Some Problems Involving Prime Numbers

D. E. Welch

April 19, 2007

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

Signed .... Date 29/09/06

STATEMENT 1

This thesis is being submitted in partial fulfillment of the requirements for the degree of ..PhD..

Signed .... Date 29/09/06

STATEMENT 2

This thesis is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by explicit references.

Signed .... Date 29/09/06

STATEMENT 3

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed .... Date 29/09/06

STATEMENT 4 - BAR ON ACCESS APPROVED

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan after expirary of a bar on access approved by the Graduate Development Committee.

Signed .... Date 29/09/06

Abstract

The first problem we consider is a variation of the Piatetski-Shapiro Prime Number Theorem. Consider a function $g(y)$, growing faster than linearly. We ask how often is the integer part of a function $g(y)$ no less than some distance j from a prime number? Using Huxley's method of exponential sums the investigation shows how the rate at which $g(y)$ increases is dependent on the size of j. The faster $g(y)$ increases, the larger the value of j.

The second problem investigates primes of arithmetic progressions, a mod q, in short intervals of the form $(x, x+x^{\theta})$, where x is sufficiently large in terms of q, $q^{\eta} < x$ for some $\eta > 0$. Such a result was proved by Fogels, for some $\theta < 1$. We explicitly determine the relationship between θ and η to establish admissible values for both.

Lastly we use our version of Fogels' theorem and a variation of Vaughan's treatment of the $||\alpha p||$ problem to investigate the following problem. Given a real number α in the interval $(0, 1)$ how many Farey fractions of the Farey sequence of order Q do we have to pass to go from α to a Farey fraction with prime denominator?

Acknowledgements

I wish to thank my supervisor, Professor Huxley, for presenting me with interesting problems, and whose endless patience I tried in their investigation. Thanks also to Nicola for her unconditional support, and to my father who awakened my interest in mathematics.

Contents

Part I

A Variation on The Piatetski-Shapiro Prime Number Theorem

Chapter 1

Introduction

Consider a function $g(y)$, growing faster than linearly so that $y = o(g(y))$. How often is the integer part of $g(n)$ prime, when n is an integer? This problem was first suggested by Gel'fond for Piatetski-Shapiro's PhD thesis. Piatecki-Shapiro [20] proved that there were infinitely many prime numbers of the form $[n^\alpha]$, where $1 < \alpha < 12/11$. This result has been improved many times since, most recently by Liu and Rivat [17] who increase the range for α to $1 < \alpha < 15/13$.

Using his method of exponential sums, Huxley [12], proved the more general result suggested by Gel'fond with certain restrictions to the function $g(y)$. The restrictions were that the function must not increase too quickly and that certain combinations of its derivatives are non-zero. The result is as follows.

Theorem 1.0.1 *(Huxley) Let $F(x)$ be a real monotone increasing function, six times continuously differentiable on a closed sub-interval I of $[1, 2]$ on which the expressions $F'(x)$, $F''(x)$ and $2F''(x) + xF^{(3)}(x)$ are nonzero. Let*

ϵ be positive, and let M be a large integer, N be a large real number with

$$N \max_I F'(x) < M \le N^{\theta-\epsilon} \tag{1.0.1}$$

with $\theta = 12/11$. Then for some constant B depending on I, $F(x)$, and on ϵ, but not M or N, there are at least $BN/\log N$ integers n such that $[g(n)]$ is a prime number p with $M \le p < 2M$, where $x = g(y)$ is the function inverse to

$$y = NF\Big(\frac{x}{M}\Big).$$

If in addition the interval I is chosen so that none of the expressions

$$F^{(3)}(x), F^{(4)}(x)$$

$$\begin{vmatrix} F^{(4)}(x) & F^{(3)}(x) \\ 3F^{(3)}(x) & F''(x) \end{vmatrix}, \begin{vmatrix} F^{(4)}(x) & F^{(3)}(x) \\ 3F^{(3)}(x) & 2F''(x) \end{vmatrix}, \begin{vmatrix} F^{(5)}(x) & F^{(4)}(x) \\ 4F^{(4)}(x) & 3F^{(3)}(x) \end{vmatrix}$$

$$\begin{vmatrix} 3F^{(3)}(x)^2 & 0 & -F''(x)^2 \\ F^{(5)}(x) & F^{(4)}(x) & F^{(3)}(x) \\ 4F^{(4)}(x) & 3F^{(3)}(x) & 2F''(x) \end{vmatrix}$$

vanishes on the interval I, then the result holds with $\theta = 3300/3019$ in (1.0.1).

As we can see this theorem includes Piatetski-Shapiro's result and a lot more.

One would like to extend the range for θ so that $g(y)$ is allowed to increase more rapidly. This can be done if we allow for primes that are further away from the values of $g(n)$. To this end we investigate the following problem: How many prime numbers are there within a distance j of $g(n)$?

Chapter 2

Preparation

We want to be able to count the number of primes p so that there exists an integer n for which

$$g(n) - j \leq p < g(n) + j \tag{2.0.1}$$

where j is a real positive number. We follow Huxley's method, for which it is more convenient to work with $f(x)$, the inverse to $x = g(y)$, where

$$f(x) = NF\left(\frac{x}{M}\right), \tag{2.0.2}$$

for $M \leq x < 2M$ and

$$0 < F'(x) < \frac{M}{N}, \tag{2.0.3}$$

for $1 \leq x \leq 2$. We also require that

$$|f^{(k)}(x)| \asymp \frac{N}{M^k} \tag{2.0.4}$$

for $M \leq x < 2M$ and all integers k. We count the prime values of m in an interval $M \leq m < 2M$, for which there exists an integer n with

$$f(m - j) < n \leq f(m + j). \tag{2.0.5}$$

We require $f(x)$ to be monotone and grow slowly enough so that

$$f(m+j) - f(m-j) < 1, \tag{2.0.6}$$

from which it follows that there is at most one n satisfying (2.0.5). It is this condition that allows us to extend the range for θ in Theorem 1.0.1.

Let $\rho(t) = [t] - t + 1/2$, then the sum

$$f(m+j) - f(m-j) + \rho(-f(m-j)) - \rho(-f(m+j)) \tag{2.0.7}$$

is one if some integer n satisfies (2.0.5), and zero if not.

Let I be a subinterval of $1 \leq x \leq 2$, chosen so that no combination of derivatives occuring in the proof vanishes on I. We sum m over integers in MI, the set of points of I multiplied by M, which we suppose has end points M_1 and M_2 respectively. We count powers of primes using von Mangoldt's weight function $\Lambda(m)$, and the prime number theorem in the following form, see Davenport [3].

Theorem 2.0.2 *(The prime number theorem) For $P \geq 2$ we have*

$$\sum_{m \leq P} \Lambda(m) = P + O\Big(\frac{P}{\log P}\Big).$$

Let Σ' denote the sum over integers m for which (2.0.5) has a solution. Then

$$\begin{aligned} \sum_{m \in MI}{}' \Lambda(m) = \quad & \sum_{m \in MI} \Lambda(m)(f(m+j) - f(m-j)) + \\ & \sum_{m \in MI} \Lambda(m)(\rho(-f(m-j)) - \rho(-f(m+j))). \end{aligned} \tag{2.0.8}$$

The Taylor series of $f(x)$ at a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots \tag{2.0.9}$$

Let $x = m + j$ and $a = m - j$ to give

$$\begin{aligned} f(m+j) - f(m-j) &= 2jf'(m-j) + \frac{(2j)^2}{2!} f''(m-j) + \dots \\ &= 2jf'(m-j) + O(j^2|f''(m-j)|) \end{aligned}$$

If in (2.0.9) we let $a = x - j$, we have

$$f'(x-j) = f'(x) + O(j|f''(x)|),$$

which together with (2.0.4) gives

$$f(m+j) - f(m-j) = 2jf'(m) + O\Big(\frac{j^2 N}{M^2}\Big), \tag{2.0.10}$$

so that in light of (2.0.4) we have

$$j \ll \frac{M}{N}, \tag{2.0.11}$$

for $j \ll M$. The first term on the right hand side of (2.0.8) is now

$$\begin{aligned} &2j \sum_{m \in MI} \Lambda(m) f'(m) + O\Big(\frac{j^2 N}{M^2} \sum_{m \in MI} \Lambda(m)\Big) \\ &= 2j \sum_{m \in MI} \Lambda(m) f'(m) + O\Big(\frac{j^2 N}{M}\Big), \end{aligned} \tag{2.0.12}$$

by the prime number theorem and (2.0.10). Using Riemann-Stieltjes integration to estimate the main term in (2.0.12) we have

$$\begin{aligned} 2j\int_{M_1}^{M_2} f'(x)d\psi(x) &= 2j\Big[f'(x)\psi(x)\Big]_{M_1}^{M_2} - 2j\int_{M_1}^{M_2} f''(x)\psi(x)dx\Big) \\ &= 2j\Big[f'(x)\psi(x)\Big]_{M_1}^{M_2} - 2j\int_{M_1}^{M_2} f''(x)xdx + O\Big(\frac{jN}{\log M}\Big) \\ &= 2j\Big[f'(x)\psi(x) - f'(x)x\Big]_{M_1}^{M_2} + 2j\int_{M_1}^{M_2} f'(x)dx \\ &\quad + O\Big(\frac{jN}{\log M}\Big) \\ &= 2j\int_{M_1}^{M_2} f'(x)dx + O\Big(\frac{jN}{\log M}\Big) \\ &= 2j(f(M_2) - f(M_1)) + O\Big(\frac{jN}{\log M}\Big) \end{aligned} \tag{2.0.13}$$

We would like now to estimate the second term on the right hand side of (2.0.8), but in order to do this we require the following exponential sum estimates.

Chapter 3

Exponential Sum Estimates

3.1 Exponential Sum Lemmas

The following Lemmas are taken from Huxley [12].

Lemma 3.1.1 *(Vaughan's identity). Let I, M and MI be as above. Let $h(x)$ be a real valued function supported on MI and u be an integer with $u+1$ a power of two, and*

$$u^3 \leq M. \tag{3.1.1}$$

If

$$S = \sum_{m \in MI} \Lambda(m) h(m),$$

then

$$S = S_0 - S_1 - S_2 - S_3,$$

where

$$\begin{aligned} S_0 &= \sum_{d \leq u} \mu(d) \sum_{m} \Lambda(m) \sum_{r} h(dmr) \\ S_1 &= \sum_{m > u} \Lambda(m) \sum_{n > u} \kappa(n) h(mn) \end{aligned}$$

$$S_2 = \sum_{s\le u}\sum_{r}\lambda(s)h(rs)$$
$$S_3 = \sum_{u<s\le u^2}\sum_{r}\lambda(s)h(rs)$$

and

$$\kappa(n) = \sum_{\substack{d|n\\ d\le u}}\mu(d)$$
$$\lambda(s) = \sum_{d\le u}\sum_{\substack{m\le u\\ dm=s}}\mu(d)\Lambda(m).$$

Proof. We write $S_0 = S + S'$ where S consists of the terms with $dr \le u$, S' the terms with $dr > u$. Then

$$\begin{aligned} S &= \sum_{m}\Lambda(m)\sum_{d\le u}\sum_{\substack{r\\ dr\le u}}\mu(d)h(dmr) \\ &= \sum_{m}\Lambda(m)\sum_{n\le u}h(mn)\sum_{d|n}\mu(d) \\ &= \sum_{m}\Lambda(m)h(m) \end{aligned}$$

by Möbius inversion. Next we split up the sum S', so that $S' = S_1 + S_1'$, where S_1 consists of the terms with $m > u$, S_1' the terms with $m \le u$. We have

$$\begin{aligned} S_1 &= \sum_{d\le u}\mu(d)\sum_{m>u}\Lambda(m)\sum_{r>u/d}h(dmr) \\ &= \sum_{m>u}\Lambda(m)\sum_{n>u}h(nm)\sum_{d\le u}\sum_{\substack{r\\ dr=n}}\mu(d) \\ &= \sum_{m>u}\Lambda(m)\sum_{n>u}h(nm)\kappa(n) \end{aligned}$$

and

$$S_1' = \sum_{m\le u}\Lambda(m)\sum_{d\le u}\mu(d)\sum_{r>u/d}h(dmr).$$

We can drop the $dr > u$ condition in S_1' since $h(dmr) = 0$ when $dmr \le u^2$ by (3.1.1). Now let $S' = S_2 + S_3$, where S_2 consists of the terms with $dm \le u$,

S_3 the terms with $dm > u$. Let $dm = s$, then we have

$$\begin{aligned} S_2 &= \sum_{s\le u}\sum_{r} h(rs) \sum_{d\le u}\sum_{\substack{m\le u\\ dm=s}} \Lambda(m)\mu(d) \\ &= \sum_{s\le u}\sum_{r} \lambda(s)h(rs) \end{aligned}$$

and

$$\begin{aligned} S_3 &= \sum_{s>u}\sum_{r} h(rs) \sum_{d\le u}\sum_{\substack{m\le u\\ dm=s}} \Lambda(m)\mu(d) \\ &= \sum_{u<s\le u^2}\sum_{r} h(rs)\lambda(s). \end{aligned}$$

which proves the Lemma.

Lemma 3.1.2 *Let $g(x)$ be a real function of bounded variation V on the closed interval $[\alpha,\beta]$, and let $f(n)$ be any sequence, then*

$$\left|\sum_{\alpha}^{\beta} g(n)f(n)\right| \le (V + |g(\alpha)|)\max_{\gamma\ge\alpha}\left|\sum_{n=\gamma}^{\beta} f(n)\right|$$

Lemma 3.1.3 *Let $f(x)$ be a real and twice continuously differentiable function on (α,β). If $|f''(x)| \asymp \lambda$ for $\alpha \le x \le \beta$, then*

$$\sum_{n=\alpha}^{\beta} e(f(n)) \ll (\beta-\alpha+1)\lambda^{1/2} + \frac{1}{\lambda^{1/2}}, \tag{3.1.2}$$

and

$$\sum_{n=\alpha}^{\beta} \rho(f(n)) \ll (\beta-\alpha+1)\lambda^{1/3} + \frac{1}{\lambda^{1/2}}, \tag{3.1.3}$$

where $\rho(t)$ is the rounding error function $\rho(t) = [t] - t + 1/2$.

Lemma 3.1.4 *Let $y_1, ..., y_M$ and $z_1, ..., z_M$ be two sequences of real numbers, with associated weights $w_1, ..., w_M$. Let H be a positive integer then*

$$\sum_{m=1}^{M} w_m(\rho(z_m) - \rho(y_y)) \ll \frac{1}{H}\sum_{m=1}^{M} |w_m| + \sum_{h=1}^{H}\left|\sum_{m=1}^{M} \frac{w_m(e(hy_m) - e(hz_m))}{2\pi ih}\right|$$

$$+\frac{1}{H}\sum_{h=1}^{H}\left|\sum_{m=1}^{M}|w_m|e(hy_m)\right|$$
$$+\frac{1}{H}\sum_{h=1}^{H}\left|\sum_{m=1}^{M}|w_m|e(hz_m)\right|.$$

Lemma 3.1.5 *Suppose that the function $F(x, y)$ is six times differentiable for $1 \le x \le 2$, $0 \le y \le 1$, and for some constants $C_1 \ge 1$, $C_2 > 0$*

$$|\partial_1^r \partial_2^s F(x,y)| \le C_1 \tag{3.1.4}$$

for $2 \le r \le 6$, $0 \le s \le 2$, $r + s \le 6$, and

$$|\partial_1^r F(x,y)| \ge \frac{1}{C_1} \tag{3.1.5}$$

for $r = 2, 3$, and

$$\left|\partial_1 \frac{\partial_1^{r-1}\partial_2 F(x,y)}{\partial_1^r F(x,y)}\right| \ge C_2 \tag{3.1.6}$$

for $r = 3$. Suppose also that either case 1 or case 2 holds:

Case 1. *$M \ll T^{1/2}$ and (3.1.5) and (3.1.6) hold with $r = 4$;*

Case 2. *$M \gg T^{1/2}$ and*

$$|3F_{111}^2 - F_{11}F_{1111}| \ge C_3, \tag{3.1.7}$$

$$|\Delta| \ge C_4, \tag{3.1.8}$$

for some positive constants C_3 and C_4, where

$$\Delta = \begin{vmatrix} 3F_{111}^2 + 4F_{11}F_{1111} & 3F_{11}F_{111} & F_{11}^2 \\ F_{11111} & F_{1111} & F_{111} \\ F_{11112} & F_{1112} & F_{112} \end{vmatrix}. \tag{3.1.9}$$

Let S_i be the sum

$$S_i = \sum_{m=M_1(i)}^{M_2(i)} e\Big(TF\Big(\frac{m}{M}, y_i\Big)\Big),$$

where $M \le M_1(i) \le M_2(i) < 2M$, and $y_1, y_2, \ldots, y_I$ lie in $0 \le y \le 1$ with

$$y_{i+1} - y_i \ge \frac{1}{J}$$

for $i < I$. Let $\epsilon > 0$ be arbitrary. Then in both cases, for

$$T^{1/3-\epsilon} \gg \frac{T^{8/57}}{I^{11/19}} + \frac{J}{I} \gg \Big(\frac{T}{M^2} + \frac{M^2}{T}\Big) \log M \tag{3.1.10}$$

we have

$$\sum_{i=1}^{I} |S_i|^5 \ll I^{16/9} M^{5/2} T^{89/114} \log^5 T + I^{8/11} J^{3/11} M^{5/2} T^{49/66} \log^5 T. \tag{3.1.11}$$

Also in case 1 with $M \ll T^{1/2}$

$$\sum_{i=1}^{I} |S_i|^5 \ll I M^{5/2} T^{5/6} E_1^{3/8} \Big(1 + \frac{JM^2}{IT}\Big)^{1/2} \log^{11/2} T \tag{3.1.12}$$

when

$$E_1 = \frac{1}{T^{8/33}} + \frac{T^{2/3}}{M^2} + \Big(J + \frac{IT}{M^2}\Big)^{-1} \ll \frac{1}{T^{\epsilon}}, \tag{3.1.13}$$

and in case 2 for $M \gg T^{1/2}$

$$\sum_{i=1}^{I} |S_i|^5 \ll I M^{5/2} T^{5/6} E_2^{3/8} \Big(1 + \frac{JT}{IM^2}\Big)^{1/2} \log^{11/2} T \tag{3.1.14}$$

when

$$E_2 = \frac{1}{T^{8/33}} + \frac{M^2}{T^{4/3}} + \Big(J + \frac{IM^2}{T}\Big)^{-1} \ll \frac{1}{T^{\epsilon}}. \tag{3.1.15}$$

When the inequalities involving ϵ in (3.1.10), (3.1.13) and (3.1.15) do not hold, then the upper bounds hold with the power of $\log T$ replaced by T^{ϵ}. The implied constants are constructed from $C_1, \ldots, C_4$, from ϵ, and from the implied order of magnitude constants in the ranges for M, I and J.

Lemma 3.1.6 *Let K be any integer, K_1 and K_2 be real numbers with $2 \le K \le K_1 \le K_2 < 2K$. For any coefficients $g(k)$ we have*

$$\sum_{k=K_1}^{K_2} g(k) = \frac{1}{2\pi i}\int_C G(s)\frac{K_2^s - K_1^s}{s}ds + O(\max|g(k)|\log K)$$

where C is the line segment $1/\log K - iK$ to $1/\log K + iK$, and

$$G(s) = \sum_{k=K}^{2K-1} \frac{g(k)}{k^s}.$$

Lemma 3.1.7 *Let $f(x)$ be a real function defined for $M \le x \le N$, and let $w(M), w(M+1), ..., w(N)$ be any coefficients. Let $D \le (N-M+1)/2$ be a positive integer. Then*

$$\Big|\sum_{m=M}^{N} w(m)e(f(m))\Big|^2 \le \frac{(N-M+2D-1)}{D}\Big(\sum_{m=M}^{N}|w(m)|^2$$
$$+2\Re\sum_{d=1}^{D-1}\Big(1-\frac{d}{D}\Big)\sum_{q=M+d}^{N-d} w(q+d)\overline{w(q-d)}e(f(q+d)-f(q-d))\Big).$$

Lemma 3.1.8

$$\Big|\sum_{m=M_1}^{M_2}\sum_{n=N_1(m)}^{N_2(m)} g(m)h(m,n)e(f(m,n))\Big|^2 \le \Big(\sum_m |g(m)|^2\Big)$$
$$\times\Big(\sum_{n_1}\sum_{n_2}\sum_{\substack{m=M_1\\N_1(m)\le n_1,n_2\\N_2(m)\ge n_1,n_2}}^{M_2} h(m,n)\overline{h(m,n)}e(f(m,n_1)-f(m,n_2))\Big).$$

Lemma 3.1.9 *(Partial summation) Let $h(x)$ be real and continuously differentiable on the interval $[M,N]$. Then we have*

$$\Big|\sum_{n=M}^{N} G(n)e(h(n))\Big| \le \Big(1+2\pi\int_M^N |h'(x)|dx\Big)\sup_{M\le x\le N}\Big|\sum_{n=x}^{N} G(n)\Big|.$$

3.2 Long Type I Exponential Sums

If we write

$$\sigma(m) = \begin{cases} \rho(-f(m-j)) - \rho(-f(m+j)) & \text{for } m \in MI, \\ 0 & \text{otherwise,} \end{cases}$$

then long type I exponential sums have the form

$$\sum_{q \le x} \alpha(q) \sum_{r} \beta(r)\sigma(rq), \tag{3.2.16}$$

where $\beta(r)$ is a monotone weight function which may also depend on q. If we let $h(m) = \sigma(m)$ in Vaughan's identity, then we see that S_0 and S_2 are long type I sums:

$$\begin{aligned} S_0 &= \sum_{d \le u} \mu(d) \sum_{m} \Lambda(m) \sum_{r} \sigma(dmr) \\ &= \sum_{d \le u} \mu(d) \sum_{e} \sigma(de) \log e, \end{aligned}$$

where $x = u$, $\alpha(q) = \mu(q)$ and $\beta(r) = \log r$;

$$S_2 = \sum_{q \le u} \lambda(q) \sum_{r} \sigma(rq),$$

where $x = u$, $\alpha(q) = \lambda(q)$ and $\beta(r) = 1$. The Sum S_1 needs to be dissected further in order to obtain a long type I sum. We introduce the new variable z so that $z + 1$ is a power of two, and

$$u^2\sqrt{2} \le z < \sqrt{M}, \tag{3.2.17}$$

and

$$uz^2 \ge 2M, \tag{3.2.18}$$

so that $w = 2M/z^2$ lies in the range $2 \le w \le u$. We write $S_1 = S_{41} + S_{42} + S_4'$, where S_{41} is the terms with $m < z$, S_{42} the terms with $n < z$ and S_4' the terms with m and $n > z$. So

$$\begin{aligned} S_{41} &= \sum_{u<m<z} \Lambda(m) \sum_{n>u} \kappa(n)\sigma(mn) \\ S_{42} &= \sum_{u<n<z} \kappa(n) \sum_{m>u} \Lambda(m)\sigma(mn) \\ S_4' &= \sum_{n>z} \kappa(n) \sum_{m>z} \Lambda(m)\sigma(mn) \end{aligned}$$

We rewrite S_4' to obtain

$$S_4' = \sum_{d\le u} \mu(d) \sum_{m>z} \Lambda(m) \sum_{r>z/d} \sigma(dmr).$$

Let S_5 be the terms in S_4' with $d \ge w$, S_6 the terms with $d < w$. Then

$$\begin{aligned} S_5 &= \sum_{w\le d\le u} \mu(d) \sum_{m>z} \Lambda(m) \sum_{r>z/d} \sigma(dmr), \\ S_6 &= \sum_{d\le w} \mu(d) \sum_{m>z} \Lambda(m) \sum_{r>z/d} \sigma(dmr). \end{aligned}$$

Next we write $S_6 = S_{71} + S_{72}$, where S_{71} consists of the terms in S_6 with $dm = q \le x$ and S_{72} the terms in S_6 with $dm = q > x$ and $x + 1$ is a power of two, which gives

$$\begin{aligned} S_{71} &= \sum_{q\le x} \sum_{d<w} \sum_{\substack{m>z \\ dm=q}} \mu(d)\Lambda(m) \sum_{r>z/d} \sigma(qr), \\ S_{72} &= \sum_{q>x} \sum_{d<w} \sum_{\substack{m>z \\ dm=q}} \mu(d)\Lambda(m) \sum_{r>z/d} \sigma(qr). \end{aligned}$$

So for $x > z$, S_{71} is also a long type I sum with

$$\alpha(q) = \sum_{d<w} \sum_{\substack{m>z \\ dm=q}} \mu(d)\Lambda(m).$$

The inner sum in (3.2.16) is

$$\sum_{r=P_1(q)}^{P_2(q)} \beta(r)(\rho(-f(rq-j)) - \rho(-f(rq+j))) \tag{3.2.19}$$

such that $P_1(q) \asymp P_2(q) \asymp M/q$. By (2.0.4) and Lemmas 3.1.2 and 3.1.3 (the bound (3.1.3)) we have that the sum in (3.2.19) is

$$\begin{aligned} &\ll \max|\beta(r)| \Big| \sum_{r=P_1(q)}^{P_2(q)} (\rho(-f(rq-j)) - \rho(-f(rq+j))) \Big| \\ &\ll \max|\beta(r)| \Big(P_2(q) \Big(\frac{q^2N}{M^2}\Big)^{1/3} + \Big(\frac{M^2}{q^2N}\Big)^{1/2} \\ &\ll \max|\beta(r)| \Big(\Big(\frac{MN}{q}\Big)^{1/3} + \frac{M}{q\sqrt{N}} \Big). \end{aligned}$$

So in all three cases our long type I sums are

$$\begin{aligned}
&\ll \sum_{q\le x}\left(\left(\frac{MN}{q}\right)^{1/3}+\frac{M}{q\sqrt{N}}\right)\max|\alpha(q)\beta(r)| \\
&\ll \left((MNx^2)^{1/3}+\frac{M}{\sqrt{N}}\log x\right)\max|\alpha(q)\beta(r)| \\
&\ll \left((MNx^2)^{1/3}+\frac{M}{\sqrt{N}}l\right)l,
\end{aligned}$$

where $l = \log M$. Contributions from the long type I sums are swallowed by the error term in (2.0.13) if

$$\left((MNx^2)^{1/3}+\frac{M}{\sqrt{N}}l\right)l \ll \frac{jN}{l},$$

so we require

$$x \ll \frac{j^{3/2}N}{\sqrt{M}l^3} \tag{3.2.20}$$

and

$$j \gg \frac{Ml^3}{N^{3/2}}, \tag{3.2.21}$$

which completes the treatment of the long type I sums in our argument.

