)k(Tp-Tr)(Tp+Tr)Tw^2}{2Ti(Tp+Tw)(TiTpTr + Tr^2Tw + Ti(Tp+Tr)Tw)}. \tag{43}$$

In (43) the first term on the RHS is the bullwhip generated by a traditional supply chain. The second term is always negative as $0 \le k \le 1$. Interestingly, the third term is; zero when $Tr=Tp$ and $Tr=-Tp$, negative when $Tr<Tp$ and positive when $Tr>Tp$. The sum of the last two terms is always negative for positive remanufacturing lead-times, Tr, which is obviously true. This leads us to investigate bullwhip when the remanufacturing lead-time (this also includes the time the product is in the hands of the user) is the same as the manufacturing lead-time. When $Tr=Tp$, (38) becomes,

$$Bullwhip_{Tr=Tp} = \frac{((k-1)^2 Ti + Tp)Tw}{2Ti^2(Tp+Tw)}. \tag{44}$$

(44) is always less than (38). Returning again to (38), differentiating (38) with respect to *k* and *Tr* yields;

$$\frac{\partial\, Bullwhip}{\partial\, k}=\frac{(k-1)Tw(Tp(Ti+Tp)Tw+TrTi(Tp+Tw))}{Ti(Tp+Tw)(Tr^2Tw+TiTpTw+TrTi(Tp+Tw))}, \tag{45}$$

$$\frac{\partial\, Bullwhip}{\partial\, Tr}=\frac{(1-k)Tw^2(2TrTp(Ti+Tp)Tw+Tr^2Ti(Tp+Tw)+TiTp^2(Tp+Tw))}{Ti(Tp+Tw)(Tr^2Tw+TiTpTw+TrTi(Tp+Tw))^2}. \tag{46}$$

(45) is monotone and always negative (or zero when *k*=1) in the return rate, *k*. (46) is monotone and positive (or zero when *k*=1) in the remanufacturing lead-time, *Tr*.

The relationship of bullwhip between return rate *k* and manufacturing lead-times, *Tr*, can be illustrated in Figure 4 when we set *Tp*=3, *Ti* = 4, and *Tw* = 8.

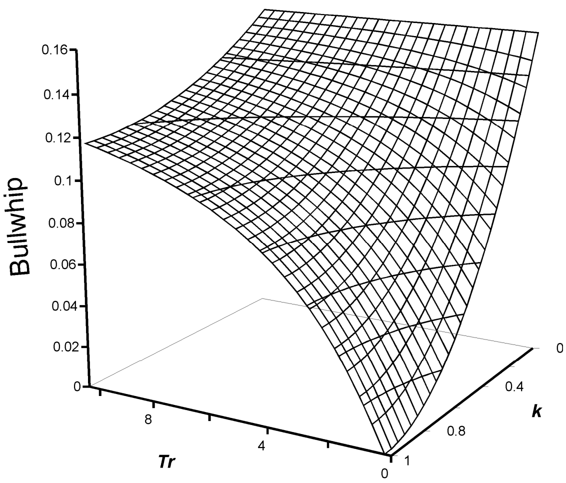

Figure 4. The effect of return rate and lead-time on bullwhip when *Tp=Tr=3* and *Tw=Ti*

Our analysis broadly supports the results of Wang (2002) who has adapted the beer game (Sterman, 1989) to include a reverse logistics scenario. Initial results from Wang (2002) also suggest that remanufacturing reduces bullwhip in a supply chain. We also find that in a remanufacturing supply chain, the inventory variance and bullwhip experienced by the manufacturer of new products is reduced when compared to a traditional supply chain. This means that returns can be used to smooth inventory variance and bullwhip. The higher the return rate and the shorter the remanufacturing lead-time, the smoother the order and inventory patterns are. However, we also notice that the parameter *Ti,* has a different impact on the inventory variance and than that on bullwhip. When *Ti* increases the inventory variance increases whilst bullwhip decreases. This leads us to the next section where we investigate the trade-off between inventory variance and bullwhip.

6. The Optimal *Ti*

Setting the objective function (OF) in our trade-off as

$$OF = \sigma_{NS}^2 + Bullwhip\,,$$

yields, after some manipulation,

$$\text{OF}=\frac{\begin{bmatrix} Tr^2Tw\left(Ti^3Tp^2+2Ti^3TpTw+\left(1+Ti^2\right)\left(Ti+Tp\right)Tw^2\right)+\\ \left(k-1\right)^2TpTw\left(Ti^4Tp^2+2Ti^4TpTw+\left(Ti+Ti^3\right)\left(Ti+Tp\right)Tw^2\right)+\\ TrTi\left(Tp+Tw\right)\left(\left(k-1\right)^2TiTw^2+TpTw^2+Ti^2TpTw^2+\left(k-1\right)^2Ti^3\left(Tp+Tw\right)^2\right)\end{bmatrix}}{2Ti^2Tw\left(Tp+Tw\right)\left(Tr^2Tw+TiTpTw+TrTi\left(Tp+Tw\right)\right)}\,. \tag{47}$$

