Limit theorems for additive functionals of stationary fields, under integrability assumptions on the higher order spectral densities*

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November 12, 2014

Abstract

We obtain central limit theorems for additive functionals of stationary fields under integrability conditions on the higher-order spectral densities. The proofs are based on the Hölder-Young-Brascamp-Lieb inequality.

AMS 2000 Classification: 60F05, 62M10, 60G15, 62M15, 60G10, 60G60
Keywords: integral of random fields, asymptotic normality, higher-order spectral densities, Hölder-Young-Brascamp-Lieb inequality.

1 Introduction

The problem. Consider a real measurable stationary in the strict sense random field X_t, $t \in \mathbb{R}^d$, with $\mathbb{E}X_t = 0$, and $\mathbb{E}|X_t|^k < \infty$, $k = 2, 3, \ldots$.

Let the random field X_t be observed over a sequence K_T of increasing dilations of a bounded convex set K of positive Lebesgue measure $|K| > 0$, containing the origin, i.e.

$$K_T = TK, \quad T \to \infty.$$

Note that $|K_T| = T^d|K|$.

We investigate the asymptotic normality of the integrals

$$S_T = \int_{t \in K_T} X_t dt \tag{1}$$

*Partly supported by the Welsh Institute of Mathematics and Computational Sciences and the Commission of the European Communities grant PIRSES-GA-2008-230804 within the programme 'Marie Curie Actions'.

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and the integrals with some weight function

$$S_T^w = \int_{t \in K_T} w(t) X_t dt \tag{2}$$

as $T \to \infty$, without imposing any extra assumption on the structure of the field such as linearity, etc.

Motivation. In the simplest cases of Gaussian or moving average processes, central limit theorems for sums (1), (2) have been often derived by the method of moments, using explicit computations involving the spectral densities.

We generalize this line of research, establishing central limit theorems for S_T and S_T^w , appropriately normalized, by the method of moments. Namely, we represent the cumulants of the integrals (1) and (2) in the spectral domain, and evaluate their asymptotic behavior using analytic tools provided by harmonic analysis. All conditions needed to prove the results will be concerned with integrability of the spectral densities of second and higher orders.

For further discussion of different approaches for derivation of CLT for stationary processes and fields, see for example [6], [20], [29].

Assumption A: We will assume throughout the existence of all order cumulants $c_k(t_1, t_2, ..., t_k) = cum_k\{X_{t_1,...,}X_{t_k}\}$ for our stationary random field X_t, and also that they are representable as Fourier transforms of “cumulant spectral densities”
$f_k(\lambda_1, ..., \lambda_{k-1}) \in L_1(\mathbb{R}^{d(k-1)})$, $k = 2, 3, ...$, i.e:

$$c_k(t_1, t_2, ..., t_k) = c_k(t_1 - t_k, .., t_{k-1} - t_k, 0) =$$

$$= \int_{(\lambda_1,...,\lambda_{k-1}) \in \mathbb{R}^{d(k-1)}} e^{i \sum_{j=1}^{k-1} \lambda_j (t_j - t_k)} f_k(\lambda_1, ..., \lambda_{k-1}) \, d\lambda_1 ... d\lambda_{k-1}.$$

Note: The functions $f_k(\lambda_1, ..., \lambda_{k-1})$ are symmetric and may be complex valued in general.

The Hölder-Young-Brascamp-Lieb (HYBL) inequality. We are able to treat here general stationary fields X_t, by making use of the powerful HYBL inequality. The computation of the cumulants of S_T (or S_T^w) in the spectral domain leads to a certain kind of convolutions of spectral densities with particular kernel functions (see formulas for the cumulants (5) and (47) below). Similar convolutions have been studied in the series of papers [1] - [6], under the name of Fejer matroid/graph integrals.

Estimates for this kind of convolutions follow from the Hölder-Young-Brascamp-Lieb inequality which, under prescribed conditions on the integrability indices for a set of functions $f_i \in L_{p_i}(S, d\mu)$, $i = 1, ..., n$, allows to write upper bounds for integrals of the form

$$\int_{S^m} \prod_{i=1}^{k} f_i(l_i(x_1, ..., x_m)) \prod_{j=1}^{m} \mu(dx_j) \tag{3}$$

with $l_i : S^m \to S$ being linear functionals, and where S may be either torus $[-\pi, \pi]^d$, Z^d, or $\mathbb{R}^d$ endowed with the corresponding Haar measure $\mu(dx)$.