3.3 Short Type I Exponential Sums

S_{72} is our only short type I sum. We divide the range $q = dm$ into blocks of the form $2^kQ \le q \le 2^{k+1}Q - 1$, where $Q = x + 1$. Implicit in this is a division of the range for r of the form $2^kP \le r \le 2^{k+1}P - 1$. This gives

$$\begin{aligned}
S_{72} &= \sum_{k=0}^{K}\sum_{q=2^kQ}^{2^{k+1}Q-1}\sum_{d<w}\sum_{\substack{m>z\\ dm=q}}\Lambda(m)\mu(d)\sum_{r>z/d}\sigma(qr) \\
&\ll K\max_k\left|\sum_{q=2^kQ}^{2^{k+1}Q-1}\sum_{d<w}\sum_{\substack{m>z\\ dm=q}}\Lambda(m)\mu(d)\sum_{r>z/d}\sigma(qr)\right| \\
&\ll K\left|\sum_{q=Q_\kappa}^{2Q_\kappa-1}\sum_{d<w}\sum_{\substack{m>z\\ dm=q}}\Lambda(m)\mu(d)\sum_{r>z/d}\sigma(qr)\right|,
\end{aligned} \tag{3.3.22}$$

where we have assumed that the maximum occurs at $k = \kappa$, and $P_\kappa Q_\kappa \asymp M$.

The sum over d and m in (3.3.22) is

$$\sum_{d<w} \sum_{\substack{m>z \\ dm=q}} |\Lambda(m)\mu(d)| \ll \sum_{m|q} \Lambda(m) = \log q,$$

where

$$\log q \asymp \log Q_\kappa \asymp K + \log Q_\kappa \asymp l,$$

which gives

$$S_{72} \ll l^2 \sum_{q=Q_\kappa}^{2Q_\kappa - 1} \max_{\substack{P_3,P_4 \\ qP_3,qP_4 \in MI}} \left| \sum_{r=P_3}^{P_4} \sigma(qr) \right|. \tag{3.3.23}$$

We now apply Lemma (3.1.4) to the inner sum over r, with

$$\sigma(qr) = \rho(-f(qr-j)) - \rho(-f(qr+j)) = \rho(z_r) - \rho(y_r),$$

and

$$w_r = \begin{cases} 1 & \text{for } P_3 \le r \le P_4. \\ 0 & \text{otherwise.} \end{cases}$$

which gives

$$\left|\sum_r w_r \sigma(qr)\right| \ll \frac{P_\kappa}{H} + \sum_{h=1}^{H} \left| \sum_r \frac{w_r(e(hy_r) - e(hz_r))}{2\pi i h} \right|$$
$$+ \frac{1}{H} \sum_{h=1}^{H} \left| \sum_r w_r e(hy_r) \right| + \frac{1}{H} \sum_{h=1}^{H} \left| \sum_r w_r e(hz_r) \right|. \tag{3.3.24}$$

The first term on the right hand side of (3.3.24) contributes

$$\frac{P_\kappa Q_\kappa l^2}{H}$$

to S_{72}, which is negligible for

$$H \gg \frac{P_\kappa Q_\kappa l^3}{jN} \asymp \frac{Ml^3}{jN}. \tag{3.3.25}$$

In the second term we have

$$\frac{e(-hf(qr+j)) - e(-hf(qr-j))}{2\pi i h} = \int_{f(qr+j)}^{f(qr-j)} e(-ht)dt = -\int_{f(qr-j)}^{f(qr+j)} e(-ht)dt,$$

where

$$-\int_{f(qr-j)}^{f(qr+j)} e(-ht)dt = -\int_0^{f(qr+j)-f(qr-j)} e(-h(t+f(qr-j)))dt = -e(-hf(qr-j)) \int_0^{f(qr+j)-f(qr-j)} e(-ht)dt.$$

By (2.0.4) and (2.0.9) we know that there exists a positive constant c such that

$$f(qr+j) - f(qr-j) \le \frac{cjN}{M},$$

which gives

$$\sum_r \frac{w_r(e(-hf(qr+j)) - e(hf(qr-j)))}{2\pi i h} = -\int_0^{cjN/M} \sum_{\substack{r \\ f(qr+j)-f(qr-j)\ge t}} w_r e(-h(t+f(qr-j)));$$

the condition $f(qr+j) - f(qr-j) \ge t$ holds on a subinterval of $[P_3, P_4]$. So the second term in (3.3.24) is

$$\ll \frac{jN}{M} \sum_{h=1}^{H} \max_{[P_5,P_6]\subset[P_3,P_4]} \left| \sum_{r=P_5}^{P_6} e(-h(f(qr-j))) \right|.$$

We have

$$\left|\sum_r w_r \sigma(qr)\right| \ll \frac{P_\kappa}{H} + \left(\frac{jN}{M} + \frac{1}{H}\right) \sum_{h=1}^{H} \max_{[P_5,P_6]\subset[P_3,P_4]} \left| \sum_{r=P_5}^{P_6} e(-h(f(qr-j))) \right|.$$

We can drop the $1/H$ term since

$$\frac{jN}{M} \gg \frac{1}{H},$$

as we will be choosing H to be no smaller then (3.3.25) will allow. So we know our short type I sum is

$$S_{72} \ll l^2 \sum_{q=Q_\kappa}^{2Q_\kappa-1} \left(\frac{P_\kappa}{H} + \frac{jN}{M} \sum_{h=1}^{H} \max_{[P_5,P_6]\subset[P_3,P_4]} \left| \sum_{r=P_5}^{P_6} e(-hf(qr-j)) \right| \right). \quad (3.3.26)$$

This first term in the sum over q is

$$\frac{P_\kappa Q_\kappa l^2}{H} \asymp \frac{jN}{l},$$

and so it is swallowed by the error term in (2.0.13). We wish now to bound the expression on the right hand side of (3.3.26) for $P \ll hN$ where $h = 1, 2, ..., H$ which breaks down into two cases. Firstly we suppose that $h = O(1)$ so that $P \ll N$. This means that the expression on the right hand side of (3.3.26) is

$$\ll \frac{l^2 QjNP}{M} \ll \frac{l^2 QjN^2}{M},$$

which is absorbed into our error term in (2.0.13) for

$$Q \ll \frac{M}{Nl^3}. \quad (3.3.27)$$

In estimating the remaining terms from the right hand side of (3.3.26) we may assume that $h \asymp H$ and consequently $P \ll HN$. We now take the sum over h outside and concentrate on the inner sum

$$\sum_{q=Q_\kappa}^{2Q_\kappa-1} \left| \sum_{r=P_5}^{P_6} e(-hf(qr-j)) \right|. \quad (3.3.28)$$

By Lemma 3.1.9 we have

$$\begin{aligned} \left| \sum_{r=P_5}^{P_6} e(-hf(qr-j)) \right| &\le \left(1 + 2\pi \int_{P_5}^{P_6} \left| \frac{\partial}{\partial r}(hf(qr-j) - hf(qr)) \right| dr \right) \\ &\quad \times \left| \sup_{P_5 \le x \le P_6} \sum_{r=x}^{P_6} e(-hf(qr)) \right| \\ &\le \left(1 + 2\pi \int_{P_5}^{P_6} \left| (hqf'(qr-j) - hqf'(qr)) \right| dr \right) \\ &\quad \times \left| \sum_{r=P_7}^{P_6} e(-hf(qr)) \right|. \end{aligned} \quad (3.3.29)$$

Taylor's theorem tells us that there exists a real number $y \in (qr - j, qr)$ such that

$$|hqf'(qr-j) - hqf'(qr))| = hqj|f''(y)|,$$

so that the sum in (3.3.28) is

$$\begin{aligned} &\ll \sum_{q=Q_\kappa}^{2Q_\kappa-1} (1 + hqj|f''(y)|P_\kappa)\Big|\sum_{r=P_7}^{P_6} e(-hf(qr))\Big| \\ &\ll \frac{jhP_\kappa Q_\kappa N}{M^2} \sum_{q=Q_\kappa}^{2Q_\kappa-1}\Big|\sum_{r=P_7}^{P_6} e(-hf(qr))\Big| \\ &\ll l^3 \sum_{q=Q_\kappa}^{2Q_\kappa-1}\Big|\sum_{r=P_7}^{P_6} e(-hf(qr))\Big| \qquad (3.3.30)\end{aligned}$$

by (3.3.25). By (2.0.2) we have

$$\sum_{q=Q_\kappa}^{2Q_\kappa-1}\Big|\sum_{r=P_5}^{P_6} e(-hf(qr))\Big| = \sum_{q=Q_\kappa}^{2Q_\kappa-1}\Big|\sum_{r=P_5}^{P_6} e\Big(-hNF\Big(\frac{q}{Q_\kappa}\frac{r}{P_\kappa}\Big)\Big)\Big|, \qquad (3.3.31)$$

to which we apply Lemma 3.1.5 with P_κ in place of M, Q_κ in place of I and $J, T \asymp HN$ and $F(x,y) = F(x(1+y))$, with y in the range $0 \le y \le 1$. The results are simpler if condition (3.1.10) in Lemma 3.1.5 does not hold. In order to negate this condition we can suppose

$$\frac{\max_h (hN)^{8/57}}{Q_\kappa^{11/19}} \ll 1,$$

which would give

$$Q_\kappa \gg \max_h (hN)^{8/33} \gg (HN)^{8/33} \asymp \Big(\frac{Ml^3}{j}\Big)^{8/33}, \qquad (3.3.32)$$

where we have taken H as large as (3.3.25) will allow. Since $Q \asymp x$, the bounds (3.2.20) and (3.3.32) give

$$\begin{aligned} \frac{j^{3/2}N}{\sqrt{Ml^3}} &\gg \Big(\frac{Ml^3}{j}\Big)^{8/33}, \\ j^{115}N^{66} &\gg M^{49}l^{246}. \qquad (3.3.33)\end{aligned}$$

We consider case 1 and case 2 of Lemma 3.1.5 which depends on the size of P_κ relative to T.

Case 1. For $P_\kappa^2 \ll T$ we have

$$\sum_{q=Q_\kappa}^{2Q_\kappa-1}\left|\sum_{r=P_5}^{P_6} e(-hf(qr))\right|^5 \ll Q_\kappa P_\kappa^{5/2} T^{5/6+\epsilon} E_1^{3/8}\left(1+\frac{Q_\kappa P_\kappa^2}{Q_\kappa T}\right)^{1/2}$$
$$\ll Q_\kappa P_\kappa^{5/2} T^{5/6+\epsilon} E_1^{3/8} \qquad (3.3.34)$$

where

$$E_1 = \frac{1}{T^{8/33}} + \frac{T^{2/3}}{P_\kappa^2} + \frac{1}{Q_\kappa + Q_\kappa T/P_\kappa^2}.$$

Since $P_\kappa^2 \ll T$ and $Q_\kappa \gg T^{8/33}$ we have

$$E_1 \ll \frac{1}{T^{8/33}} + \frac{T^{2/3}}{P_\kappa^2} + \frac{1}{Q_\kappa}$$
$$\ll \frac{1}{T^{8/33}} + \frac{T^{2/3}}{P_\kappa^2}. \qquad (3.3.35)$$

Case 2. For $P_\kappa^2 \gg T$ we have

$$\sum_{q=Q_\kappa}^{2Q_\kappa-1}\left|\sum_{r=P_5}^{P_6} e(-hf(qr))\right|^5 \ll Q_\kappa P_\kappa^{5/2} T^{5/6+\epsilon} E_2^{3/8}\left(1+\frac{Q_\kappa T}{Q_\kappa P_\kappa^2}\right)^{1/2}$$
$$\ll Q_\kappa P_\kappa^{5/2} T^{5/6+\epsilon} E_2^{3/8} \qquad (3.3.36)$$

where

$$E_2 = \frac{1}{T^{8/33}} + \frac{P_\kappa^2}{T^{4/3}} + \frac{1}{Q_\kappa + Q_\kappa P_\kappa^2/T}.$$

As in the case for E_1 we have

$$E_2 \ll \frac{1}{T^{8/33}} + \frac{P_\kappa^2}{T^{4/3}}.$$

So combining the two cases gives

$$\sum_{q=Q_\kappa}^{2Q_\kappa-1}\left|\sum_{r=P_5}^{P_6} e(-hf(qr))\right|^5 \ll Q_\kappa P_\kappa^{5/2} T^{5/6} E^{3/8}, \qquad (3.3.37)$$

where

$$E \asymp \frac{1}{T^{8/33}} + \frac{T^{2/3}}{P_\kappa^2} + \frac{P_\kappa^2}{T^{4/3}}, \tag{3.3.38}$$

and

$$P \ll HN \asymp \frac{Ml^3}{j}. \tag{3.3.39}$$

Using Holder's inequality on (3.3.37) gives

$$\begin{aligned} \sum_{q=Q_\kappa}^{2Q_\kappa-1} \left| \sum_{r=P_5}^{P_6} e(-hf(qr)) \right| &\ll \left(Q_\kappa^4 \sum_{q=Q_\kappa}^{2Q_\kappa-1} \left| \sum_{r=P_5}^{P_6} e(-hf(qr)) \right|^5 \right)^{1/5} \\ &\ll \left(Q_\kappa^5 P_\kappa^{5/2} T^{5/6+\epsilon} E^{3/8} \right)^{1/5}, \end{aligned}$$

and so we have our estimate for our short type I sum

$$S_{72} \ll \frac{jNHl^2}{M} (Q_\kappa^5 P_\kappa^{5/2} T^{5/6+\epsilon} E^{3/8})^{1/5}.$$

We require this estimate to be less than the error term in (2.0.13), which gives the following condition

$$Q_\kappa^5 P_\kappa^{5/2} T^{5/6+\epsilon} E^{3/8} \ll \left(\frac{M}{Hl^3} \right)^5.$$

We rearrange this inequality to get

$$T^{5/6+\epsilon} E^{3/8} \ll \frac{M^5}{H^5 Q_\kappa^5 P_\kappa^{5/2} l^{15}} \asymp \frac{P_\kappa^{5/2}}{H^5 l^{15}}$$

which gives us our lower bound on P_κ

$$P_\kappa \gg T^{1/3+\epsilon} l^6 H^2 E^{3/20} \asymp T^{1/3+\epsilon} l^6 H^2 \left(\frac{1}{T^{8/33}} + \frac{T^{2/3}}{P_\kappa^2} + \frac{P_\kappa^2}{T^{4/3}} \right)^{3/20}$$

by (3.3.38). Substitution of the expressions for H and T gives

$$\begin{aligned} P_\kappa \gg & \left(\frac{M}{jN}\right)^2 \left(\frac{M^{1+\epsilon}}{j}\right)^{49/165} + \left(\frac{M}{jN}\right)^{20/13} \left(\frac{M^{1+\epsilon}}{j}\right)^{1/3} \\ & + \left(\frac{M}{jN}\right)^{20/7} \left(\frac{M^{1+\epsilon}}{j}\right)^{4/21}. \end{aligned} \tag{3.3.40}$$

3.4 Type II Exponential Sums

Type II sums have the form

$$\sum_{u<s\leq z} \alpha(s) \sum_{r>u} \beta(r)\sigma(rs). \tag{3.4.41}$$

As with the short type I sum, we divide the sums into blocks of the form $2^kQ \leq q \leq 2^{k+1}Q - 1$, where Q is a power of two. Thus the range for r again is as before with $2^kP \leq r \leq 2^{k+1}P - 1$ where P is a power of two also. Let α and β satisfy

$$\begin{aligned}
\sum_{q=Q}^{2Q-1} |\alpha(q)|^2 &\leq \alpha^2 Q, \\
\sum_{r=P}^{2P-1} |\beta(r)|^2 &\leq \beta^2 P.
\end{aligned}$$

In S_3 we have $\alpha(q) = \lambda(q)$ and $\beta(r) = 1$. So in this case we have

$$\begin{aligned}
\sum_{q=Q}^{2Q-1} |\alpha(q)|^2 &= \sum_{q=Q}^{2Q-1} |\sum_{m\leq u} \sum_{\substack{d\leq u \\ q=dm}} \mu(d)\Lambda(m)|^2 \\
&\leq \sum_{q=Q}^{2Q-1} \Big(\sum_{m|q} \Lambda(m)\Big)^2 \\
&\leq l^2 Q,
\end{aligned}$$

with $\alpha = l$ and $\beta = 1$. In S_{41} we have $\alpha(q) = \Lambda(q)$ and $\beta(r) = \kappa(r)$. This gives

$$\begin{aligned}
\sum_{q=Q}^{2Q-1} |\alpha(q)|^2 &= \sum_{q=Q}^{2Q-1} |\Lambda(q)|^2 \\
&\leq l^2 Q,
\end{aligned}$$

with $\alpha = l$, and

$$\begin{aligned}
\sum_{r=P}^{2P-1} |\beta(r)|^2 &= \sum_{r=P}^{2P-1} \Big|\sum_{d\leq u} \sum_{\substack{r \\ dr=n}} \mu(d)\Big|^2 \\
&\leq \sum_{r=P}^{2P-1} d(r)^2 \leq l^3 P,
\end{aligned}$$

with $\beta = l^{3/2}$. In S_{42} we have $\alpha(q) = \kappa(q)$ and $\beta(r) = \lambda(r)$, so that $\alpha = l^{3/2}$ and $\beta = l$. The sum S_5 can be rewritten to read

$$S_5 = \sum_{w \leq d \leq u} \mu(d) \sum_{m>z} \Lambda(m) \sum_{q>z/d} \sigma(dmq).$$

If we remove the condition $dq > z$ from S_5 then we can express it in the form (3.4.41). In order to do this we use Lemma 3.1.6 on the sum over q. Let $K_1 = (z+1/2)/d$ and $K_2 = 2Q-1$. Then

$$\sum_{q=(z+1/2)/d}^{2Q-1} \sigma(dmq) = \frac{1}{2\pi i}\int_C G(s)\frac{(2Q-1)^s - ((z+1/2)/d)^s}{s}ds + O(\log Q),$$

where

$$G(s) = \sum_{q=Q}^{2Q-1} \frac{\sigma(dmq)}{q^s},$$

since $Q \leq z$. Here C is the line segment

$$s = \frac{1}{\log Q} + it.$$

where $t \in [-Q, Q]$. If we take the summation over q outside the integral we have

$$\begin{aligned} \int_C \frac{(2Q-1)^s - (z+1/2)^s}{s}ds &\ll \int_{-Q}^{Q} \frac{|idt|}{|1/\log Q + it|} \\ &\ll \int_0^{1/\log Q} \log Q dt + \int_{1/\log Q}^{Q} \frac{dt}{t} \\ &\ll \log Q. \end{aligned}$$

So we have that the block sums for S_5 are

$$\ll l \max_s \left| \sum_{r=P}^{2P-1} \sum_{w \leq d \leq u} \sum_{\substack{m>z \\ dm=r}} \mu(d)\Lambda(m) \sum_{q=Q}^{2Q-1} \frac{\sigma(qr)}{q^s} \right|$$

$$+ l \max_s \left| \sum_{r=P}^{2P-1} \sum_{w \leq d \leq u} \sum_{\substack{m>z \\ dm=r}} \frac{\mu(d)\Lambda(m)}{d^s} \sum_{q=Q}^{2Q-1} \frac{\sigma(qr)}{q^s} \right|$$

$$+O\Big(l\sum_{w\le d\le u}|\mu(d)|\sum_{\substack{m>z\\ P\le dm\le 2P}}\Lambda(m)\Big)$$

$$\ll\ l\Big|\sum_{q=Q}^{2Q-1}\sum_{r=P}^{2P-1}\alpha(q)\beta(r)\sigma(qr)\Big|$$

where

$$\beta(r)=\sum_{w\le d\le u}\sum_{\substack{m>z\\ dm=r}}\frac{\mu(d)\Lambda(m)}{d^s},$$

and $\alpha(q)=q^{-s}$, which gives $\beta=l$ and $\alpha=1$. Next we apply Lemma 3.1.4 as in the case of our short type I sum. We take z_m and y_m the same as before and

$$w_m=\begin{cases}\sum_{q=Q}^{2Q-1}\alpha(q)\sum_{r=P}^{2P-1}\beta(r) & \text{for } m\in MI,\\ 0 & \text{otherwise.}\end{cases}$$

This gives the following upper bound for the blocks of type II sums

$$\sum_{q=Q}^{2Q-1}\alpha(q)\sum_{r=P}^{2P-1}\beta(r)\sigma(qr)\ll\frac{1}{H}\sum_{q=Q}^{2Q-1}\sum_{r=P}^{2P-1}|\alpha(q)\beta(r)|$$
$$+\sum_{h=1}^{H}\Big|\sum_{r}\frac{w_r(e(hy_r)-e(hz_r))}{2\pi ih}\Big|+\frac{1}{H}\sum_{h=1}^{H}\Big|\sum_{r}w_re(hy_r)\Big|$$
$$+\frac{1}{H}\sum_{h=1}^{H}\Big|\sum_{r}w_re(hz_r)\Big|. \qquad (3.4.42)$$

The second sum on the right hand side in (3.4.42) is the same as the one we met with type I sums, except now the weight is different. We have

$$\sum_{h=1}^{H}\Big|\sum_{r}\frac{w_r(e(hy_r)-e(hz_r))}{2\pi ih}\Big|\ll$$
$$\sum_{h=1}^{H}\int_0^{cjN/M}\Big|\sum_{\substack{q=Q\\ f(qr+j)-f(qr-j)\ge t\\ qr\in MI}}^{2Q-1}\sum_{r=P}^{2P-1}\beta(r)e(-hf(qr-j))\Big|dt. \qquad (3.4.43)$$

The third and fourth terms are

$$\ll\ \frac{1}{H}\sum_{h=1}^{H}\Big|\sum_{q=Q}^{2Q-1}|\alpha(q)|\sum_{\substack{r=P\\ qr\in MI}}^{2P-1}|\beta(r)|e(-hf(qr+j))\Big|+$$

$$\frac{1}{H}\sum_{h=1}^{H}\Bigg|\sum_{q=Q}^{2Q-1}|\alpha(q)|\sum_{\substack{r=P\\ qr\in MI}}^{2P-1}|\beta(r)|e(-hf(qr-j))\Bigg|. \qquad (3.4.44)$$

We want to show that (3.4.42) is $\ll jN/l^3$. The first term on the right hand side of (3.4.42) is

$$\ll \frac{1}{H}\alpha\beta PQ \ll \alpha\beta\frac{M}{H},$$

which is small enough provided

$$H \asymp \frac{\alpha\beta M l^3}{jN}. \qquad (3.4.45)$$

It remains to estimate (3.4.43) since the sums in (3.4.44) are of the same form as (3.4.43). Let $J(t)$ be the subinterval of MI on which $f(qr+j)-f(qr-j) \geq t$. We apply Lemma 3.1.8, so that we may apply the differencing step which is given by Lemma (3.1.7).

$$\Bigg|\sum_{q=Q}^{2Q-1}\alpha(q)\sum_{\substack{r=P\\ qr\in MI}}^{2P-1}\beta(r)e(-hf(qr-j))\Bigg| \leq$$
$$\Big(\sum_{r+P}^{2P-1}|\beta(r)|^2\Big)\Big(\sum_{r=P}^{2P-1}\Bigg|\sum_{\substack{q=Q\\ qr\in J(t)}}^{2Q-1}\alpha(t)e(-hf(qr-j))\Bigg|^2\Big); \qquad (3.4.46)$$

the first factor on the right hand side is $\leq \beta^2 P$ and

$$\Bigg|\sum_{\substack{q=Q\\ qr\in J(t)}}^{2Q-1}\alpha(t)e(-hf(qr-j))\Bigg|^2 \ll l^3\Bigg|\sum_{\substack{q=Q\\ qr\in J(t)}}^{2Q-1}\alpha(t)e(-hf(qr))\Bigg|^2, \qquad (3.4.47)$$

to which we apply Lemma 3.1.7 to get

$$\begin{aligned}\Bigg|\sum_{\substack{q=Q\\ qr\in J(t)}}^{2Q-1}\alpha(t)e(-hf(qr))\Bigg|^2 \leq\ & \frac{Q+2D-1}{D}\Big(\sum_{q=Q}^{2Q-1}|\alpha(q)|^2+2\Re\sum_{d=1}^{D-1}\Big(1-\frac{d}{D}\Big)\\ & \sum_{\substack{q=Q+d\\ qr\pm dr\in J(t)}}^{2Q-d-1}\alpha(q+d)\overline{\alpha(q-d)}\\ & \times e(hf(qr+dr)-hf(qr-dr))\Big). \qquad (3.4.48)\end{aligned}$$

The first term on the right hand side of (3.4.48) is

$$\leq \frac{Q+2D-1}{D}\alpha^2 Q,$$

which contributes

$$\frac{\alpha^2\beta^2P^2Q^2}{D} \ll \frac{\alpha^2\beta^2M^2}{D}$$

to the right hand side of (3.4.46), and

$$\frac{\alpha\beta M}{\sqrt{D}}\frac{NjH}{M} \ll \frac{\alpha^2\beta^2Ml^3}{\sqrt{D}}$$

to the right hand side of (3.4.43), provided $D \ll Q$. Hence the contribution is small enough for

$$D \asymp \left(\frac{M}{Nj}\right)^2 \alpha^4\beta^4 l^{12}. \qquad (3.4.49)$$

Now we wish to estimate the second term on the right hand side of (3.4.48). By Taylor's Theorem we know that there exists $\theta \in (0,1)$ such that

$$h(f(qr+dr) - f(qr-dr)) = 2drhf'(qr+\theta dr),$$

from which we obtain the following

$$\begin{aligned}\frac{\partial^2}{\partial r^2}\Big(h(f(qr+dr) - f(qr-dr))\Big) &= 2dh(2(q+\theta d)f''(qr+\theta dr) \\ &\quad + r(q+\theta d)^2 f^{(3)}(qr+d\theta r) \\ &\asymp \frac{dhQN}{M^2} \asymp \frac{dh}{Q}\frac{N}{P^2}\end{aligned}$$

for $2F''(u) + uF^{(3)}(u) > 0$. If we take the summation over r in (3.4.46) into the sum in (3.4.48) Lemma 3.1.3 gives

$$\begin{aligned}\sum_{r=P}^{2P-1} e(hf(qr+dr) - hf(qr-dr)) &\ll \left(\frac{hdN}{QP^2}\right)^{1/2} P + \left(\frac{QP^2}{hdN}\right)^{1/2} \\ &\ll \left(\frac{HDN}{Q}\right)^{1/2} \qquad (3.4.50)\end{aligned}$$

by (3.4.49) and (3.4.45). So the right hand side of (3.4.48) becomes

$$\frac{Q}{D}\alpha^2 QD\Big(\frac{DHN}{Q}\Big)^{1/2} \asymp \alpha^2 Q^2\Big(\frac{DHN}{Q}\Big)^{1/2},$$

which contributes

$$\alpha^2\beta^2 PQ^2\Big(\frac{DHN}{Q}\Big)^{1/2}$$

to the right hand side of (3.4.46), and so we have that the right hand side of (3.4.43) is

$$\begin{aligned} &\ll \frac{HNj}{M}\sqrt{\alpha^2\beta^2 PQ^2\Big(\frac{DHN}{Q}\Big)^{1/2}} \\ &\asymp \frac{HNj}{M}\sqrt{\frac{\alpha^2\beta^2 M^2}{P}\Big(\frac{DHNP}{M}\Big)^{1/2}} \\ &\asymp \frac{HNj}{M}\frac{\alpha\beta M}{\sqrt{P}}\Big(\frac{DHNP}{M}\Big)^{1/4}. \end{aligned}$$

We require this to be $\ll Nj/l^3$, which is the case for

$$\begin{aligned} P &\gg \alpha^4\beta^4 l^{12}\frac{N}{M}DH^5 \\ &\asymp \alpha^{13}\beta^{13}l^{39}\frac{M^6}{N^6 j^7} \end{aligned} \qquad (3.4.51)$$

by (3.4.49) and (3.4.45). We also need $Q \gg D$ so we require

$$Q \gg \Big(\frac{M}{Nj}\Big)^2(\alpha\beta)^4 l^{12}.$$

We are now ready to prove the theorem.