Theoretically, the optimal *Ti* can be derived by solving for zero gradients of (47). However, as (47) is quite complex. So, in order to clarify our exposition we will study the simplified case of *Tr*=*Tp* and *Tw*=*Ti*. There (47) simplifies to (48).

$$OF=\frac{\begin{bmatrix} 2\left(k-1\right)^2Ti^4+2\left(4+3\left(k-2\right)k\right)Ti^3Tp+2Tp^2+\left(2+\left(k-2\right)k\right)TiTp\left(2+Tp^2\right)+\\ 2Ti^2\left(\left(k-1\right)^2+2\left(2+\left(k-2\right)k\right)Tp^2\right)\end{bmatrix}}{4Ti\left(Ti+Tp\right)^2} \tag{48}$$

Taking the partial derivative we have,

$$\frac{\partial OF}{\partial Ti}=\frac{\begin{bmatrix} \left(k-1\right)^2Ti^5+3\left(k-1\right)^2Ti^4Tp-3TiTp^2-Tp^3+\left(k-1\right)^2Ti^3\left(4Tp^2-1\right)+\\ Ti^2Tp\left(-3+2Tp^2+\left(k-2\right)k\left(Tp^2-1\right)\right)\end{bmatrix}}{2Ti^2\left(Ti+Tp\right)^3}\,. \tag{49}$$

Obviously, the optimal *Ti* should be the function of *Tp* and *k*. However, (49) is a high order function of *Ti* and it is hard to get an analytical solution. But for the special cases; *k*=0 and *k*=1, we have

$$Ti_{k=0}^{*}=\frac{1}{2}\left(-Tp+\sqrt{1+Tp^2}+\sqrt{1+2\,Tp\left(-Tp+\sqrt{1+Tp^2}\right)}\right) \tag{50}$$

$$Ti_{k=1}^{*}=\frac{Tp\left(3+\sqrt{1+Tp^2}\right)}{2\left(Tp^2-2\right)} \tag{51}$$

We further illustrate the more general case when *k* varies from zero to one. Figure 5 reveals that bullwhip and inventory variance (and their sum) is larger for small *k*. It verifies our conclusion that the returns can smooth both inventory and order variance. Table 2 provides some numerical results in each case. We notice that with increasing returns, it takes a larger *Ti* to minimize the sum of variances. Thus greater returns allow smoother production rates to be achieved without unduly increasing inventory requirements.

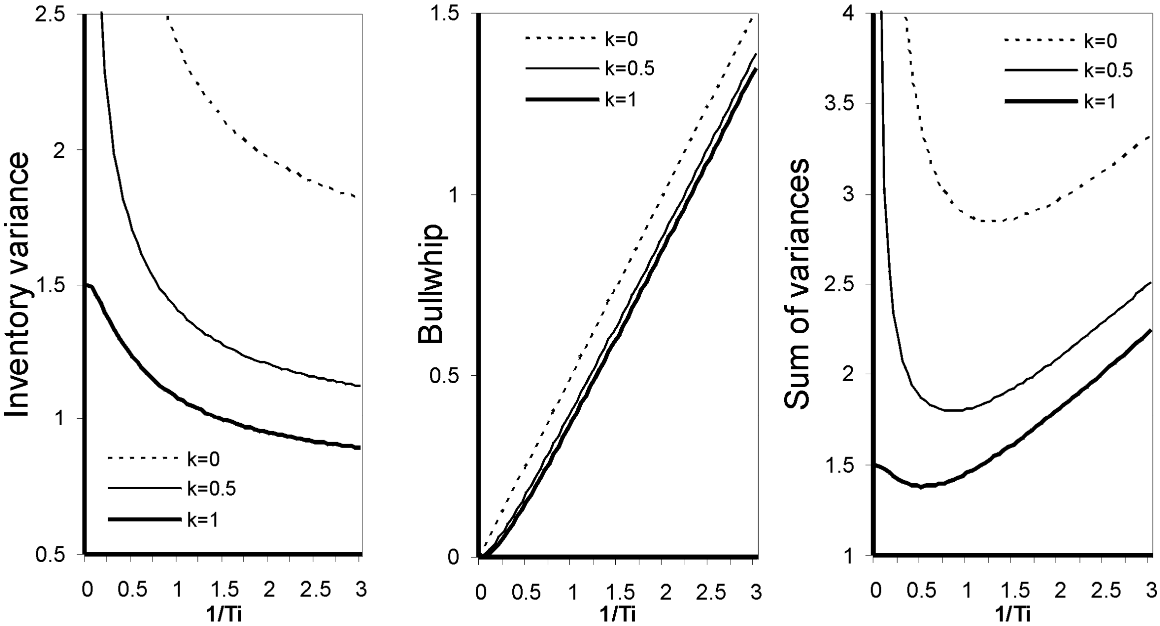

Fig 5. The optimal sum of variance ratios when *Tp=Tr=3* and *Tw=Ti*

k	T_i^*	σ^2_{NS}	**Bullwhip**	**Sum of variances**
0.0	0.78358	2.20244	0.63810	2.84054
0.2	0.91202	1.85687	0.50223	2.35909
0.4	1.08774	1.58095	0.38139	1.96234
0.6	1.33731	1.37959	0.27705	1.65663
0.8	1.68504	1.26006	0.19428	1.45433
1.0	1.94631	1.22411	0.15581	1.37992