An even more powerful tool is provided by the nonhomogeneous Hölder-Young-Brascamp-Lieb inequality, which covers the case when the above functions f_i are defined over the spaces of different dimensions: $f_i : S^{n_i} \to \mathbb{R}$ (see Appendix A).

Contents: We state limit theorems for the integrals (1) and (2) in Sections 2 and 5 respectively, with discussion of the assumptions used and of some possible applications. The example of Gaussian fields is discussed in Section 3, and an invariance principle provided in Section 4. The Hölder-Young-Brascamp-Lieb inequality used to prove our results is presented in Appendix A.

2 Main results and discussion

Given a sequence K_T of increasing dilations of a bounded convex set K of positive Lebesgue measure $|K| > 0$, containing the origin, let us consider the uniform distribution on K_T with the density

$$p_{K_T}(t) = \frac{1}{|K_T|} 1_{\{t \in K_T\}}, \ t \in \mathbb{R}^d,$$

and characteristic function

$$\phi_T(\lambda) = \int_{\mathbb{R}^d} p_{K_T}(t) e^{it\lambda} dt = \frac{1}{|K_T|} \int_{K_T} e^{it\lambda} dt, \ \lambda \in \mathbb{R}^d.$$

Define the Dirichlet type kernel

$$\Delta_T(\lambda) = \int_{t \in K_T} e^{it\lambda} dt = |K_T| \phi_T(\lambda), \ \lambda \in \mathbb{R}^d. \tag{4}$$

The cumulant of order $k \geq 2$ of the normalized integral S_T is of the form

$$\begin{aligned} I_T^{(k)} &= cum_k \left\{ \frac{S_T}{T^{d/2}}, ..., \frac{S_T}{T^{d/2}} \right\} \\ &= \frac{1}{T^{dk/2}} \int_{t_k \in K_T} ... \int_{t_1 \in K_T} c_k(t_1 - t_k, .., t_{k-1} - t_k, 0) dt_1 ... dt_k \\ &= \frac{1}{T^{dk/2}} \int_{(\lambda_1, ..., \lambda_{k-1}) \in \mathbb{R}^{d(k-1)}} f_k(\lambda_1, ..., \lambda_{k-1}) \\ &\quad \times \Delta_T(\lambda_1) ... \Delta_T(\lambda_{k-1}) \Delta_T \left(-\sum_{i=1}^{k-1} \lambda_i \right) d\lambda_1 ... d\lambda_{k-1}, \end{aligned} \tag{5}$$

where $\Delta_T(\lambda)$ is the Dirichlet type kernel (4).

We will need the following assumption on $\Delta_1(\lambda) = \int_{t \in K} e^{it\lambda} dt, \ \lambda \in \mathbb{R}^d$.

Assumption K: The bounded convex set K is such that:

$$C_p(K) := \|\Delta_1(\lambda)\|_p = \left(\int_{\mathbb{R}^d} |\Delta_1(\lambda)|^p d\lambda \right)^{1/p} < \infty, \quad \forall p > p_* \geq 1.$$

Remark 1 Assumption K and scaling imply

$$\|\Delta_T(\lambda)\|_p = T^{d(1-1/p)} C_p(K). \tag{6}$$

Remark 2 The constants $C_p(K)$ and p_* in Assumption K depend on Gaussian curvature of the set K. This fact goes back to Van der Corput when $d = 2$ – see Herz (1962), Sadikova (1966), and Stein (1986) for extensions and further references. Note that estimation of the modulus of the characteristic functions of the uniform distributions on convex sets is an active area of ongoing research. Important recent results have been obtained, e.g., in [24], [32], [33] (see also references therein). In particular, the approach developed in [33] allows to consider sets with C^1-smooth boundary (and does not use any arguments related to the curvature of the set).