Chapter 4

Proof of the Theorem

In order to state the Theorem, we say that a function $F(x)$ is a good function if it is a real monotone increasing function, six times differentiable on a closed sub-interval I of $[1,2]$ on which the expressions $F'(x)$, $F''(x)$ and $2F''(x)+xF^{(3)}(x)$,

$$F^{(3)}(x), F^{(4)}(x)$$

$$\begin{vmatrix} F^{(4)}(x) & F^{(3)}(x) \\ 3F^{(3)}(x) & F''(x) \end{vmatrix}, \begin{vmatrix} F^{(4)}(x) & F^{(3)}(x) \\ 3F^{(3)}(x) & 2F''(x) \end{vmatrix}, \begin{vmatrix} F^{(5)}(x) & F^{(4)}(x) \\ 4F^{(4)}(x) & 3F^{(3)}(x) \end{vmatrix}$$

$$\begin{vmatrix} 3F^{(3)}(x)^2 & 0 & -F''(x)^2 \\ F^{(5)}(x) & F^{(4)}(x) & F^{(3)}(x) \\ 4F^{(4)}(x) & 3F^{(3)}(x) & 2F''(x) \end{vmatrix}$$

are nonzero.

Theorem 4.0.1 *Let M be a large integer and N be a large real number. Let $x = g(y)$ be a monotone increasing function, whose inverse function can be written as*

$$y = g^{-1} = NF\Big(\frac{x}{M}\Big),$$

where $F(x)$ is a good function. Let $j \geq 1$ be a real number and let

$$\eta = \frac{\log j}{\log N}, \quad \alpha = \frac{\log M}{\log N}.$$

Then for any $\epsilon > 0$, and

$$\theta_1 + \epsilon \leq \alpha \leq \theta_2 - \epsilon,$$

where θ_1 and θ_2 are computed in terms of η by

$$\theta_1 = \frac{22\eta + 18}{17},$$

$$\theta_2 = \begin{cases} \frac{21\eta+17}{16} & \text{for } 0 \leq \eta \leq \frac{1477}{1895}, \\ \frac{3844\eta+3300}{3019} & \text{for } \frac{1477}{1895} \leq \eta < \frac{1758}{1070} \end{cases}$$

there is a constant B depending on $F(x)$ and ϵ but not M or N, such that the inequality

$$g(n) - j \leq p \leq g(n) + j$$

has at least

$$\frac{BjN}{\log N}$$

solutions in integers n and primes p in the interval MI.

The lower bound condition involving θ_1 looks inappropriate, but it enables us to keep the parameters within a range where we can apply the same combinatoric analytic lemmas, avoiding splitting cases according to the size of the parameters; we have considered what seems to be the most interesting range. For this reason the result is weaker as j tends to one than the result Theorem 1.0.1, where the dominant range is one adjacent to the range that we consider. Our range enables us to reach higher exponents α. It is of interest to note that for $\theta_2 \leq \theta$ Theorem 4.0.1 follows directly from Theorem

1.0.1, however θ_2 from Theorem 4.0.1 exceeds $\theta = 3300/3019$ from Theorem 1.0.1 for $\eta > 211/9057 = 0.0233...$

Proof. For our type II sums we have $P \asymp M/z$, so we need to choose z so that

$$\frac{M}{z} \gg \frac{M^{6+\epsilon}}{N^6 j^7}.$$

by (3.4.51), this gives

$$z \asymp \frac{N^6 j^7}{M^{5+\epsilon}}. \tag{4.0.1}$$

For our type I sums we have $P \asymp z^3/M$, which is consistent with (??) and so by (3.3.40) and (4.0.1) we need

$$\frac{z^3}{M} \gg \left(\frac{M}{jN}\right)^2 \left(\frac{M^{1+\epsilon}}{j}\right)^{49/165},$$

from which it follows that

$$j \gg \left(\frac{M^{3019+\epsilon}}{N^{3300}}\right)^{1/3844}. \tag{4.0.2}$$

Our choice of P implies the following size of Q

$$Q \asymp \frac{M^{17}}{N^{18} j^{21}},$$

so condition (3.3.27) implies

$$j \gg \left(\frac{M^{16+\epsilon}}{N^{17}}\right)^{1/21}. \tag{4.0.3}$$

By (3.3.39) and (4.0.1) we have

$$j \ll \left(\frac{M^{17+\epsilon}}{N^{18}}\right)^{1/22},$$

which, together with 4.0.2 and (4.0.3), gives

$$\max\left(\left(\frac{M^{3019+\epsilon}}{N^{3300}}\right)^{1/3844}, \left(\frac{M^{16+\epsilon}}{N^{17}}\right)^{1/21}\right) \ll j \ll \left(\frac{M^{17+\epsilon}}{N^{18}}\right)^{1/22}. \tag{4.0.4}$$

The upper bound on j in (4.0.4) is a stronger condition than that in (2.0.11).

If we now let $j \asymp N^\eta$ for some $\eta \geq 0$ we obtain

$$N^{(22\eta+18)/17+\epsilon} \ll M \ll \min_\eta(N^{(3844\eta+3300)/3019-\epsilon}, N^{(21\eta+17)/16-\epsilon}).$$

which gives the appropriate values of θ_1 and θ_2.

So far we have proved the following

$$\sum_{m\in MI}{}'\Lambda(m) = 2j(f(M_2) - f(M_2)) + O\Big(\frac{jN}{\log M}\Big)$$

by (2.0.13). We also have that

$$\sum_{m\in MI}{}'\Lambda(m) = \sum_{p\in MI}{}'\log p + O\Big(\sum_p \sum_{\substack{r\geq 2\\ p^r\leq 2m}} \log p\Big),$$

where

$$\begin{aligned} \sum_p \sum_{\substack{r\geq 2\\ p^r\leq 2m}} \log p &\leq \sum_{p\leq\sqrt{2M}} \left\lceil\frac{\log 2M}{\log p}\right\rceil \log p \\ &\leq \log 2M \sum_{p\leq\sqrt{2M}} 1 \\ &\ll \log M \frac{\sqrt{2M}}{\log M} \\ &\ll \sqrt{M}. \end{aligned}$$

So we have

$$\sum_{p\in MI}{}'\log p = 2j(f(M_2) - f(M_1)) + \Big(\frac{jN}{\log M}\Big). \tag{4.0.5}$$

The left hand side of (4.0.5) is

$$\leq \log(2M) \sum_{p\in MI}{}'1.$$

Hence

$$\sum_{p\in MI}{}'1 \geq \frac{2j}{\log 2M}(f(M_2) - f(M_1)) + O\Big(\frac{jN}{\log^2 M}\Big),$$

and

$$\frac{2j}{\log 2M}(f(M_2) - f(M_1)) \gg \frac{jN}{\log M},$$

which proves the result.

Part II

Primes in Short Segments of Arithmetic Progressions with a Large Modulus

Chapter 5

An Explicit Zero-Density Theorem

Given an arithmetic progression $l \bmod q$, Dirichlet proved that the prime numbers are uniformly distributed among the $\phi(q)$ residue classes for which $(l, q) = 1$. The next natural question to ask would be how far do you have to go into the arithmetic progression to find the first prime? Let

$$p(q, l) = \min\{p : p \equiv l \bmod q\}$$

then on the Riemann-Hypothesis for Dirichlet L-functions we have

$$p(q, l) \ll (\phi(q) \log q)^2.$$

However it is conjectured that

$$p(q, l) \ll q^{1+\epsilon}.$$

In 1944 Yu. V. Linnik proved the following theorem.

Theorem 5.0.2 *There exists absolute constants $c \geq 1$ and $L \geq 2$ such that*

$$p(q, l) \leq cq^L.$$

Linnik's original proof of his theorem was effective but complicated enough to dissuade him from calculating admissable values for c and L. Many authors since have proved the theorem with explicit values for L, most recently Heath-Brown [10] has proved Theorem 5.0.2 with $L = 5.5$.

All proofs of Linnik's Theorem rely on three main principles, based on the zeros of Dirichlet L-functions. These are the zero-free region, originally due to Landau, the log-free zero density estimate and the Deuring-Heilbronn Phenomenon, both due to Linnik. We will consider the zero free region and the Deurring-Heilbronn Phenomenon in the next section but for now we concentrate on the main ingredient of Linnik's Theorem, the log-free zero density estimate.

For $1/2 \leq \alpha \leq 1$ and $T \geq 1$ we denote by $N(\alpha, T, \chi)$ the number of zeros of $L(s, \chi)$ counted with multiplicity in the rectangle

$$R(\alpha, T) = \{s = \sigma + \imath t : \alpha < \sigma \leq 1, |t| \leq T\}$$

Then Linnik's log-free zero density Theorem is as follows

Theorem 5.0.3 *There are positive constants c_1 and c_2 (effectively computable) such that for any $1/2 \leq \alpha \leq 1$ and $T \geq 1$*

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \leq c(qT)^{c_2(1-\alpha)}.$$

In order to prove Linnik's Theorem 5.0.2 one only needs prove Theorem 5.0.3 for small rectangles, and so all explicit versions of this result are given with $T < q^{\epsilon}$. We require an explicit version with $T \geq q$ not found in the literature. Fogels [4] proved the result with $T \geq q$ but not explicitly, presumably due to the fact that the method of proof is a mixture of Linnik's original method and Turan's method and so is quite complicated. We use the method of Jutila to prove Fogels' version of Linnik's Theorem explicitly. Thus the aim of the current section is to calculate constants c_1 and c_2 such that

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \leq c_1(qT)^{c_2(1-\alpha)} \tag{5.0.1}$$

for $T \geq q$.
We will need the following Lemmas.

5.1 Lemmas

In the Lemmas that follow I have given proofs if the implied order of magnitude constants are needed, if they are not the proofs are omitted.
Let $s = \sigma + \imath t$ be a complex number, $D = q(|t| + e)$, and ϵ is a small positive constant which depends on D in the following way

$$\lim_{D \to \infty} \epsilon = 0.$$

The true value of each ϵ may change in what follows.

5.2 The Gamma Function

If we define the factorial of an integer n by the function

$$\Gamma(n+1) = n!$$

then the function $n!$ can be extended into the real and complex numbers which gives Euler's integral form

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $\Re(s) > 0$. The connection between the Gamma function and the zeta functions having long since been established, we present a few lemmas on the Gamma function required to deduce results about Dirichlet L functions.

Lemma 5.2.1 *As with the factorial function for the integers, the Gamma function has the following property*

$$\Gamma(s+1) = s\Gamma(s)$$

for all complex s.

Lemma 5.2.2 *For $0 \leq \sigma \leq 2$, we have*

$$\left|\Gamma(s) - \frac{1}{s}\right| \leq e + \frac{2}{e}. \tag{5.2.2}$$

Proof. For $\sigma \geq 0$ we have

$$\begin{aligned}
|\Gamma(s)| &= \int_0^\infty e^{-t} t^{s-1} dt \\
&= \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty e^{-t} t^{s-1} dt \\
&= \int_0^1 \sum_{k=0}^\infty \frac{(-1)^k}{k!} t^{s-1+k} dt + \int_1^\infty e^{-t} t^{s-1} dt \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{s+k} + \int_1^\infty e^{-t} t^{s-1} dt \\
&= \frac{1}{s} + \sum_{k=1}^\infty \frac{(-1)^k}{k!} \frac{1}{s+k} + \int_1^\infty e^{-t} t^{s-1} dt,
\end{aligned}$$

so that

$$\begin{aligned}
\left|\Gamma - \frac{1}{s}\right| &\leq \sum_{k=1}^\infty \frac{1}{k!k} + \int_1^\infty e^{-t} t^{\sigma-1} dt \\
&\leq \sum_{k=0}^\infty \frac{1}{k!} + \int_1^\infty e^{-t} t dt \\
&\leq e + \frac{2}{e},
\end{aligned}$$

and the lemma follows.

Lemma 5.2.3 *For $-1 \leq \sigma < 1$ and $|t| \geq 1$ we have*

$$\sqrt{2\pi} e^{-23/6} |t|^{\sigma-1/2} \exp\Big(-\frac{\pi}{2}|t|\Big) \leq |\Gamma(s)| \leq \sqrt{2\pi} e^{11/16} |t|^{\sigma-1/2} \exp\Big(-\frac{\pi}{2}|t|\Big)$$

and for $1/2 \leq \sigma \leq 2$ and $|t| \leq 1$ we have

$$|\Gamma(s)| \leq \sqrt{2\pi} 5^{3/4} \exp\Big(\frac{\pi}{2} - \frac{1}{2} + 2\Big) \leq 182$$

Proof. We begin the proof with Stirling's series, see [22].

$$\log \Gamma(z) = \frac{1}{2}\log(2\pi) + \Big(z - \frac{1}{2}\Big)\log z - z + \int_0^\infty \frac{[u] - u + 1/2}{u+z} du.$$

Let

$$\sigma(u) = \int_0^u [x] - x + 1/2 dx,$$

then $\sigma(u)$ is bounded and $\sigma(u) = \sigma(u+1)$ for integers u. We have

$$\begin{aligned} \Big|\int_0^\infty \frac{[u] - u + 1/2}{u+z} du\Big| &= \Big|\Big[\frac{\sigma(u)}{u+z}\Big]_0^\infty + \int_0^\infty \frac{\sigma}{(u+z)^2} du\Big| \\ &= \Big|\int_0^\infty \frac{\sigma}{(u+z)^2} du\Big| \\ &\leq \frac{1}{8}\int_0^\infty \frac{du}{(u+\sigma)^2 + t^2}. \end{aligned} \tag{5.2.3}$$

For $\sigma > |t|$ (5.2.3) is

$$\leq \frac{1}{8}\int_0^\infty \frac{du}{(u+\sigma)^2} = \frac{1}{8\sigma} \leq \frac{1}{|z|},$$

and for $|t| > \sigma$ (5.2.3) is

$$\leq \frac{1}{8}\int_{-\infty}^\infty \frac{dv}{v^2 + t^2} = \frac{\pi}{8|t|} \leq \frac{\pi}{8|z|/\sqrt{2}} \leq \frac{1}{|z|}.$$

We have obtained the following equation, where the implied order of magnitude constant can be taken as 1.

$$\Re(\log \Gamma(z)) = \Re\Big(\Big(\sigma - \frac{1}{2} + \imath t\Big)\Big(\log\sqrt{\sigma^2 + t^2} + \imath\theta\Big) - \sigma + \frac{1}{2}\log 2\pi + O\Big(\frac{1}{|z|}\Big)\Big).$$

For $t \geq 1$ we have

$$\begin{aligned} \theta &= \frac{\pi}{2} - \arctan\left(\frac{\sigma}{t}\right) \\ &= \frac{\pi}{2} - \left(\frac{\sigma}{t} + \frac{\phi_1}{|t|^3}\right) \end{aligned}$$

where, by the Maclaurin series expansion for $\arctan x$, $|\phi_1| \leq 1/3$. For $t \leq -1$ we have

$$\begin{aligned} \theta &= \arctan\left(\frac{\sigma}{t}\right) - \frac{\pi}{2} \\ &= \frac{\sigma}{t} + \frac{\phi_1}{|t|^3} - \frac{\pi}{2}. \end{aligned}$$

We now wish to estimate the logarithmic term in (5.2.4) as follows.

$$\begin{aligned} \sqrt{\sigma^2 + t^2} &= |t|\sqrt{\sigma^2/t^2 + 1} \\ &= |t|\left(1 + \frac{\phi_2}{|t|^2}\right), \end{aligned}$$

where $|\phi_2| \leq 1$. By the Maclaurin series for $\log x$ we have

$$\begin{aligned} \log\sqrt{\sigma^2 + t^2} &= \log\left(|t|\left(1 + \frac{\phi_2}{t^2}\right)\right) \\ &= \log|t| + \log\left(1 + \frac{\phi_2}{t^2}\right) \\ &= \log|t| + \frac{\phi_2^2}{t^2}, \end{aligned}$$

from which we obtain the relevant upper and lower bounds for $\Gamma(z)$

$$\begin{aligned} |\Gamma(z)| &\leq \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi/2|t|}\exp\left(\left(\sigma - \frac{1}{2}\right)\frac{1}{t^2} + \frac{1}{3t^2} + \frac{1}{r}\right) \\ &\leq \sqrt{2\pi}e^{11/6}|t|^{\sigma-1/2}e^{-\pi/2|t|}, \end{aligned}$$

and

$$\begin{aligned} |\Gamma(z)| &\geq \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi/2|t|}\exp\left(\left(-\frac{1}{2}0 - \frac{1}{3} - 2 - 1\right)\right. \\ &\geq \sqrt{2\pi}e^{-23/6}|t|^{\sigma-1/2}e^{-\pi/2|t|}, \end{aligned}$$

which proves the first part of the lemma. Now suppose that $1/2 \leq \sigma \leq 2$ and $|t| \leq 1$, from our treatment of the previous case we have

$$\begin{aligned}\Re(\log\Gamma(z)) &\leq \left(\sigma - \frac{1}{2}\right)\log\sqrt{\sigma^2+t^2} - t\arctan\left(\frac{t}{\sigma}\right) - \sigma + \frac{1}{2}\log 2\pi + \frac{1}{r} \\ &\leq \frac{3}{2}\log\sqrt{5} + \frac{\pi}{2} - \frac{1}{2} + \frac{1}{2}\log 2\pi + 2.\end{aligned}$$

Taking exponentials gives

$$|\Gamma(z)| \leq \sqrt{2\pi}5^{3/4}\exp\left(\frac{\pi}{2} - \frac{1}{2} + 2\right) \leq 182.$$

5.3 Dirichlet L-Functions

In order to prove Linnik's log-free zero density theorem we need to prove certain Lemmas on Dirichlet L-functions.

Lemma 5.3.1 *For χ a nontrivial character modulo q, $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_2 > \epsilon_1$ we have*

$$|L(s,\chi)| \leq \begin{cases} (e+\epsilon_1)\log D & \text{if } \sigma \geq 1 - \frac{1}{\log D}, \\ (e+\epsilon_2)D^{1-\sigma}\log D & \text{if } 1+\delta \geq \sigma \geq \frac{1}{\log D}, \end{cases}$$

where

$$\delta = \frac{\log((e+\epsilon_2)/(e+\epsilon_1))}{\log D}.$$

Proof. We treat the two ranges separately, so firstly suppose that

$$\sigma \geq 1 - \frac{1}{\log D},$$

then we estimate the L-function by cutting off the series representation of $L(s,\chi)$ at D and bounding the tail of the series. We have

$$\left|\sum_{d\leq D}\frac{\chi(d)}{d^s}\right| \leq \sum_{d\leq D}\frac{1}{d^\sigma} \leq \sum_{d\leq D}\frac{d^{1/\log D}}{d}$$

$$\leq \quad D^{1/\log D} \sum_{d \leq D} \frac{1}{d} \leq e(1 + \log D) \qquad (5.3.4)$$

by the integral test. The tail of the series is estimated in the following way

$$\begin{aligned}
\left| \sum_{d=D+1}^{\infty} \frac{\chi(d)}{d^s} \right| &\leq |s| \int_D^{\infty} \left| \sum_{D<n\leq x} \chi(n) \right| \left| \frac{dx}{x^{s+1}} \right| \\
&\leq (|t| + \sigma) q \int_D^{\infty} \frac{1}{x^{\sigma+1}} dx \\
&\leq \frac{(|t| + \sigma) q}{\sigma D^{\sigma}} \\
&\leq \frac{1}{\sigma} D^{1-\sigma} \leq \frac{e}{\sigma} \\
&\leq e + \epsilon \qquad (5.3.5)
\end{aligned}$$

for any $\epsilon > 0$, and D large enough. After putting (5.3.4) and (5.3.5) together we find that

$$|L(s, \chi)| \leq e + \epsilon + e(1 + \log D) \leq (e + \epsilon_1) \log D$$

for

$$\sigma \geq 1 - \frac{1}{\log D}.$$

Now we turn our attention to the second part of the range for σ. Suppose

$$\frac{1}{\log D} \leq \sigma \leq 1 - \frac{1}{\log D}. \qquad (5.3.6)$$

Then we have the following

$$\begin{aligned}
\left| \sum_{d \leq D} \frac{\chi(d)}{d^s} \right| &\leq \sum_{d \leq D} \frac{1}{d^{\sigma}} \\
&\leq 1 + \int_1^D x^{-\sigma} dx \\
&\leq D^{1-\sigma} \log D,
\end{aligned}$$

and for the tail of the series we have

$$\begin{aligned}
\left| \sum_{d=D+1}^{\infty} \frac{\chi(d)}{d^s} \right| &\leq |s| q \int_d^{\infty} \frac{dx}{x^{\sigma+1}} \\
&\leq (|t| + \sigma) q \int_D^{\infty} x^{-(\sigma+1)} dx \\
&\leq D^{1-\sigma} \log D.
\end{aligned}$$

So when (5.3.6) holds we have

$$|L(s,\chi)| \leq 2D^{1-\sigma}\log D.$$

However for $\sigma > 1 - 1/\log D$ we have

$$(e+\epsilon_1)\log D \leq (e+\epsilon_2)D^{1-\sigma}\log D$$

for

$$\sigma \leq 1 - \frac{\log((e+\epsilon_1)/(e+\epsilon_2))}{\log D},$$

and since $\epsilon_1 < \epsilon_2$ the condition becomes $\sigma \leq 1+\delta$ for some $\delta > 0$, and so

$$|L(s,\chi)| \leq (e+\epsilon_2)D^{1-\sigma}\log D \tag{5.3.7}$$

for

$$\frac{1}{\log D} \leq \sigma \leq 1+\delta,$$

which establishes the Lemma.

Lemma 5.3.2 *For* $0 \leq \sigma \leq 1/2$ *we have*

$$|L(s,\chi)| \ll \sqrt{D}\log D.$$

Proof. We begin the proof with the functional equation for $L(s,\chi)$, where χ is a proper character and $1/2 \leq \sigma \leq 1$, see [11]. Taking moduli, we get

$$|L(1-s,\overline{\chi})| = \left(\frac{q}{\pi}\right)^{\sigma-1/2}\left|\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))}\right||L(s,\chi)|, \tag{5.3.8}$$

where $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. We suppose that $1/2 \leq \sigma \leq 1$ and then we bound the function $L(1-s,\overline{\chi})$ whose complex argument has real part in the desired interval. First we suppose that $|t| \geq 2$

then $\Im(1/2(s+a)) \geq 1$ and $\Im(1/2(1-s+a)) \geq 1$, by Lemma (5.2.3) we have

$$\begin{aligned}\left|\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))}\right| &\leq e^{17/3}\left|\frac{t}{2}\right|^{\sigma-1/2} \\ &\ll |t|^{\sigma-1/2}.\end{aligned}$$

By Lemma 5.3.1 we have

$$\begin{aligned}|L(1-s,\overline{\chi})| &\ll (q|t|)^{\sigma-1/2}|L(s,\chi)| \\ &\ll (q|t|)^{\sigma-1/2}D^{1-\sigma}\log D \\ &\ll \sqrt{D}\log D. \end{aligned} \tag{5.3.9}$$

For $|t| < 2$ we use the following standard result on the Gamma function which holds for $a \in \{0,1\}$

$$\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))} = \frac{2^{1-s}}{\sqrt{\pi}}\Gamma(s)\sin\left(\frac{\pi}{2}(1-s+a)\right). \tag{5.3.10}$$

Thus (5.3.10) and Lemma 5.2.2 gives

$$\begin{aligned}\left|\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))}\right| &\leq \frac{2^{1-\sigma}}{\sqrt{\pi}}\left(\left|\frac{1}{s}\right|+e+\frac{2}{e}\right)\left|\sin\left(\frac{\pi}{2}(1-s+a)\right)\right| \\ &\leq 5\sinh(\frac{\pi}{2}(1-s+a)) \\ &\leq \frac{5}{2}\exp\left(\frac{\pi}{2}(1-s+a)\right) \\ &\leq \frac{5}{2}\exp\left(\frac{5\pi}{4}\right),\end{aligned}$$

from which it follows that (5.3.9) holds for all t and all proper characters $\bmod q$. The result can be extended to all non-proper characters since if ψ is a character $\bmod f$ which is induced by $\chi \bmod q$ then we have

$$\begin{aligned}|L(s,\psi)| &= \left|\prod_{p|f}\left(1-\frac{\chi(p)}{p^s}\right)\right||L(s,\chi)| \\ &\ll \left|\prod_{p|f}\left(1-\frac{\chi(p)}{p^s}\right)\right|\sqrt{q(|t|+e)}\log D\end{aligned}$$

$$\ll \sqrt{\frac{f}{q}}\sqrt{q(|t|+e)}\log D$$
$$\ll \sqrt{D}\log D.$$

The lemma follows.