Table 2. The optimal T_i^* that minimizes the sum of bullwhip and inventory variance when *Tp=Tr=3* and *Tw=Ti*

7. Conclusions

We have studied a stylized manufacturing / remanufacturing supply chain that reclaims product to as good as new, as soon as they are available and tops up serviceable inventory by production of new products. We have achieved this using block diagrams, Laplace transforms and Åström's method of calculating the integral square of a signal. Our findings are summarized in Table 3. It shows that the returned product can be used to reduce the inventory variance and bullwhip experienced by the manufacturer of original equipment compared to a manufacturer in a supply chain without remanufacturing or reverse logistics. ***This means that reverse flows can be used to improve the efficiency of supply chains***. This is in contrast to intuition and Seitz, Disney and Naim (2003) and Fleischmann *et al.* (1997) for example. Thus product recovery not only benefits the environment but may also have positive commercial effects. However, longer remanufacturing lead-times have less impact at reducing bullwhip than shorter remanufacturing lead-times.

Case	SC type	Inventory variance	Bullwhip
$Tw \neq Ti$	Traditional	Reducing *Ti* will reduce inventory variance.	Increasing *Ti* will reduce bullwhip.
	Remanufacturing	Higher return rates and shorter lead-times reduce inventory variance.	The bullwhip with returns is always less than without returns. Higher return rates and shorter lead-times smooth the order rates.
$Tw = Ti$	Traditional	Both *Tp* and *Ti* should be reduced in order to dampen inventory variance.	*Tp* has no impact on the bullwhip.
	Remanufacturing	Higher return rates and shorter lead-times reduce inventory variance.	The bullwhip with returns is always less than without returns. The higher *k* and the shorter *Tr*, the smoother the order rate. *Tp* has no impact on bullwhip.

Table 3. Summary managerial insights

Appendix A

Theorem: If the customer demand is drawn from an independently and identically distributed (i.i.d) random distribution, then the following equation holds.

$$\frac{\sigma_y^2}{\sigma_x^2} = \int_0^{+\infty} [g(t)]^2 dt = \int_0^{+\infty} [L^{-1}G(s)]^2 dt$$

where *g(t)* is the time domain response and *G(s)* is its Laplace transform in the complex frequency domain. $L^{-1}G(s)=g(t)$. *y* is output, *x* input (customer demand)

Proof: From the definition of Bullwhip, we have

$$VR = \frac{\sigma_y^2}{\sigma_x^2}. \quad \text{(A1)}$$

The definition of σ^2 is well known to be

$$\sigma_y^2 = E\big[y(t) - E[y(t)]\big]^2 = E[y(t)^2] - \big[E[y(t)]\big]^2 \text{ and} \quad \text{(A2)}$$

$$\sigma_x^2 = E\big[x(t) - E[x(t)]\big]^2 = E[x(t)^2] - \big[E[x(t)]\big]^2 \quad \text{(A3)}$$

where $E[y(t)]$ and $E[x(t)]$ are the mean values of a process's output and input respectively, denoted as $\mu(y)$ and $\mu(x)$. So

$$E[y(t)] = \mu(y) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} y(t)dt \quad and \quad E[x(t)] = \mu(x) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} x(t)dt\,. \quad \text{(A4)}$$

Suppose that the system is linear and the demand is a stationary process thus $\mu(y)=\mu(x)$. (A1) can therefore be expressed by

$$VR = \frac{\sigma_y^2}{\sigma_x^2} = \frac{E[y(t)^2]}{E[x(t)^2]}. \tag{A5}$$

We know that $y(t) = \int_{-\infty}^{+\infty} g(t) \otimes x(t) dt$ where $\otimes$ denotes convolution, then equation (A5) becomes

$$\begin{aligned} \frac{E[y(t)^2]}{E[x(t)^2]} &= \frac{\lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} y(t)^2\, dt}{\lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)^2\, dt} = \frac{\lim\limits_{T\to\infty} \int_{-T}^{+T} dt \int_{-\infty}^{+\infty} [g(t)x(t)]^2\, dt}{\lim\limits_{T\to\infty} \int_{-T}^{+T} x(t)^2\, dt} \\ &= \frac{\lim\limits_{T\to\infty} \int_{-T}^{+T} x(t)^2\, dt \int_{-\infty}^{+\infty} g(t)^2\, dt}{\lim\limits_{T\to\infty} \int_{-T}^{+T} x(t)^2 dt} = \int_{-\infty}^{+\infty} g(t)^2\, dt = \int_{-\infty}^{+\infty} [L^{-1} G(s)]^2\, dt\ . \end{aligned} \tag{A6}$$

Furthermore, we notice that $\int_{-\infty}^{0} g(t) dt = 0$. This result shows that

$$\frac{\sigma_{OUT}^2}{\sigma_{IN}^2} = \int_0^{+\infty} g(t)^2\, dt = \int_0^{+\infty} [L^{-1} G(s)]^2\, dt\ . \tag{A7}$$

This completes our proof. □

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