The explicit formula for $C_p(K)$ when K is a cube: $K = [-1/2, 1/2]^d$, is known: $C_p(K) = C_p^d$, where $C_p = \left(2\int_R |\frac{\sin(z)}{z}|^p dz\right)^{\frac{1}{p}}, \quad \forall p > 1$. Note that in this case $p_* = 1$, and $C_{p_1} > C_{p_2}$ for $p_1 < p_2$. For a ball $K_T = B_T = \{t \in R^d : \|t\| \le T/2\}$ it is known that

$$\Delta_T(\lambda) = \int_{B_T} e^{it\lambda} dt = \left(2\pi \frac{T}{2}\right)^{\frac{d}{2}} J_{d/2}\left(\|\lambda\| \frac{T}{2}\right) / \|\lambda\|^{d/2}, \quad \lambda \in \mathbb{R}^d,$$

where $J_\nu(z)$ is the Bessel function of the first kind and order ν, and

$$C_p(K) = (2\pi)^{\frac{d}{2}}\ 2^{-d(1-\frac{1}{p})} |s(1)|^{1/p} \left(\int_0^\infty \rho^{d-1} \left|\frac{J_{\frac{d}{2}}(\rho)}{\rho^{d/2}}\right|^p d\rho\right)^{1/p}, \ p > \frac{2d}{d+1},$$

where $|s(1)|$ is the surface area of the unit ball in $\mathbb{R}^d$, $d \ge 2$. In this case $p_* = \frac{2d}{d+1} > 1, d \ge 2$.

The derivation of the central limit theorem for the integrals (1) will be based on the above estimates for the norms of functions $\Delta_T(\lambda)$ and the important property stated in the next lemma.

Lemma 1 *The function*

$$\Phi_T^{(2)}(\lambda) = \frac{1}{(2\pi)^d |K| T^d} \left|\int_{t \in K_T} e^{it\lambda} dt\right|^2 = \frac{1}{(2\pi)^d |K| T^d} \Delta_T(\lambda)\Delta_T(-\lambda),\ \lambda \in \mathbb{R}^d$$

possesses the kernel properties (or is an approximate identity for convolution):

$$\int_{\mathbb{R}^d} \Phi_T^{(2)}(\lambda) d\lambda = 1, \tag{7}$$

and for any $\varepsilon > 0$ when $T \to \infty$

$$\lim \int_{\mathbb{R}^d \setminus \varepsilon K} \Phi_T^{(2)}(\lambda) d\lambda = 0. \tag{8}$$

Proof.

The first relation (7) follows from (4) and Plancherel theorem. From Hertz(1962) and Sadikova(1966) one derives the following assertion: if K is a convex set and $\partial^{(d-1)}\{K\}$ is its surface area, then for any $\epsilon > 0$

$$\int_{\|\lambda\|>\epsilon} \left| \int_{t\in K} e^{it\lambda} dt \right|^2 d\lambda \leq \frac{8}{\epsilon} \partial^{(d-1)}\{K\} \left[\int_0^{\pi} \sin^d z dz \right]^{-1}$$

is valid. This inequality and homothety properties yield the second relation (8) (see also Ivanov and Leonenko (1986), p.25).

To estimate the second-order cumulant $I_T^{(2)}$ we will need one more assumption.

Assumption B: The second-order spectral density $f_2(\lambda)$ is bounded and continuous and

$$f_2(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\mathbb{E}X_t X_0)\, dt \neq 0.$$

Under the assumption B we obtain from (5) and Lemma 1 as $T \to \infty$

$$cum_2 \left\{ \frac{S_T}{T^{d/2}}, \frac{S_T}{T^{d/2}} \right\} = Var \left\{ \frac{S_T}{T^{d/2}} \right\} = (2\pi)^d |K| \int_{\mathbb{R}^d} \Phi_T^{(2)}(\lambda) f_2(\lambda) d\lambda \to$$

$$\to (2\pi)^d |K| f_2(0). \tag{9}$$

To evaluate the integral (5) for $k \geq 3$ we consider firstly the case $d = 1$ (with K taken to be the interval $[-1/2, 1/2]$) and apply the Hölder-Young-Brascamp-Lieb inequality (see (GH), Theorem A.1 in Appendix A).

Comparing (5), with $d = 1$, and l.h.s. of (GH), we have in (5): $H = \mathbb{R}^{k-1}$ and $k + 1$ functions $g_1 = g_2 = ... = g_k = \Delta_T$ on $\mathbb{R}$, $g_{k+1} = f_k$ on $\mathbb{R}^{k-1}$; linear transformations in our case are as follows: for $x = (x_1, ..., x_{k-1}) \in \mathbb{R}^{k-1}$ $l_j(x) = x_j$, $j = 1, ..., k-1$, $l_k(x) = -\sum_{j=1}^{k-1} x_j$, $l_{k+1}(x) = \mathrm{Id}$ (identity on $\mathbb{R}^{k-1}$).