Lemma 5.3.3 *(Jensen's formula). Let C be the circle $|z| = R$. Assume $f(z)$ is regular and nonzero on C, and that C' is the open disc of radius R, then*

$$\sum_{\substack{\rho\in C'\\ f(\rho)=0}} \log\left(\frac{R}{|\rho|}\right) = \frac{1}{2\pi}\int_C \log\left|\frac{f(Re^{i\theta})}{f(0)}\right| d\theta.$$

Lemma 5.3.4 *The number of zeros of $L(s,\chi)$ in the box $0 \le \beta \le 1$, $t - 1/2 \le \gamma \le t + 1/2$ is*

$$\le \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})}\log D \le 20\log D.$$

Proof. By the functional equation for Dirichlet L-functions we know that the number of zeros in the box $B = \{\beta + i\gamma : 0 \le \beta \le 1, t-1/2 \le \gamma \le t+1/2\}$ is less than twice the number of zeros in the box $B' = \{\beta + i\gamma : 1/2 \le \beta \le 1, t - 1/2 \le \gamma \le t + 1/2\}$. Consider a circle with center $2 + it$ and radius $7/4$, our box containing at least half of the zeros is contained within this circle, so it will suffice to count the number of zeros in our circle. Now let $f(s) = L(\alpha+s,\chi)$ in Jensen's formula with $\alpha = 2+it$ and $s = 7e^{i\theta}/4$, which gives

$$\sum_{\substack{\rho\in C\\ L(\rho,\chi)=0}} \log\frac{7}{4|\rho-\alpha|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{L(s+\alpha,\chi)}{L(\alpha,\chi)}\right| d\theta.$$

The log weights of the zeros in the sum are all positive and so

$$\sum_{\substack{\rho\in B'\\ L(\rho,\chi)=0}} \log\frac{7}{4|\rho-\alpha|} \le \sum_{\substack{\rho\in C\\ L(\rho,\chi)=0}} \log\frac{7}{4|\rho-\alpha|}.$$

For $\rho \in B'$ we have

$$|\rho - \alpha| \leq \frac{\sqrt{10}}{2},$$

which gives

$$\begin{aligned}
\sum_{\substack{\rho \in B' \\ L(\rho,\chi)=0}} 1 &\leq \frac{1}{2\pi \log(7/2\sqrt{10})} \int_0^{2\pi} \log \left| \frac{L(s+\alpha,\chi)}{L(\alpha,\chi)} \right| d\theta \\
&\leq \frac{1}{2\pi \log(7/2\sqrt{10})} \int_0^{2\pi} \log \left| 2L(s+\alpha,\chi) \right| d\theta,
\end{aligned}$$

since $|L(2+it,\chi)| > 1/2$, see Prachar [21]. The term $s+\alpha$ is on the boundary of our circle and so $1/4 \leq \Re(s+\alpha) \leq 15/4$ and so

$$\begin{aligned}
\sum_{\substack{\rho \in B' \\ L(\rho,\chi)=0}} 1 &\leq \frac{1}{\log(7/2\sqrt{10})} \max_{0.25 \leq \sigma \leq 3.75} (\log |2L(s,\chi)|) \\
&\leq \frac{1}{\log(7/2\sqrt{10})} \log(2 \max(c\sqrt{D} \log D, (e+\epsilon) \log D, \\
&\qquad (e+\epsilon) D^{3/4} \log D)) \\
&\leq \frac{1}{\log(7/2\sqrt{10})} \log(2(e+\epsilon) D \log D)
\end{aligned}$$

by Lemma 5.3.1 and 5.3.2 for some constant $c > 0$ and D large enough. Hence

$$\sum_{\substack{\rho \in B' \\ L(\rho,\chi)=0}} 1 \leq \frac{1+\epsilon}{\log(7/2\sqrt{10})} \log D$$

for some $\epsilon > 0$, and therefore

$$\sum_{\substack{\rho \in B \\ L(\rho,\chi)=0}} 1 \leq \frac{2(1+\epsilon)}{\log(7/2\sqrt{10})} \log D,$$

which proves the lemma.

Lemma 5.3.5 *Let $s = 1 + \delta + i\tau$, where $\delta \in (0,1)$ and $\tau \in R$ then we have the following inequality*

$$\left|\frac{L'(s,\chi)}{L(s,\chi)} - \sum_{\substack{\rho \\ |s-\rho|\leq 1}} \frac{1}{s-\rho}\right| \leq \frac{12(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(|\tau| + 2 + e))$$

for $\epsilon > 0$.

Proof. We begin with the identity

$$\frac{L'(s,\chi)}{L(s,\chi)} = C(\chi) + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{s-\rho},$$

where χ is a proper character and $C(\chi)$ is some constant depending on χ which we do not need to know anything about, since we want to work with the following difference

$$\begin{aligned} \frac{L'(s,\chi)}{L(s,\chi)} - \frac{L'(2+it,\chi)}{L(2+it,\chi)} &= \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) \\ &= \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{2-\sigma}{(s-\rho)(2+it-\rho)}. \qquad (5.3.11) \end{aligned}$$

Let $s = 1 + \delta + i\tau$, where $\delta \in (0,1)$ and we suppose that $\tau \geq 0$, where the proof for $\tau \leq 0$ follows by symmetry. The contribution from zeros ρ with $|s - \rho| \leq 1$ on the left hand side of (5.3.11) dominate the sum, so we can estimate the contribution from zeros with $|s - \rho| > 1$ and put them into an error term. Firstly we estimate the contribution from zeros of $L(s, \chi)$ where $s = 0, -2, -4, \ldots$ or $s = -1, -3, -5, \ldots$ depending on whether χ is odd or even. Let

$$E(\rho) = \frac{2-\sigma}{(s-\rho)(2+it-\rho)}.$$

Then

$$|E(\rho)| \leq \frac{1}{(1+\delta+n)(2+n)}$$

$$\leq \frac{1}{(n+1)^2}.$$

So the real zeros on the negative real axis contribute

$$\leq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Now we want to estimate the contribution from complex zeros. Firstly consider the zeros $\rho = \beta + i\gamma$ in the region $R = \{\rho = \beta + i\gamma : 0 \leq \beta \leq 1, \tau - 1 \leq \gamma \leq \tau + 1, |\rho - (1 + \delta + i\tau)| > 1\}$, in which there are less than

$$\frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})}(\log(q(|\tau|+e)) + \log(q(|\tau|+1+e)))$$
$$\leq \frac{4(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(|\tau|+1+e))$$

zeros by Lemma 5.3.4. For $\rho \in R$ we have

$$|E(\rho)| = \frac{|2-(1+\delta)|}{|(s-\rho)(2+i\tau-\rho)|} \leq \frac{1}{|2+i\tau-\rho|} < 1,$$

since $|(1+\delta+i\tau)-\rho| > 1$. So the contribution from zeros in R to the right hand side of (5.3.11) is

$$\leq \frac{4(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(\tau+1+e)).$$

We now want to estimate the contribution of zeros $\rho = \beta + i\gamma$ where $\gamma \geq \tau + 1$. We split the sum on the right hand side of (5.3.11) into blocks of the form $\tau + n \leq \gamma \leq \tau + (n+1)$, where $n = 1, 2, \ldots.$ In each block we have

$$\begin{aligned}
|E(\rho)| &\leq \frac{1-\delta}{|(1+\delta+i\tau-\rho)(2+i\tau-\rho)|} \\
&\leq \frac{1}{\sqrt{(1+\delta-\beta)^2+(\tau-\gamma)^2}\sqrt{(2-\beta)^2+(\tau-\gamma)^2}} \\
&\leq \frac{1}{\sqrt{(1+\delta-\beta)^2+(\tau-(\tau+(n+1)))^2}} \\
&\quad \times \frac{1}{\sqrt{(2-\beta)^2+(\tau-(\tau+(n+1)))^2}}
\end{aligned}$$

$$\le \frac{1}{\sqrt{(1+\delta-\beta)^2+(n+1)^2}\sqrt{(2-\beta)^2+(n+1)^2}}$$
$$\le \frac{1}{(n+1)^2},$$

and the number of zeros in each block is

$$\le \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})}\log(q(|\tau|+(n+1)+e)).$$

Hence the contribution to the right hand side of (5.3.11) from zeros with $\gamma \ge \tau+1$ gives

$$\begin{aligned}
&\le \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})}\sum_{n=1}^{\infty}\frac{\log(q(|\tau|+(n+1)+e))}{(n+1)^2} \\
&\le \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})}\Big(\frac{1}{4}\log(q(|\tau|+2+e)) \\
&\quad + \int_1^{\infty}\frac{\log(q(|\tau|+x+1+e))}{(x+1)^2}dx\Big), \qquad (5.3.12)
\end{aligned}$$

where

$$\begin{aligned}
\int_1^{\infty}\frac{\log(q(\tau+x+1+e))}{(x+1)^2}dx &= \frac{1}{2}\log(q(\tau+2+e)) \\
&\quad + \int_1^{\infty}\frac{1}{(x+1)(x+1+\tau+e)}dx \\
&= \frac{1}{2}\log(q(\tau+2+e)) + \\
&\quad \frac{1}{\tau+e}\int_1^{\infty}\Big(\frac{1}{x+1}-\frac{1}{x+1+\tau+e}\Big)dx \\
&= \frac{1}{2}\log(q(\tau+2+e)) + \frac{1}{\tau+e}\int_1^{\tau+e}\frac{dx}{x+1} \\
&\le \frac{1}{2}\log(q(\tau+2+e)) + \frac{\log(1+\tau+e)}{\tau+e} \\
&\le \Big(\frac{1}{2}+\epsilon\Big)\log(q(\tau+2+e)).
\end{aligned}$$

Thus the right hand side of (5.3.12) is

$$\begin{aligned}
&\le \frac{2(1+\epsilon)}{\log(7/2\sqrt{10})}\Big(\frac{1}{4}\log(q(\tau+2+e)) + \Big(\frac{1}{2}+\epsilon\Big)\log(q(\tau+2+e))\Big) \\
&\le \frac{1+\epsilon}{\log(7/2\sqrt{10})}\log(q(\tau+2+e)).
\end{aligned}$$

We now estimate the contribution from the zeros $\rho = \beta + i\gamma$ where $\tau - 1 \geq \gamma \geq 0$. As before we split the sum into blocks $\tau - (n+1) \leq \gamma \leq \tau - n$ where $n = 1, 2, ..., \tau - 1$, then on each block we have

$$|E(\rho)| \leq \frac{1}{n^2}$$

and so these zeros contribute

$$\begin{aligned}
&\leq \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})} \sum_{n=1}^{\tau-1} \frac{\log(q(\tau - n + e))}{n^2} \\
&\leq \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(\tau - 1 + e)) \sum_{n=1}^{\tau-1} \frac{1}{n^2} \\
&\leq \frac{4(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(\tau - 1 + e))
\end{aligned}$$

to the right hand side of (5.3.11). It remains to estimate the contribution from zeros below the real axis. So for each block $-(n+1) \leq \gamma \leq -n$, where we have

$$|E(\rho)| \leq \frac{1}{(n+\tau)^2}$$

and

$$\leq \frac{2(1+\epsilon)}{\log(7/2\sqrt{10})} \log(q(n+1+e))$$

zeros in each block, they contribute

$$\begin{aligned}
&\leq \frac{2(1+\epsilon)}{\log(7/2\sqrt{10})} \sum_{n=0}^{\infty} \frac{\log(q(n+1+e))}{(n+\tau)^2} \\
&\leq \frac{2(1+\epsilon)}{\log(7/2\sqrt{10})} \left(\frac{\log(q(e+1))}{\tau^2} + \int_0^\infty \frac{\log(q(x+1+e))}{(x+\tau)^2} dx \right),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^\infty \frac{\log(q(x+1+e))}{(x+\tau)^2} dx &\leq \left[\frac{-\log(q(x+1+e))}{x+\tau} \right]_0^\infty + \\
&\quad \int_0^\infty \frac{1}{(x+\tau)(x+1+e)} dx \\
&\leq \frac{\log(q(1+e))}{\tau} + \int_0^\infty \frac{1}{(x+\max(\tau, 1+e))^2} dx
\end{aligned}$$

$$\leq \frac{\log(q(1+e))}{\tau} + \int_0^\infty \frac{1}{(x+1+e)^2}dx$$
$$\leq \frac{\log(q(1+e))}{\tau} + \frac{1}{e+1}.$$

We have

$$\left|\frac{L'(s,\chi)}{L(s,\chi)} - \frac{L'(2+i\tau,\chi)}{L(2+i\tau,\chi)} - \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |s-\rho|\leq 1}} \left(\frac{1}{s-\rho} - \frac{1}{2+i\tau-\rho}\right)\right| \leq$$
$$\frac{10(1+\epsilon)}{\log(7/2\sqrt{10})} \log(q(\tau+2+e)) \tag{5.3.13}$$

for $s = 1 + \delta + \imath\tau$, and

$$\left|\frac{L'(2+\imath\tau)}{L(2+\imath\tau)}\right| \leq 2\sum_{n=1}^{\infty} \frac{\log n}{n^2} \leq 4$$

since $|L(s,\chi)| \geq 1/2$. Also since $|2+\imath\tau-\rho| \geq 1$ we have

$$\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |s-\rho|\leq 1}} \frac{1}{2+\imath\tau-\rho} \leq \frac{2(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(\tau+1+e))$$

by Lemma 5.3.4. So equation (5.3.13) becomes

$$\left|\frac{L'(s,\chi)}{L(s,\chi)} - \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |s-\rho|\leq 1}}\right| \leq \frac{12(1+\epsilon_3)}{\log(7/2\sqrt{10})}\left(\log(q(\tau+2+e))\right.$$

thus proving the lemma. The next Lemma uses an argument ascribed to Linnik.

Lemma 5.3.6 *For $\delta \in (0, 1/2)$, there are*

$$\leq 4 + \frac{48\delta(1+\epsilon_3)}{\log(7/2\sqrt{10})} \log(q(|t|+2+e))$$

zeros of $L(s,\chi)$ in the region $|1+\imath t-\rho| \leq \delta$.

Proof. By Lemma 5.3.5 we have for $s = 1 + \delta + \imath t$ and χ proper

$$\begin{aligned}\Re\Big(\frac{L'(s,\chi)}{L(s,\chi)}\Big) &\geq \Re\Big(\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |s-\rho|\leq 1}} \frac{1}{s-\rho}\Big) \\ &\quad -\frac{12(1+\epsilon_3)}{\log(7/2\sqrt{10})}\log(q(\tau+2+e)). \qquad (5.3.14)\end{aligned}$$

We consider a subset $|1 + \imath t - \rho| \leq \delta$ of set of zeros in the summation in (5.3.14). Thus we have

$$\Re\Big(\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |s-\rho|\leq 1}} \frac{1}{s-\rho}\Big) \geq \Re\Big(\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |1+\imath t-\rho|\leq \delta}} \frac{1}{s-\rho}\Big)$$

and so zeros ρ in our new circle have

$$\begin{aligned}\Re\Big(\frac{1}{s-\rho}\Big) &= \Re\Big(\frac{\overline{s}-\overline{\rho}}{|s-\rho|^2}\Big) \geq \frac{1+\delta-\Re(\rho)}{(|1+\imath t-\rho|+\delta)^2} \\ &\geq \frac{\delta}{(2\delta)^2} \geq \frac{1}{4\delta}.\end{aligned}$$

Substituting this into equation (5.3.14) we have

$$\begin{aligned}\Re\Big(\frac{L'(s,\chi)}{L(s,\chi)}\Big) &\geq \frac{1}{4\delta}\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |1+\imath t-\rho|\leq 1}} 1 \\ &\quad -\frac{12(1+\epsilon_3)}{\log(7/2\sqrt{10})}\log(q(|t|+2+e)). \qquad (5.3.15)\end{aligned}$$

As for the left hand side of (5.3.15) we have

$$\begin{aligned}\Re\Big(\frac{L'(s,\chi)}{L(s,\chi)}\Big) &\leq \sum_{n\geq 2}\Big|\frac{\chi(n)\Lambda(n)}{n^s}\Big| \\ &\leq \sum_{n\geq 2}\frac{\Lambda(n)}{n^{1+\delta}} \leq \frac{1}{\delta},\end{aligned}$$

which gives the following result

$$\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |1+\imath t-\rho|\leq \delta}} 1 \leq 4 + \frac{48\delta(1+\epsilon_3)}{\log(7/2\sqrt{10})}\log(q(|t|+2+e)).$$

which holds for all characters χ mod$q$.

Lemma 5.3.7 *(Huxley) For $1/2 \leq \alpha \leq 1$, A some large positive constant and $T \geq 1$ we have*

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll (qT)^{12/5(1-\alpha)} \log^A qT.$$

Lemma 5.3.8 *For $1/2 \leq \alpha \leq 1 - v$, $v > 0$ and $\epsilon > 0$ we have*

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \leq (qT)^{(12/5+\epsilon)(1-\alpha)}.$$

Proof. This Lemma tells us that we only need to prove Linnik's log-free zero density theorem for an arbitrarily short range, $1 - v < \alpha \leq 1$. We can replace the log power in Lemma 5.3.7 until α begins to get too close to 1. Suppose the implied order of magnitude constant in Lemma 5.3.7 is $c_3 > 0$, then

$$c_3(qT)^{12/5(1-\alpha)} \log^A qT \leq (qT)^{(12/5+v)(1-\alpha)}$$

for

$$\alpha \leq 1 - \frac{\log c_3 + A \log\log qT}{v \log qT}$$

thus proving the Lemma.

We now give an explicit zero-free region for L-functions which is originally due to Landau, see [13].

Lemma 5.3.9 *$L(s, \chi)$ has at most one zero, the exceptional zero, in the region*

$$\sigma \geq 1 - \frac{1}{10 \log qT} \quad |t| \leq T.$$

The exceptional zero, if it exists, is real and simple and it is for a real, non-principal character.

By Lemma 5.3.9 and Lemma 5.3.8 we only need to prove the theorem for

$$1-\epsilon_4 \le \alpha \le 1-\frac{1}{10\log D}.$$

The next lemma is taken from Davenport [3]

Lemma 5.3.10 *Suppose* $L(\beta+\imath\gamma,\chi)=0$ *for any character* χ mod q. *Then there exists an effectively computable constant* $A>0$ *such that*

$$\beta < 1-\frac{A}{q^{1/2}\log^2 q}$$

Lemma 5.3.11 *Given a nontrivial zero* $\beta+\imath\gamma$ *of* $L(s,\chi)$ *then the number of zeros* $\beta'+\imath\gamma'$ *such that* $|\gamma-\gamma'|\le\Delta$ *and* $\alpha\le\beta'\le 1$ *is*

$$\le 478 + D^{1-\alpha}$$

where $\Delta = 1/\log D$.

Proof. Put the box $|\gamma-\gamma'|\le\Delta$, $\alpha\le\beta'\le 1$ inside a Linnik circle from Lemma 5.3.6 with radius

$$\delta=\sqrt{(1-\alpha)^2+\Delta^2}\le 1-\alpha+\Delta.$$

Then by Lemma 5.3.6 there are

$$\begin{aligned}\le \frac{48(1-\alpha+\Delta)(1+\epsilon_3)}{\log 7/2\sqrt{10}}\log D &\le 474+474(1-\alpha)\log D\\ &\le 474+D^{1-\alpha}.\end{aligned}$$

zeros inside the circle.

We now give a quantitative version of the Deuring-Heilbronn phenomenon, following Linnik, see [13].

Lemma 5.3.12 *If the exceptional zero, β_1, exists so that*

$$1 - \frac{1}{10\log qT} \leq \beta_1 < 1,$$

then the function $L(s,\chi)$ has no other zeros in the region

$$\sigma \geq 1 - \frac{|\log(1-\beta_1)\log qT|}{2\log qT} \qquad |t| < T.$$

The next Lemma is taken from Prachar [21].

Lemma 5.3.13 *Let χ be a character* mod$q$. *Then*

$$\frac{L'(-1+\imath t,\chi)}{L(-1+\imath t,\chi)} \ll \log(q(|t|+2)).$$

5.4 The Riemann Zeta Function

In counting the zeros of Dirichlet L-functions we must at some point consider the zeros of $L(s,\chi_0)$ where χ_0 is the trivial character. If we were to look at the Euler product of $L(s,\chi_0)$ we would see that you can factor $L(s,\chi_0)$ into a product of two functions, one is nonzero with no poles, and the other is $\zeta(s)$, the Riemann zeta function. So when studying the zeros of $L(s,\chi_0)$ we can use special information about $\zeta(s)$. To do this we need the following Lemmas.

Lemma 5.4.1 *The non real zeros of $L(s,\chi_0)$, where $\Re(s) > 0$ are the zeros of $\zeta(s)$ and vice versa.*

Lemma 5.4.2 *There exists some positive constant c_4 for which $\zeta(s) \neq 0$ in the region*

$$1 - \frac{c_4}{(\log t \log\log t)^{3/4}} \leq \sigma.$$

The next Lemma is taken from Huxley [11].

Lemma 5.4.3 *Let $N(\alpha, T)$ be the number of zeros of $\zeta(s)$ in the rectangle $R(\alpha, T) = \{s|\alpha \leq \sigma \leq 1\,|t| \leq T\}$ where $1/2 \leq \alpha \leq 1$ then there exists some positive constant A for which*

$$N(\alpha, T) \ll T^{12/5(1-\alpha)} \log^A T.$$

From Lemmas 5.4.2 and 5.4.3 we can deduce an implicit log-free zero density theorem for the Riemann zeta function in the following way.

Lemma 5.4.4 *For some $\epsilon > 0$ and T large enough we have*

$$N(\alpha, T) \leq T^{(12/5+\epsilon)(1-\alpha)}.$$

Proof. Suppose the implied order of magnitude constant from Lemma 5.4.3 is $c_5 > 0$ then we have

$$c_5 T^{12/5(1-\alpha)} \log^A T \leq T^{(12/5+\epsilon)(1-\alpha)}$$

for

$$\alpha \leq 1 - \frac{\log c_5 + A \log\log T}{\epsilon \log T},$$

but this range eats into our zero-free region from Lemma (5.4.2) since

$$1 - \frac{c_4}{(\log T \log\log T)^{3/4}} \leq 1 - \frac{\log c_5 + A \log\log T}{\epsilon \log T}$$

for T large enough.

5.5 Some Inequalities

The following inequality is due to Halász.

Lemma 5.5.1 *Let*

$$f(s,\chi)=\sum_{n=N_1}^{N_2}\frac{a(n)\chi(n)}{n^s}.$$

Then

$$\left(\sum_{j=1}^{J}|f(s_j,\chi_j)|\right)^2\leq\sum_{n=N_1}^{N_2}\frac{|a(n)|^2}{k(n)}\sum_{i=1}^{J}\sum_{j=1}^{J}\eta_i\overline{\eta}_jK(\overline{s}_j+s_i,\overline{\chi}_j\chi_i)$$

where $|\eta_i|=1$ *for* $i=1,2,...,J$, *and*

$$K(s,\chi)=\sum_{n=1}^{\infty}\frac{k(n)\chi(n)}{n^s}$$

where the $k(n)$ *are any nonegative numbers such that* $k(n)>0$ *whenever* $a(n)\neq 0$ *and the series* $K(s,\chi)$ *is absolutely convergent for all pairs* $(s,\chi)=(\overline{s}_j+s_i,\overline{\chi}_j\chi_i)$.

The next Lemma is an improvement due to Montgomery and Vaaler of Hilbert's inequality, see [19].

Lemma 5.5.2 *For* $i=1,2,...,I$ *let* $\rho_i=\beta_i+\imath\gamma_i$ *be complex numbers with* $\beta_i\geq 0$ *for all* i, *or* $\beta_i\leq 0$ *for all* i, *and*

$$\delta_i=\min_{\substack{j\\ i\neq j}}|\gamma_i-\gamma_j|.$$

Then

$$\left|\sum_i\sum_{\substack{j\\ i\neq j}}\frac{a_i\overline{a}_j}{\rho_i+\overline{\rho}_j}\right|<84\sum_i\frac{|a_i|^2}{\delta_i}$$

for arbitrary complex numbers a_i.

The next lemma is taken from Graham [5].

Lemma 5.5.3 *Let N, X_1 and X_2 be sufficiently large real numbers such that $N \geq X_2 > X_1$. Let*

$$f(d) = \begin{cases} \mu(d) & \text{for } d \leq X_1, \\ \dfrac{\mu(d)\log(X_2/d)}{\log(X_2/X_1)} & \text{for } X_1 \leq d \leq X_2, \\ 0 & \text{for } X_2 \leq d, \end{cases}$$

and

$$a(n) = \sum_{d|n} f(d).$$

Then for $\epsilon > 0$ we have

$$\sum_{n=1}^{N} a(n)^2 \leq \frac{(1+\epsilon)N}{\log(X_2/X_1)}.$$

5.6 Selberg's Pseudo-Characters

In 1972, Selberg announced the following result

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll (qT)^{c_6(1-\alpha)}$$

for $c_6 > 3$. Prior to this the best known result in this direction was $c_6 = 221$ due to Jutila [15], in which the height T is restricted to a small power of q. The improvement was due to Selberg's use of what he referred to as "pseudo-characters", which are sieve weights that reduce the contribution of terms indexed by numbers which have small prime factors. These weights are given by

$$b(n) = \sum_{d|n} \lambda(d),$$

where

$$\lambda(d) = d\mu(d) \sum_{\substack{m \leq M \\ d|m}} \frac{\mu^2(m)\chi_0(m)}{\phi(m)}, \tag{5.6.16}$$

and χ_0 is the trivial character mod$q$. We require the following results for Selberg's construction.

The next Lemma is taken from Prachar [21].

Lemma 5.6.1 *For $q \geq 1$ we have*

$$\frac{q}{\phi(q)} \ll \log\log q.$$

The next Lemma is taken from Jutila [16].

Lemma 5.6.2 *as $M \to \infty$ we have*

$$\sum_{\substack{m \leq M \\ (m,q)=1}} \frac{\mu^2(m)}{m} = \frac{6}{\pi^2} \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1} \log M(1 + o(1)). \tag{5.6.17}$$

Lemma 5.6.3 *For some $\epsilon > 0$ and M some large real number we have*

$$\lambda(1) \geq (1-\epsilon)\frac{6}{\pi^2}\frac{\phi(q)}{q}\log M.$$

Proof. For $m \geq 1$ we have $\phi(m) \leq m$ and so

$$\begin{aligned} \lambda(1) &= \sum_{\substack{m \leq M \\ (m,q)=1}} \frac{\mu^2(m)}{\phi(m)} \\ &\geq \sum_{\substack{m \leq M \\ (m,q)=1}} \frac{\mu^2(m)}{m} \\ &\geq (1-\epsilon)\frac{6}{\pi^2}\frac{\phi(q)}{q}\log M \end{aligned}$$

by Lemma 5.6.2.

Lemma 5.6.4 *for d an integer such that $d \geq 1$ we have*

$$\max_{d \in N} |\lambda(d)| = \lambda(1)$$

Proof. This is a property of Selberg's sieve, see Greaves [7].