Lemma 2 *Suppose there exists $z = (z_1, ..., z_{k+1}) \in [0,1]^{k+1}$ such that the condition (1) of Theorem A.1 is satisfied:*

$$z_1 + ... + z_k + (k-1) z_{k+1} = k - 1, \tag{10}$$

with

$$z_1 = ... = z_k = \frac{1}{p_1},\ z_{k+1} = \frac{1}{p_{k+1}},$$

where p_{k+1} is the integrability index of the spectral density f_k, that is, suppose $f_k(\lambda_1, ..., \lambda_{k-1}) \in L_{p_{k+1}}(\mathbb{R}^{k-1})$.

Then, the condition (2) of Theorem A.1 will be satisfied as well with such a choice of $z = (z_1, ..., z_{k+1})$.

Proof: We must check that for $\forall V \subset \mathbb{R}^{k-1}$ we have

$$\dim V \leq \sum_{j=1}^{k+1} z_j \dim(l_j(V)). \tag{11}$$

If $V \subset \mathbb{R}^{k-1}$ is such that $\dim(l_j(V)) = 1, \forall j = 1, ..., k$, then the r.h.s. of (11) becomes $\sum_{j=1}^{k} z_j + z_{k+1} \dim V$, and we need to show that

$$(1 - z_{k+1}) \dim V \leq \sum_{j=1}^{k} z_j,$$

or

$$\dim V \leq \frac{\sum_{j=1}^{k} z_j}{1 - z_{k+1}},$$

but due to (10) the r.h.s. is equal to $k-1$, and we have the inequality $\dim V \leq k-1$, which holds indeed for $\forall V \subset \mathbb{R}^{k-1}$, and, therefore, (11) holds even with all z_j different, but satisfying (10).

Let now $V \subset \mathbb{R}^{k-1}$ be such that for some indices $j_1, ..., j_l$ we have $\dim(l_{j_i}(V)) = 0,\ i = 1, ..., l$. When $l = k$, we have the trivial case $\dim V = 0$, otherwise l can be in the range from 1 to $k-2$, and in such a case we have $\dim V \leq k - 1 - l$. Consider (11): in the r.h.s. we will have l zero terms and, due to (10) and the choice $z_1 = ... = z_k$, (11) becomes

$$\dim V \leq z_1(k-l) + z_{k+1} \dim V,$$

or

$$(1 - z_{k+1}) \dim V \leq z_1(k-l). \tag{12}$$

Consider

$$\begin{aligned}(1 - z_{k+1}) \dim V &= \frac{\sum_{j=1}^{k} z_j}{k-1} \dim V = \frac{k z_1}{k-1} \dim V \\ &\leq \frac{k z_1}{k-1}(k-1-l) = z_1(k - l\frac{k}{k-1}) < z_1(k-l),\end{aligned}$$

that is, (12) holds. ■

The following central limit theorem with conditions formulated in terms of spectral densities may be proved now by the method of cumulants.

Theorem 1 *Suppose that Assumptions A and B hold for $d = 1$, $K = [-1/2, 1/2]$, and for $k \geq 3$*

$$f_k(\lambda_1, ..., \lambda_{k-1}) \in L_{p_k}(\mathbb{R}^{k-1}), \tag{13}$$

where $p_k = \frac{2(k-1)}{k-2}$. Then, as $T \to \infty$

$$\frac{S_T}{T^{1/2}} \xrightarrow{D} N(0, \sigma^2), \tag{14}$$

where $\sigma^2 = 2\pi f_2(0)$.

Proof: Taking into account (6) and Lemma 2, we apply the Hölder-Young-Brascamp-Lieb inequality and obtain for some $C > 0$ the following:

$$\left| I_T^{(k)} \right| \leq C T^{k(1-\frac{1}{p_1})-\frac{k}{2}} C_{p_1}^k(K) ||f_k||_{p_{k+1}} \tag{15}$$

for $p_1 \geq 1$ and p_{k+1} satisfying (10).