Lemma 5.6.5 *For M a large real number we have*

$$\sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d} = \lambda(1)$$

Proof. This is a standard, but instructive calculation.

$$\sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d} = \sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d_1 d_2}(d_1, d_2) \quad (5.6.18)$$

There exists some function $g(n)$ such that

$$\sum_{e|n} g(e) = n$$

then by Möbius inversion we have

$$g(n) = \sum_{e|n} e\mu\left(\frac{n}{e}\right) = \phi(n)$$

Now let $n = (d_1, d_2)$ in equation (5.6.18) to obtain

$$\begin{aligned} \sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d} &= \sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d_1 d_2} \sum_{e|(d_1,d_2)} \phi(e) \\ &= \sum_{e \le M} \phi(e) \Bigg(\sum_{\substack{d \equiv 0 \bmod e \\ d \le M}} \frac{\lambda(d)}{d} \Bigg)^2 \qquad (5.6.19) \end{aligned}$$

The inner sum on the right hand side of (5.6.19) is given by

$$\begin{aligned} \sum_{\substack{d \equiv 0 \bmod e \\ d \le M}} \frac{\lambda(d)}{d} &= \sum_{\substack{d \equiv 0 \bmod e \\ d \le M}} \mu(d) \sum_{\substack{m=1 \\ d|m}}^{M} \frac{\mu^2(m)\chi_0(m)}{\phi(m)} \\ &= \sum_{f \le M/e} \mu(fe) \sum_{\substack{m=1 \\ fe|m}}^{M} \frac{\mu^2(m)\chi_0(m)}{\phi(m)} \\ &= \sum_{f \le M/e} \mu(fe) \sum_{g \le M/ef} \frac{\mu^2(efg)\chi_0(efg)}{\phi(efg)} \\ &= \frac{\mu(e)\chi_0(e)}{\phi(e)} \sum_{f \le M/e} \mu(e) \sum_{\substack{g \le M/ef \\ (fg,e)=1}} \frac{\mu^2(fg)\chi_0(fg)}{\phi(fg)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu(e)\chi_0(e)}{\phi(e)} \sum_{\substack{h\le M/e \\ (h,e)=1}} \frac{\mu(h)\chi_0(h)}{\phi(h)} \sum_{f|h} \mu(f) \\
&= \frac{\mu(e)\chi_0(e)}{\phi(e)}
\end{aligned}
$$

after we substitute this back into equation (5.6.19) we have

$$
\begin{aligned}
\sum_{d=1}^{M^2} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1,d_2]=d}} \frac{\lambda(d_1)\lambda(d_2)}{d} &= \sum_{e\le M} \phi(e) \Big(\frac{\mu(e)\chi_0(e)}{\phi(e)}\Big)^2 \\
&= \sum_{e\le M} \frac{\mu^2(e)\chi_0(e)}{\phi(e)} \\
&= \lambda(1).
\end{aligned}
$$

5.7 Proof of the Theorem

5.7.1 Introduction

While the Generalised Riemann Hypothesis remains unproven it is natural to ask how many zeros can there be in a given region. The results produce upper bounds for the number of zeros, which decrease as the region moves away from the critical line towards the line $s = 1$, thus the zeros become "less dense"; hence the term "zero density theorem". There are various methods for counting zeros of complex functions. We have already seen one, Jensen's Theorem, our Lemma 5.3.3. We give a brief idea of the method in the hope that it will illuminate the idea behind the proof.

Let

$$M(s,\chi) = \sum_{n=1}^{X} \frac{\mu(n)\chi(n)}{n^s};$$

we will not be working with this sum but something similar. Then

$$\begin{aligned}
L(s,\chi)M(s,\chi) &= \sum_{n=1}^{\infty}\sum_{m\le X}\frac{\mu(m)\chi(n)\chi(m)}{m^s n^s} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\substack{d\le X\\ de=n}}\sum_{e}\mu(d)\chi(d)\chi(e) \\
&= \sum_{n=1}^{\infty}\frac{a(n)\chi(n)}{n^s},
\end{aligned}$$

where $a(n)$ is the incomplete Möbius inversion

$$a(n)=\sum_{\substack{d\le X\\ de=n}}\sum_{e}\mu(d).$$

So

$$L(s,\chi)M(s,\chi)=1+\sum_{n>X}\frac{a(n)\chi(n)}{n^s}.$$

Suppose that $L(s,\chi)=0$. Then

$$0=1+\sum_{n>X}\frac{a(n)\chi(n)}{n^s}. \tag{5.7.20}$$

If we split the the sum in (5.7.20) into blocks in the following way: $2^{r-1}X<n\le 2^rX$, then we have

$$\begin{aligned}
0 &= 1+\sum_{r=1}^{\infty}\sum_{n>2^{r-1}X}^{2^rX}\frac{a(n)\chi(n)}{n^s} \\
&= 1+\sum_{r=1}^{\infty}B_r(s),
\end{aligned}$$

where

$$B_r(s)=\sum_{n>2^{r-1}X}^{2^rX}\frac{a(n)\chi(n)}{n^s}.$$

Now suppose that

$$|B_r(s)|<\frac{1}{r(r+1)}$$

for each r. Then

$$\left|\sum_{r=1}^{\infty} B_r(s)\right| \leq \sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1.$$

So

$$|L(s,\chi)M(s,\chi) - 1| < 1,$$

and therefore $L(s,\chi) \neq 0$. So if $L(s,\chi) = 0$ at $s = \rho$, then for some r we have

$$|B_r(\rho)| \geq \frac{1}{r(r+1)}.$$

In practice the series $B_r(s)$ is truncated so it becomes a Dirichlet polynomial, and the problem of counting zeros reduces to counting large values of Dirichlet polynomials.

We prove the following theorem.

Theorem 5.7.1 *Let $N(\alpha, T, \chi)$ be the number of zeros of $L(s,\chi)$ in the rectangle $R(\alpha, T) = \{\sigma + it \mid \alpha \leq \sigma \leq 1, \ |t| \leq T\}$, where $1 - v \leq \alpha \leq 1$ and $\epsilon > 0$ and $v > 0$ can be taken as small as we like for T large enough. Then for any $m > 0$*

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \leq \frac{12\pi^2(604 + 4m)}{m^2}(qT)^{3(1+4m+\epsilon)(1-\alpha)}. \qquad (5.7.21)$$

5.7.2 The Proof

Now we have the idea behind the proof we can proceed. We choose a far more complicated $M(s,\chi)$ given by

$$M(s,\chi) = \sum_{d \leq X_2} \frac{f(d)}{d^s} \sum_{e \leq M} \frac{\lambda(e)}{e^s}(d,e)^s \chi([d,e]),$$

where $f(d)$ is the function from Lemma 5.5.3, $\lambda(e)$ is the function from (5.6.16), $X_1 = D^{x_1}$ and X_2 are large real numbers such that $X_1 < X_2$ and $M = X_2/X_1 = D^m$. We now multiply $M(s,\chi)$ by $L(s,\chi)$ to obtain

$$\begin{aligned}
L(s,\chi)M(s,\chi) &= \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{d\leq X_2} \frac{f(d)}{d^s} \sum_{e\leq M} \frac{\lambda(e)}{e^s}(d,e)^s\chi([d,e]) \\
&= \sum_{d\leq X_2} f(d) \sum_{e\leq M} \sum_{m=1}^{\infty} \frac{\lambda(e)\chi([d,e])\chi(m)}{([d,e]m)^s} \\
&= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \sum_{d|n} f(d) \sum_{e|n} \lambda(e) \\
&= \sum_{n=1}^{\infty} \frac{a(n)b(n)\chi(n)}{n^s} \\
&= \sum_{n\leq X_1} \frac{a(n)b(n)\chi(n)}{n^s} + \sum_{n>X_1} \frac{a(n)b(n)\chi(n)}{n^s} \\
&= \sum_{n\leq X_1} \frac{b(n)\chi(n)}{n^s} \sum_{d|n} \mu(d) + \sum_{n>X_1} \frac{a(n)b(n)\chi(n)}{n^s} \\
&= \lambda(1) + \sum_{n>X_1} \frac{a(n)b(n)\chi(n)}{n^s}.
\end{aligned}$$

By a well known Mellin transform we have the identity

$$e^{-1/Y_1}\lambda(1) + \sum_{n>X_1} \frac{a(n)b(n)\chi(n)}{e^{n/Y_1}n^s} =$$
$$\frac{1}{2\pi\imath}\int_{1-\imath\infty}^{1+\imath\infty} L(s+z,\chi)M(s+z,\chi)Y_1^z\Gamma(z)dz, \qquad (5.7.22)$$

for some large real number Y_1, We take $Y_1 = D^{y_1}$ and we assume that $Y_1 > X_2$. As before the expression in (5.7.22) should be small for most s except where $s = \rho$ is a nontrivial zero of $L(\rho,\chi)$. We evaluate (5.7.22) at a zero, ρ, of $L(s,\chi)$ to obtain

$$e^{-1/Y_1}\lambda(1) + \sum_{n>X_1} \frac{a(n)b(n)\chi(n)}{e^{n/Y_1}n^\rho} =$$
$$\frac{1}{2\pi\imath}\int_{1-\imath\infty}^{1+\imath\infty} L(\rho+z,\chi)M(\rho+z,\chi)Y_1^z\Gamma(z)dz,$$

so that

$$\left|\frac{1}{2\pi\imath}\int_{1-\imath\infty}^{1+\imath\infty} L(\rho+z,\chi)M(\rho+z,\chi)Y_1^z\Gamma(z)dz - \sum_{n>X_1}\frac{a(n)b(n)\chi(n)}{e^{n/Y_1}n^\rho}\right|$$

$$= e^{-1/Y_1}\lambda(1) > \frac{2\lambda(1)}{3} \qquad (5.7.23)$$

for Y_1 large enough. Given (5.7.23) there are two possibilities, either the zero ρ forces the sum above a certain bound

$$\left|\sum_{n>X_1}\frac{a(n)b(n)\chi(n)}{e^{n/Y_1}n^\rho}\right| > \frac{\lambda(1)}{6}; \qquad (5.7.24)$$

we call such zeros ρ case 1 zeros, or the zero ρ forces the integral in (5.7.23) above a certain bound

$$\left|\frac{1}{2\pi\imath}\int_{1-\imath\infty}^{1+\imath\infty} L(\rho+z,\chi)M(\rho+z,\chi)Y_1^z\Gamma(z)dz\right| > \frac{\lambda(1)}{6}; \qquad (5.7.25)$$

these are case 2 zeros. The method is to count the number of times (5.7.24) happens, thus counting case 1 zeros, and bound the integral in (5.7.25) so that case 2 zeros do not occur.

Case 2 Zeros

We begin with the treatment of case 2 zeros. Move the contour in (5.7.25) to the line $\Re(z+\rho) = 0$, so that $z = -\beta + it$ where $\alpha \le \beta \le 1$. There is a pole of the L-function in the integrand at $z + \rho = 1$ if $\chi = \chi_0$, the trivial character. First we suppose that $\chi \ne \chi_0$, then we estimate each term in the integral as follows.

$$\begin{aligned} |M(\rho+z,\chi)| &= \left|\sum_{d\le X_2}\frac{f(d)}{d^{i(t+\gamma)}}\sum_{e\le M}\frac{\lambda(e)}{e^{i(t+\gamma)}}(d,e)^{i(t+\gamma)}\chi([d,e])\right| \\ &\ll \sum_{d\le X_2}|f(d)|\sum_{e\le M}|\lambda(e)| \\ &\ll \lambda(1)X_2M \end{aligned}$$

by Lemma 5.6.4. By Lemma 5.3.2 we have

$$|L(\rho + z, \chi)| \ll \sqrt{D} \log D.$$

So our integral in (5.7.25) is

$$\ll \lambda(1) X_2 M Y_1^{-\alpha} \sqrt{D} \log D \int_{-\infty}^{+\infty} |\Gamma(-\beta + it)| dt,$$

where by Lemmas 5.2.1 and 5.2.3 we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\Gamma(-\beta + it)| dt &\ll \frac{1}{\beta}\Big(\int_1^{\infty} + \int_{-\infty}^{-1}\Big) |\Gamma(1 - \beta + it)| dt \\
&\quad + \frac{1}{\beta} \int_{-1}^{1} |\Gamma(1 - \beta + it)| dt \\
&\ll \Big(\int_1^{\infty} + \int_{-\infty}^{-1}\Big) |t|^{-\beta + 1/2} \exp\Big(-\frac{\pi}{2}|t|\Big) dt \\
&\quad + \int_{-1}^{1} |\Gamma(1 - \beta + it)| dt \\
&\ll \int_1^{\infty} t^{-\beta + 1/2} \exp\Big(-\frac{\pi}{2} t\Big) dt \\
&\quad + \int_{-1}^{1} |\Gamma(1 - \beta + it)| dt \\
&\ll \int_1^{\infty} \exp\Big(-\frac{\pi}{2} t\Big) dt \\
&\quad + \int_{-1}^{1} |\Gamma(1 - \beta + it)| dt \\
&\ll 1 + \int_{-1}^{1} |\Gamma(1 - \beta + it)| dt. \qquad (5.7.26)
\end{aligned}$$

In order to estimate the remaining integral in (5.7.26) we use lemmas 5.3.10 and 5.2.2 as follows.

$$\begin{aligned}
\int_{-1}^{1} |\Gamma(1 - \beta + it)| dt &\ll \int_0^1 \frac{dt}{\sqrt{(1-\beta)^2 + t^2}} \\
&\ll \Big(\int_0^{1-\beta} + \int_{1-\beta}^{1}\Big)\Big(\frac{dt}{\sqrt{(1-\beta)^2 + t^2}}\Big) \\
&\ll \frac{1-\beta}{\sqrt{(1-\beta)^2}} + \int_{1-\beta}^{1} \frac{dt}{t} \\
&\ll 1 + \log \frac{1}{1-\beta} \\
&\ll 1 + \log q
\end{aligned}$$

Hence the integral in (5.7.25) is

$$\ll \lambda(1)X_2MY_1^{-\alpha}\sqrt{D}\log^2 D \tag{5.7.27}$$

for $\chi \neq \chi_0$. Suppose the implied order of magnitude constant in (5.7.27) is A then we eliminate A by substituting in an extra logarithmic factor of D so that we have a strict inequality. Hence the integral in (5.7.25) is now

$$\leq \lambda(1)X_2MY_1^{-\alpha}\sqrt{D}\log^3 D, \tag{5.7.28}$$

for D large enough. We can now rule out case 2 zeros for all L-functions modulo q except the one with the trivial character, if we bound expression (5.7.28) away from $\lambda(1)/3$. So suppose

$$\lambda(1)X_2MY_1^{-\alpha}\sqrt{D}\log^3 D \leq \frac{\lambda(1)}{6}$$

which forces

$$X_2 \leq \frac{Y_1^{\alpha}}{M\sqrt{D}\log^3 D}. \tag{5.7.29}$$

Now in the special case $\chi = \chi_0$ then the integral in (5.7.25) has a pole at $z + \rho = 1$ with residue

$$\left|\frac{\phi(q)}{q}M(1,\chi_0)Y_1^{1-\rho}\Gamma(1-\rho)\right|, \tag{5.7.30}$$

and so this also needs to be bounded away from $\lambda(1)/6$ if possible. We have

$$\begin{aligned}
|M(1,\chi_0)| &\ll \left|\sum_{d\leq X_2}\frac{f(d)}{d}\sum_{e\leq M}\frac{\lambda(e)}{e}(d,e)\chi_0([d,e])\right| \\
&\ll \lambda(1)M\sum_{d\leq X_2}\frac{|f(d)|}{d} \\
&\ll \lambda(1)M\sum_{d\leq X_2}\frac{1}{d} \\
&\ll \lambda(1)M(1+\log X_2) \\
&\ll \lambda(1)M\log X_2
\end{aligned} \tag{5.7.31}$$

for X_2 large enough. To estimate the Γ term in (5.7.30) we need to shift it again, so that the real part is such that, we do not have a situation where the real and imaginary parts could be simultaneously zero. For $|\gamma| \le 1$ we have

$$\begin{aligned}|\Gamma(1-\beta-i\gamma)| &= \frac{|\Gamma(2-\beta-i\gamma)|}{|1-\beta+i\gamma|} \\ &\ll \frac{1}{1-\beta}|\Gamma(2-\beta-i\gamma)| \\ &\ll \sqrt{q}\log^2 q, \end{aligned} \tag{5.7.32}$$

by lemma 5.3.10, and for $|\gamma| > 1$ we have

$$\begin{aligned}|\Gamma(1-\rho)| &\ll |\gamma|^{1-\beta-1/2}\exp\Big(-\frac{\pi}{2}|\gamma|\Big) \\ &\ll \exp\Big(-\frac{\pi}{2}|\gamma|\Big) \end{aligned} \tag{5.7.33}$$

by Lemma 5.2.3. The inequalities (5.7.32) and (5.7.33) give the following

$$|\Gamma(1-\rho)| \ll \exp\Big(-\frac{\pi}{2}|\gamma|\Big)\sqrt{D}\log^2 D \tag{5.7.34}$$

for all γ. So by equations (5.7.31) and (5.7.34) the residue is

$$\begin{aligned}&\ll 20\frac{\phi(q)}{q}\lambda(1)MY_1^{1-\alpha}\sqrt{D}\log^2 D\exp\Big(\frac{-\pi}{2}|\gamma|\Big)\log X_2 \\ &\ll \frac{\phi(q)}{q}\lambda(1)MY_1^{1-\alpha}\exp\Big(\frac{-\pi}{2}|\gamma|\Big)\sqrt{D}\log^3 D. \end{aligned} \tag{5.7.35}$$

Suppose the implied order of magnitude constant from (5.7.35) is A, we can obtain a strict inequality in (5.7.35), without the need to calculate A by substituting in another logarithmic factor of D. As before we wish to bound the contribution from the trivial character so that case 2 zeros do not occur. However this is not quite possible as the bound depends on the height of the zeros in other words

$$\frac{\phi(q)}{q}\lambda(1)MY_1^{1-\alpha}\exp\Big(\frac{-\pi}{2}|\gamma|\Big)\sqrt{D}\log^4 D \le \frac{\lambda(1)}{6},$$

for

$$\frac{\phi(q)}{q} M Y_1^{1-\alpha} \sqrt{D} \log^4 D \ \le \ \exp\left(\frac{\pi}{2}|\gamma|\right).$$

So case 2 zeros do not occur for

$$|\gamma| \ge \frac{2}{\pi} \log\left(\frac{\phi(q)}{q} M Y_1^{1-\alpha} \sqrt{D} \log^4 D\right).$$

Lemma (5.4.1) tells us that counting zeros of $L(s, \chi_0)$ is equivalent to counting zeros of $\zeta(s)$ so by Lemma (5.4.4) the number of case 2 zeros is

$$\le \left(\frac{2}{\pi} \log\left(\frac{\phi(q)}{q} M Y_1^{1-\alpha} \sqrt{D} \log^4 D\right)\right).^{(12/5+\epsilon)(1-\alpha)} \tag{5.7.36}$$

Case 1 Zeros

We now turn our attention to the main portion of the proof of Linnik's density theorem, which is counting case 1 zeros, which are nontrivial zeros ρ of $L(s, \chi)$ such that

$$\left| \sum_{n > X_1} \frac{a(n) b(n) \chi(n)}{n^{\rho} e^{n/Y_1}} \right| > \frac{\lambda(1)}{6}. \tag{5.7.37}$$

We wish to truncate this sum since it is easier to work with Dirichlet polynomials than Dirichlet series. We cut off the sum at Y_2 where $Y_2 = Y_1 \log D$. The tail of the series now has to be bounded to prove we can do this. Cauchy's inequality gives

$$\left| \sum_{n > Y_2} \frac{a(n) b(n) \chi(n)}{n^{\rho} e^{n/Y_1}} \right|^2 \le \sum_{n > Y_2} \frac{|a(n)|^2}{n^{1+1/\log Y_1}} \sum_{m > Y_2} \frac{|b(m)|^2 m^{1+1/\log Y_1}}{e^{2m/Y_1} m^{2\alpha}}. \tag{5.7.38}$$

For $\alpha \ge 1/2 + 1/2 \log Y_1$ the second sum on the right hand side of (5.7.38) is

$$\sum_{m > Y_2} \frac{|b(m)|^2 m^{1+1/\log Y_1}}{e^{2m/Y_1} m^{2\alpha}} \ \le \ \sum_{m > Y_2} \frac{|b(m)|^2}{e^{2m/Y_1}}$$

$$\begin{aligned}
&\le \sum_{m>Y_2} \frac{1}{e^{2m/Y_1}} \Big|\sum_{e|m} \lambda(e)\Big|^2 \\
&\le \sum_{m>Y_2} \frac{1}{e^{2m/Y_1}} \sum_{\substack{d_1 \; d_2 \\ [d_1,d_2]|m}} \sum \lambda(d_1)\lambda(d_2) \\
&\le \sum_d \sum_{\substack{d_1 \; d_2 \\ [d_1,d_2]=d}} \sum \lambda(d_1)\lambda(d_2) \sum_{\substack{m\equiv 0 \bmod d \\ m>Y_2}} \frac{1}{e^{2m/Y_1}} \\
&\le \sum_d \sum_{\substack{d_1 \; d_2 \\ [d_1,d_2]=d}} \sum \lambda(d_1)\lambda(d_2) \int_{Y_2/d}^{\infty} e^{-2xd/Y_1} dx \\
&\le \frac{Y_1 e^{-2Y_2/Y_1}}{2} \sum_d \sum_{\substack{d_1 \; d_2 \\ [d_1,d_2]=d}} \sum \frac{\lambda(d_1)\lambda(d_2)}{d} \\
&\le \frac{\lambda(1)}{2} Y_1 e^{-2Y_2/Y_1} \\
&\le \frac{\lambda(1)}{2} Y_1 \exp\left(\frac{-2Y_1 \log D}{Y_1}\right) \\
&\le \frac{\lambda(1)}{2D^{2-y_1}} \qquad (5.7.39)
\end{aligned}$$

by Lemma 5.6.5. The first sum on the right hand side of (5.7.38) is

$$\begin{aligned}
\sum_{n>Y_2} \frac{|a(n)|^2}{n^{1+1/\log Y_1}} &\le \sum_{r=0}^{\infty} \sum_{n>2^r Y_2}^{2^{r+1}Y_2} \frac{|a(n)|^2}{n^{1+1/\log Y_1}} \\
&\le \sum_{r=0}^{\infty} \left(\frac{1}{2^r Y_2}\right)^{1+1/\log Y_1} \sum_{n>2^r Y_2}^{2^{r+1}Y_2} |a(n)|^2 \\
&\le 2\sum_{r=0}^{\infty} \left(\frac{1}{2^r Y_2}\right)^{1+1/\log Y_1} \frac{(1+\epsilon)2^{r+1}Y_2}{\log X_2/X_1} \\
&\le \frac{8(1+\epsilon)}{Y_2^{1/\log Y_1} \log M} \sum_{r=0}^{\infty} \left(\frac{1}{2^{1/\log Y_1}}\right)^r \qquad (5.7.40)
\end{aligned}$$

by Lemma 5.5.3. The sum on the right hand side of (5.7.40) is a geometric progression, so we have

$$\sum_{r=0}^{\infty} \left(\frac{1}{2^{1/\log Y_1}}\right)^r = \frac{1}{1-\frac{1}{2^{1/\log Y_1}}} \le \log Y_1$$

for Y_1 large enough. Hence (5.7.40) becomes

$$\sum_{n>Y_2} \frac{|a(n)|^2}{n^{1+1/\log Y_1}} \le \frac{8(1+\epsilon)\log Y_1}{Y_2^{1/\log Y_1} \log M},$$

where

$$Y_2^{1/\log Y_1} > e,$$

since $Y_1 < Y_2$. So

$$\sum_{n>Y_2} \frac{|a(n)|^2}{n^{1+1/\log Y_1}} \leq \frac{8(1+\epsilon)\log Y_1}{e\log M}. \tag{5.7.41}$$

Using equation (5.7.39) and (5.7.41) we have

$$\begin{aligned}\Big|\sum_{n>Y_2} \frac{a(n)b(n)\chi(n)}{n^\rho e^{n/Y_1}}\Big|^2 &\leq \frac{4(1+\epsilon)\lambda(1)\log Y_1}{eD^{2-y_1}\log M} \\ &\leq \epsilon\lambda(1)\end{aligned}$$

for some $\epsilon > 0$. Condition (5.7.37) becomes

$$\Big|\sum_{X_1<n\leq Y_2} \frac{a(n)b(n)\chi(n)}{n^\rho e^{n/Y_1}}\Big| > \lambda(1)\Big(\frac{1}{6}+\epsilon\Big) \tag{5.7.42}$$

for some $\epsilon > 0$. The sum on the left of (5.7.42) is our zero detecting Dirichlet polynomial, serving our aim to bound the number of times that the inequality in (5.7.42) can occur.