We can choose $p_1 = 2$ and come to the bound

$$\left| I_T^{(k)} \right| \leq C C_2^k(K) ||f_k||_{p_{k+1}}, \tag{16}$$

for such a choice of p_1, the corresponding index p_{k+1} we obtain from (10):

$$p_{k+1} = \frac{2(k-1)}{k-2}, \ k \geq 3. \tag{17}$$

However, we are able to prove that, in fact, $I_T^{(k)} \to 0$ as $T \to \infty$ (that is, bound in (16) can be strengthen to the form $o(1)$), requiring still $f_k(\lambda_1, ..., \lambda_{k-1}) \in L_{p_{k+1}}(\mathbb{R}^{k-1})$ with the same p_{k+1} given by (17).

Let us choose in (10) $\tilde{p}_1 = ... = \tilde{p}_{k-2} = 2$ (that is, $\tilde{z}_1 = ... = \tilde{z}_{k-2} = \frac{1}{2}$) and $\tilde{p}_{k-1} = \tilde{p}_k$ be close but less than 2 ($\tilde{z}_{k-1} = \tilde{z}_k$ close but more than $\frac{1}{2}$).

Note that with the above choice of integrability indices the condition (2) in Theorem A.1 still holds in the case under consideration. Indeed, as it has been just shown, (11) holds with all z_j different, but satisfying (10), for all $V \subset \mathbb{R}^{k-1}$ such that $\dim(l_j(V)) = 1, \forall j = 1, ..., k$. Next, if for some l indices $j_1, ..., j_l$ we have $\dim(l_{j_i}(V)) = 0, \ i = 1, ..., l, \ 1 \leq l \leq k-2$, then $\dim V \leq k - 1 - l$. Let us write (11) in the form

$$(1 - z_{k+1}) \dim V \leq \sum_{j=1}^{k} z_j \dim(l_j(V)).$$

or

$$\frac{\sum_{j=1}^{k} z_j}{k-1} \dim V \leq \sum_{j=1}^{k} z_j \dim(l_j(V)).$$

The above will hold if we show that

$$\frac{\sum_{j=1}^{k} z_j}{k-1} (k-1-l) \leq \sum_{j=1}^{k} z_j \dim(l_j(V)). \tag{18}$$

There are only $k - l$ non-zero terms in the r.h.s., choose for two those z_j, which correspond to non-zero terms, the values $\tilde{z}_1 + \tilde{\varepsilon} = \frac{1}{2} + \tilde{\varepsilon}$, and for the rest of z_j we take values $\tilde{z}_1 = \frac{1}{2}$. Then the inequality (18) becomes

$$\frac{\frac{1}{2}(k-2) + 2(\frac{1}{2} + \tilde{\varepsilon})}{k-1} (k-1-l) \leq \frac{1}{2}(k-l-2) + 2(\frac{1}{2} + \tilde{\varepsilon}),$$

or

$$\frac{(k-1-l)}{k-1} (k\tilde{z}_1 + 2\tilde{\varepsilon}) \leq \tilde{z}_1 (k-l) + 2\tilde{\varepsilon}.$$

The l.h.s. can be written in the form $\tilde{z}_1(k - l\frac{k}{k-1}) + 2\tilde{\varepsilon}(1 - \frac{l}{k-1})$, which shows that the last inequality holds.

Therefore, we can use the HYBL inequality with the above chosen set of indices of integrability $\tilde{p}_j$, $j = 1, .., k$.

Then the bound (15) becomes

$$\begin{aligned} \left| I_T^{(k)} \right| &\leq CT^{1-\frac{2}{\tilde{p}_k}} C_2^{k-2}(K) C_{\tilde{p}_k}^2(K) ||f_k||_{\tilde{p}_{k+1}} \\ &= CT^{-\varepsilon} C_2^{k-2}(K) C_{\tilde{p}_k}^2(K) ||f_k||_{\tilde{p}_{k+1}}, \end{aligned} \tag{19}$$

where $\varepsilon = \frac{2}{\tilde{p}_k} - 1 > 0$ and corresponding $\tilde{p}_{k+1}$, obtained from (10), will be such that

$$\tilde{p}_{k+1} > p_{k+1} = \frac{2(k-1)}{k-2}, \ k \geq 3 \tag{20}$$

(note that we do not need here the exact expressions for $\tilde{p}_k$ and $\tilde{p}_{k+1}$).