Now we want to split the rectangle $R(\alpha, T)$ into smaller ones. Let $\Delta = 1/\log D$, and consider the region

$$\alpha \leq \sigma \leq 1,\ \max(-T, k\Delta) \leq t \leq \min(T, (k+1)\Delta) \tag{5.7.43}$$

for $k = 0, \pm1, \pm2, ...$ For each $L(s,\chi)$ having a zero in the region (5.7.43) we choose an arbitrary one of the zeros to represent the whole group. Considering the even and odd numbers k separately, we get two Δ-well spaced systems. Let J denote the cardinality of the system containing at least half of the zeros. In view of Linnik's Lemma, Lemma 5.3.6, it is sufficient to estimate the number J in order to prove the zero density theorem for the number J. We wish to apply the Halász inequality to our Dirichlet polynomial in (5.7.42). Let

$$g(s,\chi) = \sum_{X_1<n\leq Y_2} \frac{a(n)b(n)\chi(n)}{n^{\rho}e^{n/Y_1}},$$

and in Lemma 5.5.1 let $f(s,\chi) = g(s,\chi)$, then sum over pairs $(\rho_i - \alpha, \chi_i)$, which run over the zeros with the corresponding characters in the larger of the two Δ-well spaced systems. So in Lemma 5.5.1 let $s_i = \rho_i - \alpha$,

$$k(n) = \frac{b(n)^2}{n}(e^{-n/Y_2} - e^{-n/X_1}) = \frac{b(n)^2}{n}h(n).$$

$N_1 = X_1$ and $N_2 = Y_2$. Then by Lemma 5.5.1 we have

$$\begin{aligned}\Big(\sum_{i\leq J}|g(s_i+\alpha,\chi_i)|\Big)^2 &\leq \sum_{X_1<n\leq Y_2}\left|\frac{a(n)b(n)}{n^{\alpha}e^{n/Y_1}}\right|^2\frac{n}{b^2(n)h(n)}\\ &\quad\sum_{i\leq J}\sum_{j\leq J}|\eta_i\overline{\eta_j}B(s_i+\overline{s_j},\chi_i\overline{\chi_j})|,\end{aligned}$$

so that

$$\begin{aligned}\Big(\sum_{i\leq J}|g(\rho_i,\chi_i)|\Big)^2 &\leq \sum_{X_1<n\leq Y_2}\frac{|a(n)|^2}{n^{2\alpha-1}}\frac{1}{e^{n/Y_1}h(n)}\\ &\quad\sum_{i\leq J}\sum_{j\leq J}\left|\sum_{n=1}^{\infty}\eta_i\overline{\eta_j}\frac{\chi_i(n)\overline{\chi_j}(n)b^2(n)h(n)}{n^{\rho_i+\overline{\rho_j}+1-2\alpha}}\right| \qquad (5.7.44)\end{aligned}$$

We now expand out the second part of the right hand side of (5.7.44) to get

$$\begin{aligned}&\sum_{i\leq J}\sum_{j\leq J}\sum_{n=1}^{\infty}\eta_i\overline{\eta_j}\frac{\chi_i(n)\overline{\chi_j}(n)b^2(n)h(n)}{n^{\rho_i+\overline{\rho_j}+1-2\alpha}}\\ =&\sum_{d\leq M^2}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\lambda(d_1)\lambda(d_2)\sum_{i\leq J}\sum_{j\leq J}\eta_i\overline{\eta_j}\sum_{n\equiv 0 \bmod d}\frac{h(n)\chi_i(n)\overline{\chi_j}(n)}{n^{\rho_i+\overline{\rho_j}+1-2\alpha}}\\ =&\sum_{d\leq M^2}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\lambda(d_1)\lambda(d_2)\times\\ &\sum_{i\leq J}\sum_{j\leq J}\eta_i\overline{\eta_j}\sum_{m=1}^{\infty}\frac{h(md)\chi_i(md)\overline{\chi_j}(md)}{(md)^{\rho_i+\overline{\rho_j}+1-2\alpha}}. \qquad (5.7.45)\end{aligned}$$

Now

$$\begin{aligned} h(md) &= \exp\left(\frac{-m}{Y_2/d}\right) - \exp\left(\frac{-m}{X_1/d}\right) \\ &= h\Big(\frac{X_1}{d}, \frac{Y_2}{d}; m\Big). \end{aligned}$$

Let $s_{ij} = \rho_i + \overline{\rho}_j - 2\alpha + 1$, $\chi_{ij} = \chi i\overline{\chi_j}$ and

$$H\Big(\frac{X_1}{d}, \frac{Y_2}{d}, s_{ij}, \chi_{ij}\Big) = \sum_{m=1}^{\infty} h\Big(\frac{X_1}{d}, \frac{Y_2}{d}; m\Big) \frac{\chi_{ij}(m)}{m^{s_{ij}}},$$

so that the expression in (5.7.45) becomes

$$\sum_{i \le J} \sum_{j \le J} \eta_i \overline{\eta_j} \sum_{d \le M^2} \frac{\chi_{ij}(d)}{d^{s_{ij}}} \sum_{d_1} \sum_{\substack{d_2 \\ [d_1, d_2] = d}} \lambda(d_1)\lambda(d_2) H\Big(\frac{X_1}{d}, \frac{Y_2}{d}, s_{ij}, \chi_{ij}\Big). \quad (5.7.46)$$

Expressing the Dirichlet series in (5.7.46) in terms of its corresponding integral gives us the equation

$$\begin{aligned} &H\Big(\frac{X_1}{d}, \frac{Y_2}{d}, s_{ij}, \chi_{ij}\Big) = \\ &\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} L(s_{ij} + z)\bigg(\Big(\frac{Y_2}{d}\Big)^z - \Big(\frac{X_1}{d}\Big)^z\bigg)\Gamma(z) dz. \quad (5.7.47) \end{aligned}$$

We estimate the integral in (5.7.47) by moving the contour back to the line $\Re(z) = -1$ with a semicircle radius Δ at $z = -1$ to avoid the pole of Γ. If $\chi_i = \chi_j$ $\chi_{ij} = \chi_0$ and there is a pole of our L-function at $z = 1 - s_{ij}$. First we assume $\chi_i \neq \chi_j$ then split the contour into the following five parts

$$\begin{aligned} C_1 &= \{z = \sigma + it \in C | \sigma = -1, t < -1\}, \\ C_2 &= \{z = \sigma + it \in C | \sigma = -1, -1 \le t < -\Delta\}, \\ C_3 &= \{z = \Delta e^{i\theta} - 1 \in C | \theta \in [-\pi/2, \pi/2]\}, \\ C_4 &= \{z = \sigma + it \in C | \sigma = -1, \Delta \le t \le 1\}, \\ C_5 &= \{z = \sigma + it \in C | \sigma = -1, t > 1\}. \end{aligned}$$

On each contour C_i we integrate approximations of the Γ function and bound the other terms in the integral for the corresponding range. First we consider the integral on C_3. Shifting the Γ function twice gives

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z(z+1)}\Gamma(z+2),$$

so that on C_3 we have

$$\begin{aligned} |z| &= |\Delta e^{i\theta} - 1| \geq 1 - \Delta, \\ |z+1| &= \Delta, \\ |\Gamma(z)| &\leq \frac{|\Gamma(z+2)|}{\Delta(1-\Delta)}. \end{aligned}$$

We have $\Re(z+2) \in [1, 1+\Delta]$ and $|\Im(z+2)| \leq \Delta < 1$ so by Lemma 5.2.3 we have

$$|\Gamma(z)| \ll \frac{1}{\Delta(1-\Delta)},$$

which gives

$$\int_{C_3} |\Gamma(z)||dz| \ll \frac{\Delta}{\Delta(1-\Delta)} \ll 1 \tag{5.7.48}$$

for D large enough. On C_2 and C_4 we have, as above $|\Gamma(z+2)| \ll 1$ and

$$\begin{aligned} |z(z+1)| &= |(it-1)it| = |-t^2 - it| \\ &= \sqrt{t^4 + t^2} = |t|\sqrt{t^2+1} \geq t^2, \end{aligned}$$

so

$$\begin{aligned} \left(\int_{C_2} + \int_{C_4}\right)|\Gamma(z)||dz| &\ll \int_{\Delta}^{1} \frac{1}{t^2} dt \\ &\ll \frac{1}{\Delta} - 1 \ll \frac{1}{\Delta}. \end{aligned} \tag{5.7.49}$$

On C_1 and C_5 we have

$$\left|\frac{\Gamma(z+2)}{z(z+1)}\right| \ll \frac{1}{t^2}|t|^{1-1/2}\exp\left(-\frac{\pi}{2}|t|\right)$$

by Lemma (5.2.3), and so we have

$$\begin{aligned}\Big(\int_{C_1}+\int_{C_5}\Big)|\Gamma(z)||dz| &\ll \int_1^\infty \frac{e^{-\frac{\pi}{2}t}}{t^{3/2}}dt \\ &\ll \int_1^\infty t^{-3/2}dt \\ &\leq 1. \end{aligned} \tag{5.7.50}$$

By (5.7.48), (5.7.49) and (5.7.50) we have

$$\begin{aligned}\Big|\int_{\Re(z)=1}\Gamma(z)dz\Big| &\ll 1+\frac{1}{\Delta} \\ &\ll \log D,\end{aligned}$$

for D large enough. We now estimate the other terms of the integrand in (5.7.47), for z on any one of our contours C_i for $i=1,2,3,4,5$, we have

$$\begin{aligned}\Big|\Big(\frac{Y_2}{d}\Big)^z-\Big(\frac{X_1}{d}\Big)^z\Big| &\leq \Big(\frac{Y_2}{d}\Big)^{\Delta-1}+\Big(\frac{X_1}{d}\Big)^{\Delta-1} \\ &\leq d^{1-\Delta}(Y_2^{\Delta-1}+X_1^{\Delta-1}) \\ &\leq d^{1-\Delta}Y_2^{\Delta}\Big(\frac{1}{Y_2}+\frac{1}{X_1}\Big) \\ &\ll \frac{d^{1-\Delta}Y_2^{\Delta}}{X_1}\end{aligned}$$

for $Y_2 > BX_1$ where B is some constant which can be taken as large as we require. Now we want to bound the L-function term of the integrand in (5.7.47). We use

$$\beta_i+\beta_j-2\alpha+1-1\leq \Re(s_{ij}+z)\leq \beta_i+\beta_j-2\alpha+1-1+\Delta,$$

so that $0\leq \Re(s_{ij}+z)\leq 2(1-\alpha)+\Delta$. Hence by Lemma 5.3.2

$$|L(s_{ij}+z,\chi_{ij})|\ll \sqrt{D}\log D.$$

So if $\chi_i\neq\chi_j$ then

$$H\Big(\frac{X_1}{d},\frac{Y_2}{d},s_{ij},\chi_{ij}\Big)\ll \frac{d^{1-\Delta}Y_2^{\Delta}}{X_1}\sqrt{D}\log^2 D,$$

which makes the expression in (5.7.46)

$$\begin{aligned}
&\ll \frac{Y_2^{\Delta}}{X_1}\sqrt{D}\log^2 D\Big|\sum_{i\le J}\sum_{j\le J}\eta_i\overline{\eta_j}\sum_{d\le M^2}\frac{\chi_{ij}(d)}{d^{s_{ij}+\Delta-1}}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\lambda(d_1)\lambda(d_2)\Big| \\
&\ll \frac{Y_2^{\Delta}}{X_1}\sqrt{D}\log^2 D\sum_{i\le J}\sum_{j\le J}\Big|\sum_{d\le M^2}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\lambda(d_1)\lambda(d_2)\Big| \\
&\ll \frac{Y_2^{\Delta}}{X_1}\sqrt{D}J^2M^2\lambda(1)^2\log^2 D. \qquad (5.7.51)
\end{aligned}$$

Now we need to consider the case when $\chi_i = \chi_j$ and there is a pole coming from our L-function. The residue is

$$\frac{\phi(q)}{q}\left(\left(\frac{Y_2}{d}\right)^{1-s_{ij}} - \left(\frac{X_1}{d}\right)^{1-s_{ij}}\right)\Gamma(1-s_{ij}),$$

and so the residue terms contribute

$$\begin{aligned}
&\frac{\phi(q)}{q}\Big|\sum_{i\le J}\sum_{j\le J}\eta_i\overline{\eta_j}\sum_{d\le M^2}\frac{\chi_{ij}}{d^{s_{ij}}}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\lambda(d_1)\lambda(d_2) \\
&\times\left(\left(\frac{Y_2}{d}\right)^{1-s_{ij}} - \left(\frac{X_1}{d}\right)^{1-s_{ij}}\right)\Gamma(1-s_{ij})\Big| \\
\le\ &\frac{\phi(q)}{q}\Big|\sum_{i\le J}\sum_{j\le J}\eta_i\overline{\eta_j}(Y_2^{1-s_{ij}} - X_1^{1-s_{ij}})\Gamma(1-s_{ij})\Big|\sum_{d\le M^2}\sum_{d_1}\sum_{\substack{d_2\\ [d_1,d_2]=d}}\frac{\lambda(d_1)\lambda(d_2)}{d} \\
\le\ &\frac{\phi(q)}{q}\lambda(1)\Big|\sum_{i\le J}\sum_{j\le J}\eta_i\overline{\eta_j}(Y_2^{1-s_{ij}} - X_1^{1-s_{ij}})\Gamma(1-s_{ij})\Big| \qquad (5.7.52)
\end{aligned}$$

by Lemma 5.6.5. Now

$$\begin{aligned}
1-s_{ij} &= 1-\rho_i-\overline{\rho_j}+2\alpha-1 \\
&= 2\alpha-\beta_i-\beta_j+i(\gamma_j-\gamma_i),
\end{aligned}$$

so that $\Re(1-s_{ij}) \in [-1/2, 0]$ for $\alpha > 3/4$ and $\Im(1-s_{ij}) = \gamma_j - \gamma_i$. As before the estimates involving the Γ function break down into two cases, one where the imaginary part is greater than one in modulus and the other where the imaginary part is less than one in modulus. First we suppose that

$|\gamma_j - \gamma_i| \geq 1$. Then we can use Lemma (5.2.3) to bound the right hand side of (5.7.52) with the following expression

$$\frac{\phi(q)}{q}\lambda(1)\Big|\sum_{\substack{i\leq J\, j\leq J\\ |\gamma_i-\gamma_j|\geq 1}} \frac{\sqrt{2\pi}e^{11/16}|Y_2^{1-s_{ij}} - X_1^{1-s_{ij}}|}{|\gamma_j-\gamma_i|^{1/2}\exp(\pi/2|\gamma_j-\gamma_i|)}\Big|. \quad (5.7.53)$$

By Lemma 5.3.4 in a unit interval $\gamma_i + n \leq \gamma_j \leq \gamma_i + n + 1$ there are $\leq 20\log D$ imaginary parts γ_j of $|L(s,\chi)|$. The expression (5.7.53) is

$$\begin{aligned}
&\leq 20\sqrt{2\pi}e^{11/16}\frac{\phi(q)}{q}\lambda(1)\log D\sum_{i\leq J} e^{-\pi/2}\\
&\leq 5\sqrt{2\pi}e^{11/16}\frac{\phi(q)}{q}\lambda(1)J\log D. \quad (5.7.54)
\end{aligned}$$

Now suppose that $|\gamma_j - \gamma_i| < 1$. By the proof of Lemma 5.2.2 we have the following identity

$$\begin{aligned}
\Gamma(1-s_{ij}) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{1-s_{ij}+k} + \int_1^{\infty} e^{-t}t^{-s_{ij}}dt\\
&= \frac{1}{1-s_{ij}} - \frac{1}{1-s_{ij}+1} + \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\frac{1}{1-s_{ij}+k} +\\
&\quad \int_1^{\infty} e^{-t}t^{-s_{ij}}dt, \quad (5.7.55)
\end{aligned}$$

where

$$\Big|\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\frac{1}{1-s_{ij}+k} + \int_1^{\infty} e^{-t}t^{-s_{ij}}dt\Big| \leq e + \frac{1}{e}. \quad (5.7.56)$$

By (5.7.55) and (5.7.56) we have

$$\begin{aligned}
\Big|\Gamma(1-s_{ij}) - \frac{1}{1-s_{ij}}\Big| &\leq \frac{1}{|1-s_{ij}+1|} + e + \frac{1}{e}\\
&\leq 2 + e + \frac{1}{e}, \quad (5.7.57)
\end{aligned}$$

then we have the following upper bound to (5.7.52)

$$\leq \frac{\phi(q)}{q}\lambda(1)\Big(\Big|\sum_{\substack{i\leq J\, j\leq J\\ |\gamma_j-\gamma_i|<1}} \eta_i\overline{\eta_j}(Y_2^{1-s_{ij}} - X_1^{1-s_{ij}})\frac{1}{1-s_{ij}}\Big|$$

$$
\begin{aligned}
& +20\Big(2+\frac{1}{e}+e\Big)J\log D\Big) \\
\leq\ & \frac{\phi(q)}{q}\lambda(1)\Big(\Big|\sum_{i\leq J}\sum_{\substack{j\leq J \\ |\gamma_j-\gamma_i|<1}}\frac{\eta_i\overline{\eta_j}(Y_2^{1-s_{ij}}-X_1^{1-s_{ij}})}{1-s_{ij}}\Big|+ \\
& 120J\log D\Big). \qquad (5.7.58)
\end{aligned}
$$

For $i=j$ we have

$$
\begin{aligned}
\Big|\sum_{i\leq J}\sum_{\substack{j\leq J \\ i=j}}\frac{Y_2^{1-s_{ij}}-X_1^{1-s_{ij}}}{1-s_{ij}}\Big| &= \Big|\sum_{i\leq J}\int_{X_1}^{Y_2}\frac{1}{x^{s_{ii}}}dx\Big| \\
&\leq J\int_{X_1}^{Y_2}\frac{1}{x}dx \\
&= J\log\frac{Y_2}{X_1}. \qquad (5.7.59)
\end{aligned}
$$

Now we suppose that $i\neq j$, let

$$
S=\Big|\sum_{i\leq J}\sum_{\substack{j\leq J \\ i\neq j}}\frac{Y_2^{1-s_{ij}}-X_1^{1-s_{ij}}}{1-s_{ij}}\Big|.
$$

Then

$$
\begin{aligned}
S &\leq \Big|\sum_{i\leq J}\sum_{\substack{j\leq J \\ i\neq j}}\frac{Y_2^{1-s_{ij}}}{1-s_{ij}}\Big|+\Big|\sum_{i\leq J}\sum_{\substack{j\leq J \\ i\neq j}}\frac{X_1^{1-s_{ij}}}{1-s_{ij}}\Big| \\
&= S(Y_2)+S(X_1).
\end{aligned}
$$

We now invoke Lemma 5.5.2 with $a_r=Y_2^{-\rho_i+\alpha}$ and $a_s=Y_2^{-\rho_j+\alpha}$, then $a_r\overline{a_s}=Y_2^{1-s_{ij}}$. By Lemma 5.5.2 we have

$$
\begin{aligned}
S(Y_2) &< \frac{84}{\Delta}\sum_{i\leq J}|Y_2^{\alpha-\rho_i}|^2<\frac{84J}{\Delta} \\
S(X_1) &< \frac{84}{\Delta}\sum_{i\leq J}|X_1^{\alpha-\rho_i}|^2<\frac{84J}{\Delta}
\end{aligned}
$$

which means that

$$
S\leq\frac{168J}{\Delta}. \qquad (5.7.60)
$$

Hence we have the following upper bound for (5.7.52)

$$\leq \frac{\phi(q)}{q} J\lambda(1)\Big(578 \log D + \log \frac{Y_2}{X_1}\Big) \tag{5.7.61}$$

So by (5.7.54) and (5.7.61) our integral in (5.7.47) is

$$\begin{aligned}
&\leq J\lambda(1)\frac{\phi(q)}{q}\Big((5\sqrt{2\pi}e^{11/16} + 578)\log D + \log \frac{Y_2}{X_1}\Big) \\
&\leq J\lambda(1)\frac{\phi(q)}{q}\Big((603 \log D + \log \frac{Y_2}{X_1}\Big) \qquad (5.7.62)
\end{aligned}$$

if $\chi_i = \chi_j$. By (5.7.51) and (5.7.62), (5.7.46) is

$$\begin{aligned}
\leq\ & \frac{aY_2^{\Delta}}{X_1}\sqrt{D}J^2M^2\lambda(1)^2 \log^2 D + \lambda(1)\frac{\phi(q)}{q}J\Big(603 \log D + \\
& \log \frac{Y_2}{X_1},\Big) \qquad (5.7.63)
\end{aligned}$$

where a is some positive constant and

$$\begin{aligned}
Y_2^{\Delta} &= \exp\Big(\frac{\log(Y_1 \log D)}{\log D}\Big) \\
&= \exp\Big(\frac{\log(D^{y_1} \log D)}{\log D}\Big) \\
&= \exp\Big(\frac{y_1 \log D + \log\log D}{\log D}\Big) \\
&= e^{y_1+\epsilon} \qquad (5.7.64)
\end{aligned}$$

for some $\epsilon > 0$. Also

$$\log \frac{Y_2}{X_1} = (y_2 - x_1)\log D$$

and the expression (5.7.63) becomes

$$\begin{aligned}
\leq\ & \frac{a}{X_1}\sqrt{D}J^2M^2\lambda(1)^2 \log^2 D + \\
& \lambda(1)\frac{\phi(q)}{q}J \log D(603 + y_2 - x_1). \qquad (5.7.65)
\end{aligned}$$

So by (5.7.44) we have

$$\Big(\sum_{i \leq J} |g(\rho_i, \chi_i)|\Big)^2 \leq \Big(\frac{a}{X_1}\sqrt{D}J^2M^2\lambda(1)^2 \log^2 D +$$

$$\lambda(1)\frac{\phi(q)}{q}J\log D(630+y_2-x_1)\Big)$$
$$\sum_{X_1<n\le Y_2}\frac{|a(n)|^2}{n^{2\alpha-1}e^{n/Y_1}h(n)}. \tag{5.7.66}$$

It remains to estimate the sum on the right hand side of (5.7.66). First we note that for $n>X_1$ we have

$$\begin{aligned} e^{n/Y_1}h(n) &= \exp\Big(n\Big(\frac{1}{Y_1}-\frac{1}{Y_2}\Big)\Big)-\exp\Big(-n\Big(\frac{1}{X_1}-\frac{1}{Y_1}\Big)\Big)\\ &\ge e^0-\exp\Big(-X_1\Big(\frac{1}{X_1}-\frac{1}{Y_1}\Big)\Big)\\ &\ge e^0-\exp\Big(\frac{X_1}{Y_1}-1\Big)\\ &\ge 1-\exp\Big(\epsilon-1\Big)\end{aligned}$$

for any $\epsilon>0$, so let ϵ be small enough so that

$$e^{n/Y_1}h(n)\ge\frac{1}{3}, \tag{5.7.67}$$

say. Now the remainder of the summand in (5.7.66) is estimated by partial summation, Lemma 5.5.3 and Lemma 5.3.9.

$$\begin{aligned}\sum_{X_1<n\le Y_2}\frac{|a(n)|^2}{n^{2\alpha-1}} &\le Y_2\sum_{n\le Y_2}|a(n)|^2-\int_{1-\epsilon}^{Y_2}(1-2\alpha)t^{-2\alpha}\sum_{n\le t}|a(n)|^2dt\\ &\le \frac{(1+\epsilon_7)Y_2}{\log X_2/X_1}+(2\alpha-1)\int_{1-\epsilon}^{Y_2}\frac{(1+\epsilon_7)t^{1-2\alpha}}{\log X_2/\log X_1}dt\\ &\le \frac{(1+\epsilon_7)Y_2}{\log X_2/X_1}+\frac{(2\alpha-1)(1+\epsilon_7)Y_2^{2-2\alpha}}{(2-2\alpha)\log X_2/X_1}\\ &\le \frac{Y_2}{\log X_2/X_1}\Big(1+\epsilon+\frac{2\alpha-1}{2-2\alpha}\Big)\\ &\le \frac{Y_2}{\log M}\Big(1+\epsilon+\frac{10\log D}{2}\Big)\\ &\le \frac{(5+\epsilon)Y_2\log D}{\log M}\\ &\le \frac{5+\epsilon}{m}Y_2^{2-2\alpha}.\end{aligned} \tag{5.7.68}$$

So by (5.7.66), (5.7.67) and (5.7.68) we have

$$\Big(\sum_{i\le J}|g(\rho_i,\chi_i)|\Big)^2\le\Big(\frac{a}{X_1}\sqrt{D}J^2M^2\lambda(1)^2\log^2 D+$$

$$\lambda(1)\frac{\phi(q)}{q}J\log D(603+y_2-x_1)\Big)\frac{15+\epsilon}{m}Y_2^{2-2\alpha}. \qquad (5.7.69)$$

We now require a lower bound for the left hand side of (5.7.69). By (5.7.42) we have

$$|g(\rho_i,\chi_i)| > \lambda(1)\Big(\frac{1}{3}+\epsilon\Big),$$

so that

$$\Big(\sum_{i\le J}|g(\rho_i,\chi_i)|\Big)^2 > J^2\lambda(1)^2\frac{2}{9},$$

and therefore (5.7.69) becomes

$$J^2\lambda(1)^2 \le \Big(\frac{a}{X_1}\sqrt{D}J^2M^2\lambda(1)^2\log^2 D +$$
$$\lambda(1)\frac{\phi(q)}{q}J\log D(603+y_2-x_1)\Big)\frac{15+\epsilon}{m}Y_2^{2-2\alpha}. \qquad (5.7.70)$$

Once we have rearranged (5.7.70) so that all the terms involving J^2 are together, we will need the coefficient in J^2 to be positive on the left hand side of (5.7.70). Suppose we want the coefficient of J^2 to be greater than 1/2 then this gives the condition

$$X_1 \ \ge \ 2a\sqrt{D}M^2Y_2^{2-2\alpha}\log^2 D,$$

which will be satisfied if

$$X_1 \ge \sqrt{D}M^2Y_2^{2-2\alpha}\log^3 D. \qquad (5.7.71)$$

Condition (5.7.29) gives

$$X_1 \le \frac{Y_1^\alpha}{M^2\sqrt{D}\log^3 D}. \qquad (5.7.72)$$

We need

$$\sqrt{D}M^2Y_2^{2-2\alpha}\log^2 D < \frac{Y_1^\alpha}{M^2\sqrt{D}\log^3 D}, \qquad (5.7.73)$$

or

$$\sqrt{D}M^2Y_2^{2-2\alpha}\log^2 D < \frac{Y_2^\alpha}{M^2\sqrt{D}\log^4 D},$$

which yields the following condition

$$Y_2 \geq (M^4D\log^6 D)^{1/(3\alpha-2)}. \tag{5.7.74}$$

By Lemma 5.3.8 we only need to prove the zero density theorem for $\alpha \geq 1-v$, so we choose

$$\begin{aligned} Y_2 &= (M^4D\log^6 D)^{1/(1-3v)}, \\ &= (M^4D\log^6 D)^{1+\epsilon} \end{aligned} \tag{5.7.75}$$

from which it follows that

$$y_1 = 4m + 1 + \epsilon, \tag{5.7.76}$$

since $Y_2 = Y_1 \log D$. Substitution of this into (5.7.70) gives

$$\begin{aligned} J &\leq \frac{31(603 + y_2 - x_1)\phi(q)}{mq\lambda(1)}(D^{1+4m+\epsilon})^{2-2\alpha}\log D \\ &\leq \frac{31(603 + y_2 - x_1)\phi(q)}{mq\lambda(1)}D^{(2+8m+\epsilon)(1-\alpha)}\log D \end{aligned} \tag{5.7.77}$$

Lemma 5.6.3 gives

$$\frac{\phi(q)\log D}{\lambda(1)q} = \frac{\pi^2\log D}{6m(1-\epsilon)\log D} = \frac{\pi^2}{6m(1-\epsilon)},$$

so that (5.7.77) becomes

$$\begin{aligned} J &\leq \frac{31\pi^2(603 + y_2 - x_1)}{6m^2(1-\epsilon)}D^{(2+8m+\epsilon)(1-\alpha)} \\ &\leq \frac{6\pi^2(4697 + y_2 - x_1)}{m^2(1-\epsilon)}D^{2(1+4m+\epsilon)(1-\alpha)}, \end{aligned}$$

which is true for $1 - v \leq \alpha \leq 1$. Now J is the well spaced system containing at least half of the zeros, so we need to multiply J by 2 and also by the number of zeros in our spaced regions, which we have by Lemma 5.3.11. Then we must add in the zeros of $\zeta(s)$, and so by (5.7.36)

$$\begin{aligned}\sum_{\chi \bmod q} N(\alpha,T,\chi) &\leq 2(478+D^{1-\alpha})\Big(\frac{6\pi^2(603+y_2-x_1)}{m^2(1-\epsilon)}D^{2(1+4m+\epsilon)(1-\alpha)}\Big) \\ &\quad +\Big(\frac{2}{\pi}\log\Big(\frac{\phi(q)}{q}MY_1^{1-\alpha}\sqrt{D}\log^4 D\Big)\Big)^{(12/5+\epsilon)(1-\alpha)} \\ &\leq \frac{12\pi^2(603+y_2-x_1)}{m^2(1-\epsilon)}D^{3(1+4m+\epsilon)(1-\alpha)}. \end{aligned} \tag{5.7.78}$$

It remains to choose the powers of the parameters. Expressions (5.7.29) and (5.7.71) are together implied by

$$2y_1\epsilon+\frac{1}{2}+2m \leq x_1 \leq y_1(1-\epsilon)-2m-\frac{1}{2}. \tag{5.7.79}$$

For simplicity we choose the mid point of the interval (5.7.79) and let $x_1 = y_1(1+\epsilon)/2$. Due to the fact that $X_2 \ll Y_1$ we must have $x_1+m < y_1$ and therefore

$$m < \frac{y_1}{2}(1-\epsilon), \tag{5.7.80}$$

thus by virtue of equation (5.7.76) the inequality (5.7.80) will be satisfied for all $m > 0$.

Chapter 6

Fogels' Theorem

Fogels [4] was the first to prove Linnik's log-free zero-density theorem in rectangles which are taller than the value of the modulus of the L-function. As an application Fogels proves the following theorem.

Theorem 6.0.2 *There exist constants $\theta < 1$ and η such that if $q > q_0$ and $(q, l) = 1$, then for any $x > q^\eta$ in the interval $(x, x + x^\theta)$ there is a prime $p \equiv l \bmod q$.*

In his paper Fogels does not calculate the constants in his zero density theorem and therefore fails to give admisable values for θ or η. Versions of this theorem have been proved since, but their aim is to reduce the size of θ which costs you in your size of η, these theorems take the following form.

Theorem 6.0.3 *For $q \le \log^A x$ where A is some large positive constant there exists a prime $p \equiv l \bmod q$ in the interval $(x, x + x^\theta)$*

The most recent version of this theorem is due to Baker, Harman and Pintz [2] who prove $\theta > 0.55$. We prove Fogels' original result with θ and η given explicitly. We follow Fogels' method for calculating θ, with the difference being we have the advantage of a much simpler proof of the Linnik log-free zero density theorem, which makes calculating constants much easier. The following lemma is taken from Fogels [4].

Lemma 6.0.4 *Let q and l be integers, y and x be large positive real numbers Then*

$$S = \phi(q) \sum_{n \equiv l \bmod q} \Lambda(n) \exp\left(\frac{-1}{4y} \log^2\left(\frac{n}{x}\right)\right) =$$
$$i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y) ds. \tag{6.0.1}$$

We begin the calculation of Fogels' constant by moving the contour in (6.0.1) to the line $\Re(s) = -1$. This gives the following

$$S = i\sqrt{\frac{y}{\pi}} \int_{-1-i\infty}^{-1+i\infty} \sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y)\, ds$$
$$+2\pi i \sum_{-1<\sigma<2} Res\Big(\sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y)\Big). \tag{6.0.2}$$

Our sum has a residue from zeros and poles of the L-function, since there is a pole at $s = 1$, and we have also passed all nontrivial zeros in the critical strip $s = \rho$. These are given by the following

$$\underset{s=1}{Res}\left(\sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y)\right) = -x \exp(y)$$

and

$$\underset{s=\rho}{Res}\left(\sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y)\right) = \sum_{\chi \bmod q} \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ -1<\Re(\rho)<1}} \overline{\chi}\, x^\rho \exp(\rho^2 y).$$

We aim to show that the integral in (6.0.2) is negligible, and that the residue terms give a main term in the formula for S. Let

$$I = \int_{-1-i\infty}^{-1+i\infty} \Big| \sum_{\chi \bmod q} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} x^s \exp(s^2 y) ds \Big|$$

then by Lemma 5.3.13 we have

$$\begin{aligned} I &\ll \int_{-\infty}^{+\infty} \frac{\exp(y(1-t^2))}{x} q \log(q(|t|+2))dt \\ &\ll \int_{0}^{+\infty} \frac{\exp(y(1-t^2))}{x} q \log(q(t+2))dt \\ &\ll \frac{\exp(y)q}{x\sqrt{y}} \log 2q \\ &\ll x^{-1/2(1+\theta_1)} q \log q, \end{aligned}$$

where we have chosen $y = x^{\theta_1 - 1}$ for some $\theta_1 \in (0,1)$. This gives the following formula for S:

$$\begin{aligned} S &= 2\sqrt{\pi y}\Big(x\exp(y) - \sum_{\chi \bmod q} \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ -1<\Re(\rho)<1}} \overline{\chi}(l)\, x^{\rho} \exp(\rho^2 y)\Big) \\ &\quad + O(x^{-1/2(1+\theta_1)} q \log q). \end{aligned}$$

We estimate the sum in S by taking a subset of the zeros that are in a rectangle $G = \{s = \sigma + it \mid \sigma \in [0,1],\ |t| \le T\}$ into the main term and the contribution from the remaining zeros into the error term. Firstly we estimate the contribution from the zeros outside the rectangle G. This is given by the following sum

$$\sum_{\chi \bmod q} \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |\Im(\rho)|>T}} \overline{\chi}(l) x^{\rho} \exp(\rho^2 y). \tag{6.0.3}$$

We estimate the inner sum in (6.0.3) with the use of Lemma 5.3.9 which tells us that since the zeros under consideration in (6.0.3) have $\Im(\rho) > 0$ we have

$$\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ |\Im(\rho)|>T}} \overline{\chi}(l) x^{\rho} \exp(\rho^2 y) \ll x^{1-\delta_0} \sum_{\substack{\rho \\ |t|>T}} \exp(-t^2 y), \tag{6.0.4}$$

where

$$\delta_0 = \frac{1}{10 \log qT}.$$

We estimate the sum on the right hand side of (6.0.4) with Lemma 5.3.4 as follows

$$\sum_{\substack{\rho \\ |t|>T}} \exp(-t^2 y) = \Big(\sum_{\substack{\rho \\ T<|t|\le m}} + \sum_{\substack{\rho \\ m<|t|\le m+1}} + \ldots \Big) \exp(-t^2 y), \quad (6.0.5)$$

where $T < m \le [T] + 1$ and m is an integer. So by Lemma 5.3.4 we know that the sum in (6.0.4) is

$$\begin{aligned} & O\Big(\exp(-T^2 y) \log(q(m+2)) + \sum_{k \ge m} \exp(-k^2) \log(q(k+2)) \Big) \\ = \; & O(\exp(T + 1 - T^2 y) \log q). \end{aligned}$$

When we substitute this back into the the expression (6.0.3), we get an extra factor $\phi(q)$ from the summation over the characters mod$q$. So that the contribution to S from zeros outside the rectangle G is

$$\ll \phi(q) x^{1-\delta_0} \exp(T + 1 - T^2 y) \log q. \quad (6.0.6)$$

We now make the following substitutions. Let

$$T = x^{1/c_1}, \quad (6.0.7)$$

$$x \ge q^{c_1}, \quad (6.0.8)$$

$$y = x^{\theta_1 - 1}, \quad (6.0.9)$$

so that (6.0.4) becomes

$$\begin{aligned} & O(x^{1/c_1 + \epsilon + 1 - \delta_0} \exp(x^{1/c_1} + 1 - x^{2/c_1 + \theta_1 - 1})) \\ = \; & O(x^2 \exp(x^{1/c_1} - x^{2/c_1 + \theta_1 - 1})). \quad (6.0.10) \end{aligned}$$

We require the contribution from (6.0.10) to be $\ll x^{-2}$ which yields the following condition on θ_1:

$$\theta_1 > 1 - \frac{1}{c_1}. \tag{6.0.11}$$

The following representation of S has now been established

$$\begin{aligned} S &= 2\sqrt{\pi y}\Big(x\exp(y) - \sum_{\chi \bmod q} \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ \rho\in G}} \overline{\chi}(l)x^{\rho}\exp(\rho^2 y) + O(x^{-2})\Big) \\ &\quad + O(x^{-1/2(1+\theta_1)}q\log qx). \end{aligned} \tag{6.0.12}$$

Now we need to estimate the contribution from the zeros ρ with $\rho \in G$. There are two possibilities: either the exceptional zero exists or it does not. We have to treat each case separately as the theory will not allow us to assume that the exceptional zero might exist. Firstly we assume that the exceptional zero does not exist, and that the region in Lemma 5.3.9 is zero free. Secondly we assume that the region in Lemma 5.3.9 does contain a real zero, and then Lemma 5.3.12 implies that other zeros are even further into the critical strip than in the non-exceptional case.

6.1 The Nonexceptional Case

Here we assume that the exceptional zero does not exist. Hence by Lemma 5.3.9 there are no zeros in the region

$$1 - \frac{1}{10\log qt} \le \sigma \le 1, \quad |t| \le T.$$

By Theorem 5.7.1 there are

$$\le c_2(qT)^{c_3(n+1)/10\log qT} = c_2\exp\left(\frac{c_3(n+1)}{10}\right)$$

zeros, where we will choose the values of c_2 and c_3 within their constraints later, in the region

$$1-\frac{n+1}{10\log qT}\leq\sigma\leq 1-\frac{n}{10\log qT},\quad |t|\leq T.$$

We use this to bound the sum on the right hand side of (6.0.12). Let

$$S'=\Bigg|\sum_{\chi \bmod q}\sum_{\substack{\rho\\ L(\rho,\chi)=0\\ \rho\in G}}\overline{\chi}(l)x^{\rho}\exp(\rho^2 y)\Bigg|.$$

Then

$$\begin{aligned} S' &\leq x\exp(y)\sum_{\chi \bmod q}\sum_{\substack{\rho=1-\delta+it\\ L(\rho,\chi)=0\\ \rho\in G}}x^{-\delta} \\ &\leq c_2 x\exp(y)\sum_{n=1}^{\infty}\exp\left(\frac{-n}{10\log qT}\log x+\frac{c_3(n+1)}{10}\right). \end{aligned} \quad (6.1.13)$$

By (6.0.7) and (6.0.8) we have

$$x\geq (qT)^{c_1/2}\quad\Rightarrow\quad -\log x\leq -\frac{c_1}{2}\log qT,$$

which after substitution into the expression for S' gives

$$\begin{aligned} S' &\leq c_2 x\exp(y)\sum_{n=1}^{\infty}\exp\left(\frac{-c_1 n}{20}+\frac{c_3(n+1)}{10}\right) \\ &\leq c_2 x\exp\left(y+\frac{c_3}{10}\right)\sum_{n=1}^{\infty}\exp\left(n\Big(\frac{2c_3-c_1}{20}\Big)\right). \end{aligned} \quad (6.1.14)$$

The sum over n on the right hand side of (6.1.14) is a geometric progression with common ratio

$$\exp\left(\frac{2c_3-c_1}{20}\right).$$

Hence

$$S'\leq c_2 x\exp\left(y+\frac{c_3}{10}\right)\frac{\exp\left(\frac{2c_3-c_1}{20}\right)}{1-\exp\left(\frac{2c_3-c_1}{20}\right)},$$

and

$$\exp\left(\frac{2c_3 - c_1}{20}\right) < \frac{1}{2}$$

for

$$c_1 > 2c_3 + 20\log 2, \tag{6.1.15}$$

which we will arrange to be the case on our choice of c_1. So

$$S' \leq 2c_2 x \exp\left(y + \frac{c_3}{5} - \frac{c_1}{20}\right). \tag{6.1.16}$$

We require

$$S' < \frac{1}{2}x\exp(y),$$

so that the expression in brackets in (6.0.12) is greater than something positive in absolute value, ie

$$S \geq \sqrt{\pi y}x\exp(y) + O(x^{\theta_1 - 3} + x^{-1/2(1+\theta_1)}q\log qx).$$

In order to satisfy this condition we require

$$c_1 > 4c_3 + 20\log 4c_2. \tag{6.1.17}$$

In the sum S on the left hand side of (6.0.1), the exponential term in the summand gives weight to an interval of the form $(x - f(x), x + f(x))$ where

$$f(x) = \sqrt{b_1 x^{1+\theta_1}\log x} \tag{6.1.18}$$

for some constant $b_1 > 0$ to be made explicit later. To see the result of this weighting, we consider the size of S inside and outside the interval. Firstly suppose $n \in [x + f(x), 2x]$. Then $n = x + m$ where $m \in [f(x), x]$. By the Macluarin expansion of $\log(1+w)$ for $w \in (-1,1)$, we have

$$\begin{aligned}\log^2\left(\frac{n}{x}\right) &= \log^2\left(1 + \frac{m}{x}\right) \geq \frac{1}{2}\left(\frac{m}{x}\right)^2 \\ &\geq \left(\frac{\sqrt{b_1 x^{1+\theta_1}\log x}}{x}\right)^2 \\ &\geq b_1 x^{\theta_1 - 1}\log x. \end{aligned} \tag{6.1.19}$$

So the contribution to S for $n \in [x + f(x), 2x]$ is

$$\begin{aligned}
&\le \frac{\phi(q)}{q} \sum_{n \in [x+f(x),2x]} \log(n) \exp\left(\frac{-b_1}{4} x^0 \log x\right) \\
&\le x^{-b_1/4} \log x \sum_{n \in [x+f(x),2x]} 1 \\
&\le x^{1-b_1/4} \log x \\
&\le \frac{1}{x^2} \qquad (6.1.20)
\end{aligned}$$

for $b_1 > 12$. Now we consider the contribution to S from the terms with $n \in [1, x - f(x)]$, so $n = x - m$ where $m \in [f(x), x]$, in this case we have

$$\log^2\left(\frac{n}{x}\right) = \log^2\left(1 - \frac{m}{x}\right) \ge \frac{1}{2}\left(\frac{m}{x}\right)^2;$$

hence the argument follows the same as before. So the contribution to S for $n \in [1, x - f(x)] \cap [x + f(x), 2x]$ is

$$\le 2x^{1-b_1/4} \log x. \qquad (6.1.21)$$

It remains to estimate the contribution made by the interval $(2x, \infty)$. On this interval we have

$$\log\left(\frac{n}{x}\right) \ge b_2 x^{\theta_1 - 1} \log n$$

for x large enough, and some $b_2 > 0$, so that the contribution to S on the interval $(2x, \infty)$ is

$$\begin{aligned}
&\le \sum_{n > 2x} \log n \exp\left(\frac{-b_2}{4} x^0 \log n\right) \\
&\le \sum_{n > 2x} \frac{\log n}{n^{b_2}} \\
&\le \int_{2x}^{\infty} u^{-b_2/4+\epsilon} du \\
&\le \frac{(2x)^{1-b_2/4+\epsilon}}{b_2/4 - 1 - \epsilon} \le \frac{1}{x^2}
\end{aligned}$$

for $b_2 > 12$. Let $I = (x - f(x), x + f(x))$ then we have the following result

$$\phi(q) \sum_{\substack{n\in I \\ n\equiv l \bmod q}} \Lambda(n) \exp\left(\frac{-1}{4y}\log^2\left(\frac{n}{x}\right)\right) > b_3 x^{1/2+\theta_1/2} \tag{6.1.22}$$

for some $b_3 > \sqrt{\pi}$. The left hand side of (6.1.22) is

$$\leq x^{1/c_1} \sum_{\substack{n\in I \\ n\equiv l \bmod q}} \Lambda(n).$$

Hence

$$\sum_{\substack{n\in I \\ n\equiv l \bmod q}} \Lambda(n) > b_3 x^{1/2+\theta_1/2-1/c_1},$$

and so

$$\sum_{\substack{p\in I \\ p\equiv l \bmod q}} 1 > \frac{x^{1/2+\theta_1/2-1/c_1}}{\log x}, \tag{6.1.23}$$

where

$$I = (x - \sqrt{12x^{1+\theta_1}\log x}, x + \sqrt{12x^{1+\theta_1}\log x}).$$

Now if we shift the interval I so that we are looking at the interval

$$(x, x + 2\sqrt{12x^{1+\theta_1}\log x}),$$

then our new interval is of the right form for Fogels' Theorem with

$$x^\theta \geq \sqrt{12x^{1+\theta_1}\log x}, \tag{6.1.24}$$

which is satisfied for

$$\theta > 1/2 + \theta_1/2. \tag{6.1.25}$$

Thus we have explicitly proved Fogels' theorem under the assumption that there are no exceptional zeros.

6.2 The Exceptional Case

Now we suppose that the exceptional zero $\beta_1 = 1-\delta_1$ does exist. By Lemma 5.3.9 we know that β_1 is real and that

$$1 - \frac{1}{10\log qT} \leq \beta_1 < 1. \tag{6.2.26}$$

We take the contribution from the exceptional zero in (6.0.12) into the main term, M, and contributions from all other zeros go into the error term, E.

$$\begin{aligned}
M &= |x\exp(y) - \overline{\chi}_{\beta_1}(l)x^{\beta_1}\exp(\beta_1^2 y)| \\
&\geq x\Big(1 - \frac{x^{-\delta_1}}{\beta_1}\Big) \\
&\geq x(\beta_1 - x^{-\delta_1}) \\
&\geq x(1 - \delta_1 - (qT)^{-\delta_1 c_1/2}) \\
&\geq x\Big(\frac{\frac{\delta_1 c_1}{2}\log qT}{1 + \frac{\delta_1 c_1}{2}\log qT} - \delta_1\Big)
\end{aligned}$$

by (6.0.7) and (6.0.8). Also by (6.2.26) we have

$$\delta_1 \leq \frac{1}{10\log qT},$$

which gives

$$\begin{aligned}
M &\geq x\Big(\frac{\delta_1 c_1 \log qT}{2 + \delta_1 c_1 \log qT} - \delta_1\Big) \\
&\geq x\Big(\frac{\delta_1 c_1 \log qT}{20 + c_1} - \delta_1\Big) \\
&\geq \delta_1 x\Big(\frac{\log qT}{\frac{20}{c_1} + 1} - 1\Big) \\
&\geq \delta_1 x \frac{\log qT}{2(\frac{20}{c_1} + 1)}.
\end{aligned} \tag{6.2.27}$$

The contribution from all other zeros in (6.0.12) is estimated using the Deuring-Heilbronn phenomenon, Lemma 5.3.12. So by Lemma 5.3.12 there are no other zeros in the region

$$\sigma \geq 1 - \frac{|\log(1-\beta_1)\log qT|}{2\log qT} \qquad |t| < T.$$

If we put this result into the zero density theorem, Theorem 5.7.1 we find that for $n = 1, 2, 3, \ldots$ there are at most

$$\begin{aligned}
&\leq \; c_2 \exp\left(\frac{c_3(n+1)|\log(1-\beta_1)\log qT|}{2\log qT}\log qT\right) \\
&\leq \; c_2 \exp\left(\frac{c_3(n+1)|\log(1-\beta_1)\log qT|}{2}\right) \qquad (6.2.28)
\end{aligned}$$

zeros $\rho = \sigma + \imath t$ in the intervals

$$1 - \frac{(n+1)|\log(1-\beta_1)\log qT|}{2\log qT} \leq \sigma \leq 1 - \frac{n|\log(1-\beta_1)\log qT|}{2\log qT}, \quad |t| \leq T.$$

We wish to estimate the sum over all nonreal zeros in (6.0.12), call it S'', using (6.2.28) as follows:

$$\begin{aligned}
S'' &= \left|\sum_{\chi \bmod q} \sum_{\substack{\rho \\ \rho \in G\ R \\ L(\rho,\chi)=0}} x^\rho \exp(\rho^2 y)\overline{\chi}(l)\right| \\
&\leq \; x\exp(y) \sum_{\chi \bmod q} \sum_{\substack{\rho = 1-\delta+\imath t \\ \rho \in G\ R \\ L(\rho,\chi)=0}} x^{-\delta} \\
&\leq \; c_2 x\exp(y) \sum_{n=1}^{\infty} \exp\left(\frac{c_3(n+1)|\log(1-\beta_1)\log qT|}{2}\right. \\
&\quad \left. - \frac{n|\log(1-\beta_1)\log qT|}{2\log qT}\log x\right) \\
&\leq \; c_2 x \exp\left(y + \frac{c_3|\log(1-\beta_1)\log qT|}{2}\right) \times \\
&\quad \sum_{n=1}^{\infty} \exp\left(\frac{n|\log(1-\beta_1)\log qT|}{2}\left(c_3 - \frac{\log x}{\log qT}\right)\right).
\end{aligned}$$

From (6.1.17) we see that

$$\begin{aligned}
S'' \;\leq\; & c_2 x \exp\left(y + \frac{c_3|\log(1-\beta_1)\log qT|}{2}\right) \times \\
& \sum_{n=1}^{\infty} \exp\left(\frac{n|\log(1-\beta_1)\log qT|}{2}\left(c_3 - \frac{c_1}{2}\right)\right). \qquad (6.2.29)
\end{aligned}$$

We know that

$$\frac{1}{10\log qT} \geq \delta_1 \Rightarrow \frac{1}{10} \geq \delta_1 \log qT,$$

so

$$\log(1-\beta_1)\log qT = \log(\delta_1 \log qT) < 0,$$

which means that

$$|\log(1-\beta_1)\log qT| = -\log(1-\beta_1)\log qT.$$

Hence the expression on the right hand side of (6.2.29) is a geometric progression with common ratio

$$\begin{aligned}\exp\left(\frac{\log(1-\beta_1)\log qT}{2}\left(\frac{c_1}{2}-c_3\right)\right) &= (\delta_1 \log qT)^{c_1/4-c_3/2}\\ &\leq \left(\frac{1}{10}\right)^{c_1/4-c_3/2}\\ &\leq \frac{1}{2}\end{aligned}$$

for

$$c_1 \geq 2c_3 + \frac{4\log 2}{\log 10}, \tag{6.2.30}$$

and so

$$\begin{aligned}S'' &\leq 2c_2 x \exp\left(y + \frac{\log(\delta_1 \log qT)}{2}\left(\frac{c_1}{2} - 2c_3\right)\right)\\ &\leq 2c_2 x \exp(y)(\delta_1 \log qT)^{c_1/4-c_3}\\ &\leq c_2(2+\epsilon)x(\delta_1 \log qT)^{c_1/4-c_3}\end{aligned}$$

for some $\epsilon > 0$ and for x large enough. So in the light of (6.2.27) we require

$$\frac{\delta_1 \log qT}{2(2+20/c_1)} > c_2(2+\epsilon)(\delta_1 \log qT)^{c_1/4-c_3}. \tag{6.2.31}$$

We ensure the inequality (6.2.31) by imposing two separate inequalities which when satisfied will satisfy (6.2.31) for c_1. The first inequality is given by

$$\delta_1 \log qT > (\delta_1 \log qT)^{c_1/4-c_3}, \tag{6.2.32}$$

and this is satisfied when

$$c_1 > 4(c_3 + 1) \tag{6.2.33}$$

since $\delta \log qT < 1$. The second inequality is

$$\frac{1}{2(2 + 20/c_1)} > (2 + \epsilon)c_2,$$

which is satisfied for

$$c_1 > \frac{40(2 + \epsilon)}{1 - 4(2 + \epsilon)c_2}. \tag{6.2.34}$$

For c_1 satisfying (6.2.33) and (6.2.34) we have the inequality (6.2.31). Hence

$$S \geq b_4 x \delta_1 \log qT,$$

where

$$b_4 \leq \frac{1}{\frac{20}{c_1} + 1}.$$

So by the same argument that was used for the nonexceptional case we have

$$\sum_{\substack{p \in I \\ p \equiv l \bmod q}} 1 > \frac{b_5 x^{1-1/c_1} \delta_1 \log qT}{\log x}.$$

So the same result follows as in the nonexceptional case, but with different conditions on c_1, which yield a different θ. We choose the strongest of the conditions on θ so that both the nonexceptional and the exceptional cases hold.

6.3 Fogels' Constant

We need now to calculate a value for θ for which Theorem 6.0.2 holds. We have the following theorem.

Theorem 6.3.1 *For q large enough and any $x > q^{328}$ the interval $(x, x + x^{655/656})$ contains a prime $p \equiv l \bmod q$ as long as $(q, l) = 1$*

Proof. Out of (6.1.22), (6.1.17), (6.2.30), (6.2.33) and (6.2.34) the strongest condition on c_1 is (6.1.17). By our zero density Theorem 5.7.1 we have

$$c_2 > \frac{12\pi^2(604 + 2m)}{m^2}$$

and

$$c_3 > 3(1 + 4m)$$

substitution of these into condition (6.1.17) gives the following

$$c_1 > 12(1 + 4m) + 20(\log 48\pi^2 + \log(604 + 2m) - 2\log m). \quad (6.3.35)$$

We must find m so that c_1 can be chosen as small as possible. The expression in (6.3.35) is minimal when the derivative is zero, so we must solve the following equation

$$48 + 20\Big(\frac{2}{4698 + 2m} - \frac{2}{m}\Big) = 0, \quad (6.3.36)$$

which gives us the quadratic

$$12m^2 + 3619m - 1020 = 0 \quad (6.3.37)$$

Thus we find there is a zero of (6.3.37) for $m \approx 0.281582916$. Substitution of this value for m into (6.3.35) gives us $c_1 = 328$. then by (6.0.11) and (6.1.25) we have $\theta = 655/656$.

Part III

Farey Fractions With Prime Denominator

Chapter 7

Introduction

Dirichlet showed that any real number α can be approximated by a rational number a/q with

$$\left|\frac{a}{q} - \alpha\right| \leq \frac{1}{qQ},$$

for $1 \leq q \leq Q$. It is natural to ask whether there are results of this type with q restricted to certain sequences, such as the sequence of primes. However if $\alpha = 1/4$ and q is an odd prime

$$\left|\frac{a}{q} - \alpha\right| \geq \frac{1}{4q}.$$

We have a trivial obstacle, that there are no Farey fractions sufficiently near to $1/4$. This problem is due to the fact that the Farey sequence has some areas where Farey fractions are more dense than in others. When we are in an area where there are lots of Farey fractions, rationals are easily approximated We call such areas the minor arcs. When we are in an area where Farey fractions are less dense, for example near $1/4$, rationals are not as well approximated, these are the so called major arcs. The problem described above when we are near $1/4$ can be avoided if we redefine what we mean by distance. Let $\mathcal{F}(Q)$ be the Farey sequence of order Q and let $N_Q(\alpha, \beta)$ be the number of fractions of $\mathcal{F}(Q)$ in the interval $[\alpha, \beta]$ (if $\alpha \le \beta$) or $[\beta, \alpha]$ (if $\beta \le \alpha$). Then the above problem can be reworded in the following way, given a real number $\alpha \in (0, 1)$ and Q an integer, if we travel along the real line from α how many Farey fractions do we have to pass to be sure we have passed at least one with prime denominator? In other words what is the size of

$$\min_p N_Q(\alpha, a/p)? \tag{7.0.1}$$

We use Fogels' reformation of Linnik's theorem on small primes in arithmetic progressions and a special case of Vaughan's theorem on the distribution of αp modulo one for p prime.

7.1 Heuristics and Trivial Arguments

To my knowledge this problem has not before been considered, so it is necessary that we first try to construct some idea of what our answer should look like. I wrote a program which calculated the maximum number of Farey fractions one would have to pass to find one with prime denominator. The first version had complexity in the order of magnitude Q^4 for the Farey sequence of order Q, and as a result only went up to the Farey sequence of order 500. The second program I wrote had complexity in the order of magnitude Q^2, but at the current time the program contains too many bugs to have provided any results worth mentioning.

We begin by considering what we know to be true, then we construct a heuristic argument to find the best possible conjecture. We require the following Lemma taken from Huxley [12].