Therefore, for the functions $f_k \in L_{\tilde{p}_{k+1}}$ we have $I_T^{(k)} \to 0$ as $T \to \infty$, for $k \geq 3$.

Remembering that we are interested in evaluating (5) for the functions f_k which are in L_1 (as being spectral densities), we summarize the above reasonings as follows:

(i) for $f_k \in L_1 \cap L_{p_{k+1}}$ we have obtained the bound (16);

(ii) for $f_k \in L_1 \cap L_{\tilde{p}_{k+1}}$ we have obtained the convergence $I_T^{(k)} \to 0$ as $T \to \infty$.

It is left to note that

(iii) $L_1 \cap L_{\tilde{p}_{k+1}}$ is dense in $L_1 \cap L_{p_{k+1}}$ (see (20))

to conclude that the convergence $I_T^{(k)} \to 0$ as $T \to \infty$ holds for functions from $L_1 \cap L_{p_{k+1}}$ as well.

Indeed, for $f_k \in L_1 \cap L_{p_{k+1}}$ and $g_k \in L_1 \cap L_{\tilde{p}_{k+1}}$ we can write

$$\left| I_T^{(k)}(f_k) \right| \leq \left| I_T^{(k)}(f_k - g_k) \right| + \left| I_T^{(k)}(g_k) \right|,$$

where the first term can be made arbitrarily small with the choice of g_k in view of (i) and (iii), and the second term tends to zero in view of (ii) ■

To cover the case of general $d \geq 1$ we state the next theorem.

Theorem 2 *Suppose that Assumptions A, K with $p_* < 2$, and B hold, and for $k \geq 3$*

$$f_k(\lambda_1, ..., \lambda_{k-1}) \in L_{p_k}(\mathbb{R}^{d(k-1)}), \tag{21}$$

where $p_k = \frac{2(k-1)}{k-2}$. Then, as $T \to \infty$

$$\frac{S_T}{T^{d/2}} \xrightarrow{D} N(0, \sigma^2), \tag{22}$$

where $\sigma^2 = (2\pi)^d |K| f_2(0)$.

Proof: We can write for $k \geq 3$:

$$\begin{aligned}
|I_T^{(k)}| &\leq \frac{1}{T^{dk/2}} \left(\int_{\mathbb{R}^{d(k-1)}} |f_k(\lambda_1, ..., \lambda_{k-1})|^p \, d\lambda_1 ... d\lambda_{k-1} \right)^{1/p} \\
&\quad \times \left(\int_{\mathbb{R}^{d(k-1)}} \left| \Delta_T(\lambda_1) ... \Delta_T(\lambda_{k-1}) \Delta_T \left(-\sum_{i=1}^{k-1} \lambda_i \right) \right|^q \, d\lambda_1 ... d\lambda_{k-1} \right)^{1/q} \\
&\leq K_H \, \frac{1}{T^{dk/2}} ||f_k||_p \left(||\Delta_T||_{qk/(k-1)} \right)^k \\
&= K_H \left(C_{qk/(k-1)}(K) \right)^k T^{dk(1-\frac{k-1}{qk}) - \frac{dk}{2}} ||f_k||_p, \qquad (23)
\end{aligned}$$

where we applied the Hölder inequality for the first step, the homogenous HYBL inequality for the second step (with K_H being the constant coming from this inequality), and then we applied (6). Here we suppose $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{qk}{k-1} > p^*$ (see Assumption K), $f_k \in L_p$.

Now, with the choice $q = \frac{2(k-1)}{k}$ we get the bound:

$$\left| I_T^{(k)} \right| \leq K_H C_2^k(K) ||f_k||_p,$$

where the corresponding index $p = \frac{2(k-1)}{k-2}$.

From this point we can derive the convergence to zero for $I_T^{(k)}$ under the assumption $f_k \in L_p$, $p = \frac{2(k-1)}{k-2}$, with the reasonings analogous to those used to strengthen the bound (16) to the form $o(1)$ (see the proof of Theorem 1 above), the reasonings will be even simpler as now we do not need to check the conditions of Theorem A.1, but can just rewrite (23) in the form

$$|I_T^{(k)}| \leq K_H (C_{\tilde{q}k/(k-1)}(K))^k \, T^{dk(1-\frac{k-1}{\tilde{q}k}) - \frac{dk}{2}} ||f_k||_{\tilde{p}},$$

where $1/\tilde{q} + 1/\tilde{p} = 1$, $\tilde{p} > p$, $\tilde{q} < q$ and the rest of the proof is preserved. ■

Remark 3 For balls and cubes the condition $p_* < 2$ holds.