Lemma 7.1.1 *Let I be an interval of length Δ. Then*

$$\sum_{a/q\in\mathcal{F}(Q)} 1 \leq \Delta Q^2 + 1.$$

We constrict ourselves to considering primes $p \asymp Q$ otherwise our results do not work. For Q large enough there exists a prime $p \in (Q/2, Q]$, which appears in $\mathcal{F}(Q)$ as a denominator $\phi(p) = p - 1$ times. The Farey fractions with p as a denominator are spaced at a distance

$$\frac{1}{p} \asymp \frac{1}{Q}$$

apart. By Lemma 7.1.1 we can see that in between these Farey fractions with prime denominators there are at most N Farey fractions where

$$N \ll \frac{Q^2}{Q} \asymp Q.$$

So trivially we know that the answer is at most order of magnitude Q. But we should be able to do much better, as we only considered a very small subset set of the primes. The actual number of Farey fractions with prime denominator in $\mathcal{F}(Q)$ is given by

$$\begin{aligned}
\sum_{p\leq Q} \phi(p) &= \sum_{p\leq Q} p - \pi(Q) \\
&= \int_1^Q x d\pi(x) + O\Big(\frac{Q}{\log Q}\Big) \\
&= \Big[x\pi(x)\Big]_1^Q - \int_1^Q \pi(x)dx + O\Big(\frac{Q}{\log Q}\Big) \\
&= \frac{Q^2}{\log Q} - \int_1^Q \frac{x}{\log x} + O\Big(\int_1^Q \Big|\frac{x}{\log^2 x}\Big|dx + \frac{Q}{\log Q}\Big) \\
&= \frac{Q^2}{2\log Q} + O\Big(\frac{Q^2}{\log^2 Q}\Big)
\end{aligned}$$

by Riemann-Stieltjes integration and the prime number theorem. The best possible result would be obtained if all Farey fractions with prime denominators were equally spaced by some distance Δ, which is clearly not the case, but it helps to assume it true for our heuristic argument. Then if there were K Farey fractions with prime denominator we would have

$$\Delta(K+1) = 1.$$

Since we know the order of magnitude for K we can approximate Δ as follows

$$\Delta \asymp \frac{\log Q}{Q^2},$$

and so in light of Lemma 7.1.1 we can say that

$$N \asymp \log Q.$$

So we now have some idea of the range our answer should be in, we are looking for a result that is of the order of magnitude Q^θ where $\theta < 1$ but we know we can not do any better than $\log Q$. What we prove is the following theorem.

Theorem 7.1.2 *Let α be a real number in the interval $(0, 1)$. There exists a Farey fraction a/p such that $p \in (Q/2, Q)$ is prime and*

$$N_Q(\alpha, a/p) \ll Q^{1-1/1312} \log^8 Q.$$

7.1.1 Selberg's Sieve on the Farey Fractions

An alternative wording of the problem suggests a new approach. Given $\alpha \in [0, 1]$ can we find a number $\beta \in [0, 1]$ such that there is at least one Farey fraction with prime denominator in $[\alpha, \beta] \cap \mathcal{F}(Q)$ if $\alpha \le \beta$, or $[\beta, \alpha] \cap \mathcal{F}(Q)$ if $\alpha > \beta$? We would require an approximation of the number of Farey fractions with prime denominator in a sub-interval of $[0, 1]$. Let $N_{Q,p}(\alpha, \beta)$ be the number of Farey fractions between α and β that have prime denominators. If we knew $N_{Q,p}(\alpha,\beta)$ we could adjust β until our approximation of $N_{Q,p}(\alpha,\beta)$ was positive. Then we could use Lemma 7.1.1 to find an upper bound for $N_Q(\alpha,\beta)$. The best we can hope for in this direction is an upper bound for $N_{Q,p}(\alpha,\beta)$ which can be done using Selberg's sieve. We need the following Lemma.

Lemma 7.1.3 *For integers $a \geq 1$, $d \geq 1$ and $b \geq 1$, such that $(a,b)=1$ we have*

$$\sum_{\substack{x\leq X \\ ax+by\equiv 0 \bmod d}}\sum_{y\leq Y} 1 = \frac{XY}{d} + O(X+Y).$$

Proof. Solutions of $ax+by \equiv 0 \bmod d$ form a lattice of determinant d. Suppose this lattice has the basis $\underline{w}_1$, $\underline{w}_2$, where $\underline{w}_1 = (u_1, v_1)$ and $\underline{w}_2 = (u_2, v_2)$. So the solutions of $ax+by \equiv 0 \bmod d$ are lattice points with coordinates $(x,y) = m\underline{w}_1 + n\underline{w}_2$ for some integers m and n. One of our requirements is that

$$1 \leq x = mu_1 + nu_2 \leq X$$

and

$$1 \leq y = mu_1 + nu_2 \leq Y.$$

This represents a parallelogram in (m,n) space in which we need to count the number of vectors with integer coordinates. Divide up (m,n) into unit squares to form the fundamental lattice. Suppose N_1 squares are inside the parallelogram and N_2 squares are partly inside the parallelogram. Then if A is the area of the parallelogram we have $N_1 \leq A \leq N_1 + N_2$. We shall adopt the convention that each square belongs to the point in the lower left corner of the square. Suppose N_3 of the N_2 squares have their lattice points inside the parallelogram and N_4 do not. So $N_3 + N_4 = N_2$. The number of lattice points inside the parallelogram is therefore $N_1 + N_3$ and we have $N_1 \le A \le N_1 + N_3 + N_4$. If we imagine an arrow on the boundary of the parallelogram with an anticlockwise direction then we see that the boundary of the parallelogram enters each of the N_2 boxes on one side and leaves usually by another side. The number of vertical lines that get cut are $\le [B] + 1 \le B + 1$ where B is the breadth of the parallelogram. Each line gets cut at most twice so the number of lines the curve enters a new square along a vertical side is $\le 2B + 2$. Similarly at most $2H + 2$ squares are entered by the curve across a horizontal side, where H is the height of the parallelogram. Hence $N_2 \le 2B + 2H + 4$. Let $N = N_1 + N_3$, then we have

$$N - N_3 = N_1 \le A \le N + N_4,$$

which gives

$$|N - A| \le 2B + 2H + 4. \tag{7.1.2}$$

The area of the parallelogram is

$$A = \frac{XY}{|v_1 u_2 - u_1 v_2|}.$$

by a standard calculation. The area of the parallelogram in the lattice of determinant d is $|v_1 u_2 - u_1 v_2|$, so we have

$$A = \frac{XY}{d}. \tag{7.1.3}$$

By equations (7.1.2) and (7.1.3) we have

$$N = \frac{XY}{d} + O(X + Y)$$

which proves the result.

Now we can use Selberg's sieve to calculate an upper bound for $N_{Q,p}(\alpha, \beta)$. Given α and β, where we shall suppose that $\alpha < \beta$, we find Farey fractions a/b and c/d in $\mathcal{F}(Q)$, such that

$$\frac{a}{b} \le \alpha \le \beta \le \frac{c}{d}$$

and a/b and c/d are consecutive in $\mathcal{F}(\max(b,d))$. Farey fractions in the set $\mathcal{F}(Q) \cap [a/b, c/d]$ are of the form

$$\frac{ax+cy}{bx+dy},$$

where $(x, y) = 1$, and $bx + dy \le Q$. We use Selberg's sieve to sieve the set $A = \{bx + dy \le Q | (x, y) = 1\}$ for primes. The result is as follows.

Lemma 7.1.4 *The number of Farey fractions with prime denominator of $\mathcal{F}(Q)$ in the interval $[a/b, c/d]$, where a/b and c/d are consecutive in $\mathcal{F}(\max(b, d))$, is at most*

$$\frac{(2+\epsilon)Q^2}{bd\log Q},$$

for $\epsilon > 0$.

Now if we take the interval $[a/b, c/d]$ and assume that there are as many Farey fractions with prime denominator as lemma 7.1.4 will allow then, by lemma 7.1.1, roughly speaking the ratio of the number of Farey fractions to the number of Farey fractions with prime denominator is

$$\frac{Q^2/bd}{Q^2/bd\log Q} \asymp \log Q.$$

So that for every interval containing $\log Q$ Farey fractions, one should be prime.

Chapter 8

The Minor Arcs

The following theorem is due to Vaughan [23].

Theorem 8.0.5 *(Vaughan) Suppose that* $(a, q) = 1$, $|\alpha - a/q| \leq 1/q^2$, $H \geq 1$, $N \geq 1$. *Let*

$$S = \sum_{h \leq H} \Big| \sum_{n \leq N} \Lambda(n) e(\alpha h n) \Big|.$$

Then

$$S \ll HN \log^8 HN \Big(\frac{1}{q^{1/2}} + \frac{1}{N^{1/4}} + \Big(\frac{q}{HN} \Big)^{1/2} + \Big(\frac{1}{H^2 N} \Big)^{1/5} \Big),$$

where Λ *is von Mangoldt's function, and* $e(x) = exp(2\pi i x)$.

Using this result Vaughan proved the following corollary.

Corollary 8.0.6 *Suppose that* α *is irrational and* $||\gamma||$ *denotes the distance of* γ *from a nearest integer. Then there are infinitely many prime numbers* p *such that*

$$||\alpha p - \beta|| \ll p^{-1/4} \log^8 p.$$

We require a similar result to this, given in the following Corollary to Theorem 8.0.5.

Corollary 8.0.7 *Suppose α is a real number in $(0,1)$, Q is a large positive number, c_2, c_3 and c_4 are positive constants and a/q is a convergent to the continued fraction expansion for α with q in the range*

$$Q_1 = \frac{c_2 \log^{16} Q}{\delta^2} \le q \le \frac{c_3 \delta Q}{\log^{16} Q} = Q_2$$

where

$$\frac{c_4 \log^8 Q}{Q^{1/4}} \le \delta < \frac{1}{2}$$

Then there is a prime $p \in (Q/2, Q]$ for which

$$||\alpha p|| \ll \delta.$$

We will need the following lemma which is essentially Lemmas 8a, and 8b of Chapter 1 of Vinogradov [24].

Lemma 8.0.8 *Let $Y \ge 2$ be an integer and $|\alpha - a/q| \le 1/q^2$, where $(a,q) = 1$. Then for all real numbers β and positive integers N we have*

$$\sum_{n \le N} \min\left(Y, \frac{1}{||\alpha n + \beta||}\right) \le \left(1 + \frac{N}{q}\right)(3Y + 4q \log Y).$$

We can now proceed to the proof of Corollary (8.0.7).

Let $\rho(t) = [t] - t + 1/2$ be the rounding error function which is well known to have Fourier expansion

$$\rho(t) = \sum_{1 \le h \le |H|} \frac{e(ht)}{2\pi i h} + O\Big(\min\left(1, \frac{1}{H||t||}\right)\Big),$$

for t not an integer. Then for $\delta < 1/2$, the sum

$$\sum_{Q/2 < p \le Q} (\rho(\alpha p + \delta) - \rho(\alpha p - \delta) + 2\delta)$$

counts the exact number of primes in the range $(Q/2, Q]$ for which

$$||\alpha p|| < \delta.$$

Following Vaughan we shall count powers of primes with a logarithmic weight. So we require an estimate for the sum

$$\sum_{Q/2<n\le Q} \Lambda(n)(\rho(\alpha n+\delta)-\rho(\alpha n-\delta)+2\delta).$$

By the prime number theorem this sum reduces to

$$\begin{aligned} & 2\delta Q+\sum_{Q/2<n\le Q} \Lambda(n)(\rho(\alpha n+\delta)-\rho(\alpha n-\delta))+O\Big(\frac{\delta Q}{\log Q}\Big) \\ = & \ 2\delta Q+\Sigma_1+O\Big(\frac{\delta Q}{\log Q}\Big). \end{aligned}$$

We estimate Σ_1 using the Fourier expansion of the rounding error function, which gives

$$\Sigma_1=\sum_{Q/2<n\le Q}\ \sum_{1\le h\le |H|}\Big(\frac{e(h(\alpha n+\delta))-e(h(\alpha n-\delta))}{2\pi i h}\Big)+O\Big(\Sigma_2\Big),$$

where by Lemma 8.0.8 we have

$$\begin{aligned} \Sigma_2 &= \frac{1}{H}\sum_{n\le Q}\min\Big(H,\frac{1}{||\alpha n+\delta||}\Big) \\ &\le \frac{1}{H}\Big(1+\frac{Q}{q}\Big)(3H+4q\log H). \end{aligned}$$

Now

$$\begin{aligned} \Sigma_1+O(\Sigma_2) &= \sum_{Q/2<n\le Q}\Big(\sum_{1\le h\le |H|}\frac{\Lambda(n)e(\alpha h n)}{2\pi i h}(e(h\delta)-e(-h\delta))\Big) \\ &= \sum_{Q/2<n\le Q}\Big(\sum_{1\le h\le |H|}\frac{\Lambda(n)e(\alpha h n)}{2\pi i h}\int_{-\delta}^{\delta}2\pi i h e(ht)dt\Big) \\ &\ll \delta\sum_{h\le H}\Big|\sum_{Q/2<n\le Q}\Lambda(n)e(\alpha h n)\Big| \\ &\ll \delta HQ\log^8(HQ)\Big(\frac{1}{q^{1/2}}+\frac{1}{Q^{1/4}}+\Big(\frac{q}{HQ}\Big)^{1/2}+\frac{1}{(H^2Q)^{1/5}}\Big) \end{aligned}$$

by Theorem 8.0.5. On choosing $H=1/\delta$ we obtain the following

$$\Sigma_1\ll Q\log^8 Q\Big(\frac{1}{q^{1/2}}+\frac{1}{Q^{1/4}}+\Big(\frac{\delta q}{Q}\Big)^{1/2}+\Big(\frac{\delta^2}{Q}\Big)^{1/5}\Big),$$

and

$$\Sigma_2 \ll 1 + \frac{Q}{q} + \delta^{1-\epsilon}\Big(q + \frac{Q}{q}\Big),$$

for some $\epsilon > 0$. This gives

$$\sum_{\substack{Q/2<n\leq Q \\ ||\alpha n||\leq\delta}} \Lambda(n) \;=\; 2\delta Q + O\Big(\frac{\delta Q}{\log Q} + \Sigma_1\Big).$$

Hence we need to choose δ so that $\Sigma_1 \ll \delta Q$, the order of magnitude of the expected number of primes. This yields the three following conditions

$$\begin{aligned} q &\geq \frac{c_2 \log^{16} Q}{\delta^2}, \\ q &\leq \frac{c_3 \delta Q}{\log^{16} Q}, \\ \delta &\geq \frac{c_4 \log^8 Q}{Q^{1/4}}, \end{aligned}$$

stated in the Corollary.

To obtain his Corollary Vaughan chose the least δ, which is

$$\delta = \frac{c_4 \log^8 Q}{Q^{1/4}}$$

We would also like δ to be as small as possible but this gives us a much smaller measure for the set of α for which Corollary 8.0.7 holds. We defer our choice of δ until we have treated all such $\alpha \in (0, 1)$.

We have the following corollary.

Corollary 8.0.9 *Suppose α and δ satisfy the conditions of Corollary (8.0.7) and a/p is a Farey fraction of $\mathcal{F}(Q)$ with p a prime in $(Q/2, Q]$. Then*

$$N_Q(\alpha, a/p) \ll \delta Q + 1.$$

Proof. From Corollary 8.0.7 we have $||\alpha p|| < \delta$ which means there exists some integer a such that

$$|\alpha p - a| \;<\; \delta$$

$$\Rightarrow \left|\alpha - \frac{a}{p}\right| \ < \ \frac{\delta}{p} \leq \frac{2\delta}{Q}$$

and by Lemma 7.1.1 we have

$$\begin{aligned} N_Q(\alpha, a/p) \ &\leq \ \frac{2\delta}{Q}Q^2 + 1 \\ &\leq \ 2\delta Q + 1. \end{aligned}$$

Chapter 9

The Major Arcs

Lemma 9.0.10 *Suppose $\alpha \in (0,1)$ is a real number with two succesive convergents in its continued fraction expansion a_r/q_r and a_{r+1}/q_{r+1} where*

$$q_r \ll \frac{\log^{16} Q}{\delta^2}, \tag{9.0.1}$$

and

$$Q > q_{r+1} \geq \frac{\delta Q}{\log^{16} Q}, \tag{9.0.2}$$

where

$$\frac{1}{2} > \delta \gg \frac{\log^8 Q}{Q^{1/1312}},$$

Then there is a prime number $p \in (Q/2, Q]$ for which

$$||\alpha p|| \leq \delta.$$

Proof. For simplicity we shall assume that r is even, then the proof for r odd follows similarly.

Since a_r/q_r and a_{r+1}/q_{r+1} are succesive convergents of α, we have

$$\frac{a_r}{q_r} < \alpha < \frac{a_{r+1}}{q_{r+1}}.$$

Farey fractions in

$$\left(\frac{a_r}{q_r}, \frac{a_{r+1}}{q_{r+1}}\right) \cap \mathcal{F}(Q),$$

are of the form

$$\frac{a_r x + a_{r+1} y}{q_r x + q_{r+1} y},$$

where $(x, y) = 1$ and $q_r x + q_{r+1} y \leq Q$, (simple property of the Farey sequence, see Huxley [12] Lemma 1.2.2). We choose $y = y_0$ so that

$$\tfrac{Q}{2} \leq q_{r+1} y_0 < Q. \tag{9.0.3}$$

Now we consider fractions of the form

$$\frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0},$$

with $(x, y_0) = 1$ and $q_r x + q_{r+1} y_0 \leq Q$. The denominators form a short segment of an arithmetic progression within which we require a prime number. As x increases we could be moving further away from α so we require an upper bound for x which forces the following inequality

$$\left|\alpha - \frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0}\right| \ll \frac{\delta}{Q}.$$

Firstly we need to consider the case where

$$\frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0} \in \left(\frac{a_r}{q_r}, \alpha\right). \tag{9.0.4}$$

We have the following inequality

$$\begin{aligned}
\left|\alpha - \frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0}\right| &\leq \left|\frac{a_{r+1}}{q_{r+1}} - \frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0}\right| \\
&= \frac{x}{q_{r+1}(q_r x + q_{r+1} y_0)} \\
&\leq \frac{2x}{q_{r+1} Q} \\
&\ll \frac{x \log^{16} Q}{\delta Q^2},
\end{aligned}$$

by (9.0.2) and (9.0.3). So we must have

$$\frac{x \log^{16} Q}{\delta Q^2} \ll \frac{\delta}{Q},$$

which gives the following upper bound for x

$$x \ll \frac{\delta^2 Q}{\log^{16} Q}. \tag{9.0.5}$$

Now we suppose that

$$\frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0} \in \left(\alpha, \frac{a_{r+1}}{q_{r+1}}\right).$$

Then we have

$$\begin{aligned} \left|\alpha - \frac{a_r x + a_{r+1} y_0}{q_r x + q_{r+1} y_0}\right| &\leq \left|\alpha - \frac{a_{r+1}}{q_{r+1}}\right| \\ &\leq \frac{1}{q_{r+1}^2} \ll \frac{1}{Q^2} \end{aligned}$$

by (9.0.3). Then

$$\frac{1}{Q^2} \ll \frac{\delta}{Q}$$

for $\delta \gg Q^{-1}$, which is the case, so we only need to consider the case where (9.0.4) holds. By (9.0.5) we have an arithmetic progression $l \bmod q_r$ of length L, where

$$L \asymp \frac{\delta^2 Q}{\log^{16} Q}. \tag{9.0.6}$$

Theorem 5.7.1 states that this arithmetic progression will contain a prime if $q_r^{328} \leq Q$ and $L \ll Q^{655/656}$. The condition $q_r^{328} \leq Q$ forces

$$\delta \gg \frac{\log^8 Q}{Q^{1/656}},$$

by (9.0.1 and the condition $L \gg Q^{655/656}$ becomes

$$\delta \gg \frac{\log^8 Q}{Q^{1/1312}},$$

by (9.0.6). We choose

$$\begin{aligned} \delta &= \max\left(\frac{\log^8 Q}{Q^{1/656}}, \frac{\log^8 Q}{Q^{1/1312}}\right) \\ &= \frac{\log^8 Q}{Q^{1/1312}}. \end{aligned}$$

So we have the following Corollary equivalent to Corollary 8.0.9.

Corollary 9.0.11 *Suppose α satisfies the conditions of Lemma* 9.0.10 *and a/p is a Farey fraction of $\mathcal{F}(Q)$ with p a prime in $(Q/2, Q]$. Then*

$$N_Q(\alpha, a/p) \ll \delta Q + 1,$$

for

$$\delta \gg \frac{\log^8 Q}{Q^{1/1312}}.$$

Proof. The proof follows from Lemma 9.0.10 along the same lines as Corollary 8.0.9.

9.1 A subcase of the Major Arcs

Now we need to consider the case where $Q \leq q_{r+1}$. We have the following Lemma.

Lemma 9.1.1 *If α has two successive convergents a_r/q_r and a_{r+1}/q_{r+1} where*

$$q_r \ll \frac{\log^{16} Q}{\delta^2}, \tag{9.1.7}$$

and $q_{r+1} \geq Q$ then the interval (α, β) or (β, α) contains a Farey fraction, a/p where p is a prime, of $\mathcal{F}(Q)$ where

$$N_Q(\alpha, \beta) \ll \delta Q + 1,$$

for

$$\delta \gg \frac{\log^8 Q}{Q^{1/1312}}$$

Proof. Instead of looking at Farey fractions of $\mathcal{F}(Q)$ between a_r/q_r and a_{r+1}/q_{r+1} as we did in Lemma 9.0.10, we now look at Farey fractions of $\mathcal{F}(Q)$ between a_r/q_r and a_{r-1}/q_{r-1}. Consider the sequence of rationals

$$\frac{a_r x + a_{r-1}}{q_r x + q_{r-1}};$$

it is tending towards the closest fraction of $\mathcal{F}(Q)$ to a_{r+1}/q_{r+1}. Choose X so that

$$\begin{aligned} q_r X + q_{r-1} &\le Q, \\ q_r(X+1) + q_{r-1} &\ge Q; \end{aligned}$$

then

$$\begin{aligned} X &= \frac{Q - q_{r-1}}{q_r} + O(1) \\ &= \frac{Q}{q_r} + O(1) \\ &\asymp \frac{\delta^2 Q}{\log^{16} Q}, \end{aligned} \tag{9.1.8}$$

by (9.1.7), and

$$N_Q\Big(\alpha, \frac{a_r X + a_{r-1}}{q_r X + q_{r-1}}\Big) = 0.$$

Now we choose x so that

$$\frac{Q}{2} \le q_r x + q_{r-1} \le \frac{3Q}{4}$$

and $(X, x) = 1$. We have

$$\left|\frac{a_r X + a_{r-1}}{q_r X + q_{r-1}} - \frac{a_r x + a_{r-1}}{q_r x + q_{r-1}}\right| = \frac{|X - x|}{(q_r X + q_{r-1})(q_r x + q_{r-1})}$$

$$\begin{aligned}
&\le \frac{|X-x|}{(q_r x + q_{r-1})^2}\\
&\le \frac{4|X-x|}{Q^2}\\
&\ll \frac{\delta^2}{Q\log^{16} Q}\\
&\ll \frac{\delta}{Q}.
\end{aligned}$$

So by Theorem 5.7.1 there is a Farey fraction with prime denominator in the interval

$$\left(\frac{a_r X + a_{r-1}}{q_r X + q_{r-1}}, \frac{a_r x + a_{r-1}}{q_r x + q_{r-1}}\right)$$

provided that

$$\frac{\delta^2 Q}{\log^{16} Q} \gg Q^{655/656},$$

by (9.1.8), and

$$\left(\frac{\log^{16} Q}{\delta^2}\right)^{328} \ll Q,$$

by (9.1.7). As in the previous major arcs case we have

$$\delta \ll \frac{\log^8 Q}{Q^{1/1312}}.$$

Now we see that

$$N_Q\left(\alpha, \frac{a_r x + a_{r-1}}{q_r x + q_{r-1}}\right) = N_Q\left(\frac{a_r x + a_{r-1}}{q_r x + q_{r-1}}, \frac{a_r X + a_{r-1}}{q_r X + q_{r-1}}\right), \tag{9.1.9}$$

and the proof follows.

Bibliography

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer-Verlag, New York, 1976.

[2] R. C. Baker, G. Harman, and J. Pintz. The exceptional set for Goldbach's problem in short intervals. *Sieve Methods, Exponential Sums, and their Applications in Number Theory. London Math. Soc. Lecture Note Ser.*, 237:1–54, 1995.

[3] H. Davenport. *Multiplicative Number Theory.* Markham, Chicago, 1967.

[4] E. Fogels. On the zeroes of L-functions. *Acta Arith.*, 11:67–96, 1965.

[5] S. W. Graham. *Applications of Sieve Methods.* PhD thesis, 1977.

[6] S. W. Graham and G. Kolesnik. *van der Corput's Method of Exponential Sums.* Cambridge Univ. Press, 1991.

[7] G. Greaves. *Sieves in Number Theory.* Springer-Verlag, Ergebnisse der Mathematik und ihrer Grenzgebiete, 2001.

[8] D. R. Heath-Brown. Prime numbers in short intervals and a generalized Vaughan identity. *Canad. J. Math.*, 34:1365–1377, 1982.

[9] D. R. Heath-Brown. The Piatecki-Shapiro prime number theorem. *Journal of Number Theory*, 16:242–266, 1983.

[10] D. R. Heath-Brown. Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression. *Proc. London Math. Soc.*, 64:265–338, 1992.

[11] M. N. Huxley. *The Distribution of Prime Numbers.* Clarendon Press, Oxford, 1972.

[12] M. N. Huxley. *Area, Lattice Points and Exponential Sums.* Oxford Science Publications, New York, 1996.

[13] H. Iwaniec and E. Kowalski. *Analytic Number Theory.* American Mathematical Society, Rhode Island, 2004.

[14] M. Jutila. On two theorems of Linnik concerning the zeroes of Dirichlet's L-functions. *Ann. Acad. Sci. Fennicae*, 458:?, 1969.

[15] M. Jutila. A new estimate for Linnik's constant. *Ann. Acad. Sci. Fennicae*, 471:1–32, 1970.

[16] M. Jutila. On Linnik's constant. *Math. Scand.*, 41:45–62, 1977.

[17] H. Q. Liu and J. Rivat. On the Piatecki-Shapiro Prime Number Theorem. *Bull.London Math. Soc.*, 24:143–217, 1992.

[18] H. L. Montgomery. *Topics in Multiplicative Number Theory.* Springer-Verlag, Berlin, 1971.

[19] H. L. Montgomery and J. D. Vaaler. A further generalization of Hilbert's inequality. *Mathematika*, 45:35–39, 1999.

[20] I. I. Piatecki-Shapiro. On the distribution of prime numbers in sequences of the form $[f(n)]$. *Mat. Sbornik*, 33:559–566, 1953.

[21] K. Prachar. *Primzahlverteilung*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[22] E. C. Titchmarsh. *The Theory of Functions*. Oxford Univ. Press, 1932.

[23] R. C. Vaughan. On the distribution of αp modulo one. *Mathematatika*, 24:135–141, 1977.

[24] I. M. Vinogradov. *The Method of Trigonometrical Sums in the Theory of Numbers, translated from the Russian, revised and annotated by K. F. Roth and A. Davenport.* Interscience Publishers, London, 1954.