Remark 4 As a consequence of the above theorem we can state that the CLT (22) holds under Assumptions A, K with $p_* < 2$, and B, if the spectral densities $f_k \in L_4(\mathbb{R}^{d(k-1)})$, $k \geq 3$. However, Theorem 2 provides more refined conditions, showing that for the central limit theorem to hold the index of integrability of higher order spectral densitiesf_k can become smaller and smaller, approaching to 2 as k grows.

Remark 5 Although the application of the nonhomogeneous HYBL inequality allows to bound the integral $I_T^{(k)}$ and write the relation for the integrability indices more straightforwardly, it requires the verification of condition (2) of Theorem A.1, which we were able to provide here only for the case $d = 1$. Two steps procedure with the use of the Hölder and the homogeneous HYBL inequalities in the proof of Theorem 2 gives the same results concerning the values of p_k. Our conjecture is that in some particular situation the general condition (2) of Theorem A.1 can be relaxed in some way. For example, in the case under consideration we have a particular "symmetric" situation, where arguments of all the functions belong to spaces $\mathbb{R}^d$ ($g_i(\lambda)$, $\lambda \in \mathbb{R}^d$, $f_k(\lambda_1, ..., \lambda_{k-1})$, $\lambda_i \in \mathbb{R}^d$), that is, H_i, $i = 1, ..., k$ are all equal to $\mathbb{R}^d$, $H_{k+1} = (\mathbb{R}^d)^{k-1}$, $H = (\mathbb{R}^d)^{k-1}$; some kind of "symmetry" is also present in our special set of functions and linear transformations. In such a situation we may suppose that it is sufficient to check (11) not for all possible subsets $V \subset \mathbb{R}^{d(k-1)}$ but for only particular case of sets "equidimensional in projections" in the sense that we consider only such sets V for which either $\dim(l_{j_i}(V)) = d$, $i = 1, ..., k-1$, or if for some $1 \le j_0 \le k-1$ $\dim(l_{j_0}(V)) = d' < d$, then also $\dim(l_{j_i}(V)) = d' < d$ for all $i = 1, ..., k-1$. For such sets the validity of condition (2) of Theorem A.1 can be derived by induction. We believe that this condition (2) for the nonhomogeneous HYBL inequality requires deeper exploration and address this topic for further analysis.

The next remark is about a possible condition of the convex sets in form of the kernel property.

Remark 6 One can assume that the function

$$\Phi_T^{(k)}(\lambda_1, ..., \lambda_{k-1}) = \frac{1}{(2\pi)^{d(k-1)} |K|^{k-1} T^d} \Delta_T(\lambda_1)...\Delta_T(\lambda_{k-1})\Delta_T\left(-\sum_{i=1}^{k-1}\lambda_i\right)$$

has the kernel property on $\mathbb{R}^{d(k-1)}$ for $k \ge 2$, i.e.:

$$\int_{\mathbb{R}^{d(k-1)}} \Phi_T^{(k)}(\lambda_1, ..., \lambda_{k-1}) d\lambda_1...d\lambda_{k-1} = 1, \tag{24}$$

and for any $\varepsilon > 0$ when $T \to \infty$

$$\lim \int_{\mathbb{R}^{d(k-1)} \setminus \varepsilon K^{k-1}} \Phi_T^{(k)}(\lambda_1, ..., \lambda_{k-1}) d\lambda_1...d\lambda_{k-1} = 0. \tag{25}$$

Note that (24), (25) hold for the rectangle $K = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ (see, for instance, Bentkus and Rutkauskas (1973) or Avram, Leonenko and Sakhno (2010) and the references therein). If the higher-order spectral densities $f_k(\lambda_1, ..., \lambda_{k-1}), k \ge 2$ are continuous and bounded and $f_k(0, ..., 0) \ne 0$, then

$$I_T^{(k)} = \frac{(2\pi)^d |K|^{k-1}}{T^{d\left(\frac{k}{2}-1\right)}} \int_{\mathbb{R}^{d(k-1)}} \Phi_T^{(k)}(\lambda_1, ..., \lambda_{k-1}) f_k(\lambda_1, ..., \lambda_{k-1}) d\lambda_1...d\lambda_{k-1} \sim$$

$$\sim \frac{(2\pi)^d |K|^{k-1}}{T^{d\left(\frac{k}{2}-1\right)}} f_k(0, ..., 0),$$

as $T \to \infty$, thus tend to zero for $k \ge 3$, and the central limit theorem, Theorem 1, follows.

3 Gaussian fields

Let us consider how the above method for deriving Theorem 2 can be used in the situation when the field $X(t)$ is a nonlinear transformation of a Gaussian field. Note that this kind of limit theorems, often called in the literature Breuer-Major theorems, have been addressed by many authors. Recently, powerful theory based on Malliavin calculus was exploited in the series of papers by Nualart, Ortiz-Lattore, Nourdin, Peccati, Tudor and others to develop CLTs in the framework of Wiener Chaos via remarkable fourth moment approach (see, for example, [38], [39] and references therein). We show how CLT can be stated quite straightforwardly with the use of the Hölder-Young-Brascamp-Lieb inequality.

For a stationary Gaussian filed $X(t), t \in \mathbb{R}^d$, consider the field $Y(t) = G(X(t)), t \in \mathbb{R}^d$. For a quite broad class of functions G, evaluation of asymptotic behavior of the normalized integrals $S_T = \int_{t \in K_T} Y(t)dt$ reduces to consideration of the integrals $\int_{t \in K_T} H_m(X(t))dt$, with a particular m, where $H_m(x)$ is the Hermite polynomial, m is Hermite rank of G (see, e.g., Ivanov and Leonenko (1986), p.55).

To demonstrate the approach based on the use of the Hölder-Young-Brascamp-Lieb inequality, we consider here only the case of integrals

$$S_T = S_T(H_2(X(t))) = \int_{t \in K_T} H_2(X(t))dt, \tag{26}$$

where $H_2(x) = x^2 - 1$.

Suppose that the centered Gaussian field $X(t), t \in \mathbb{R}^d$, has a spectral density $f(\lambda), \lambda \in \mathbb{R}^d$. Then we can write the following Wiener-Itô integral representation:

$$H_2(X(t)) = \int_{\mathbb{R}^{2d}} e^{it(\lambda_1+\lambda_2)} \sqrt{f(\lambda_1)}\sqrt{f(\lambda_2)} W(d\lambda_1)W(d\lambda_2), \tag{27}$$

where $W(\cdot)$ is the Gaussian complex white noise measure (with integration on the hyperplanes $\lambda_i = \pm\lambda_j, i, j = 1, 2, i \neq j$, being excluded). Applying the formulas for the cumulants of multiple stochastic Wiener-Itô integrals (see, e.g., [40] and references therein), we have that the spectral density of the second order of the field (27) is given by

$$g_2(\lambda) = \int_{\mathbb{R}^d} f(\lambda_1) f(\lambda + \lambda_1) d\lambda_1,$$

which is well defined if $f(\lambda) \in L_2(\mathbb{R}^d)$, and this condition guarantees also that the Assumption B holds.

Next, the cumulants of the normalized integral (26) can be written in the form

$$\begin{aligned} I_T^{(k)} &= cum_k \left\{ \frac{S_T}{T^{d/2}}, ..., \frac{S_T}{T^{d/2}} \right\} \\ &= \frac{1}{T^{dk/2}} \int_{(\lambda_1,...,\lambda_{k-1}) \in \mathbb{R}^{d(k-1)}} \Delta_T(\lambda_1)...\Delta_T(\lambda_{k-1})\Delta_T\left(-\sum_{i=1}^{k-1}\lambda_i\right) \end{aligned}$$

$$\times \int_{\mathbb{R}^d} f(\lambda)f(\lambda+\lambda_1)...f(\lambda+\lambda_1+...+\lambda_{k-1})d\lambda\, d\lambda_1...d\lambda_{k-1}. \tag{28}$$

Now we can repeat the same reasonings as those for the proof of Theorem 2 to conclude that $I_T^{(k)} \to 0$ as $T \to \infty$, for $k \geq 3$, under the condition $f(\lambda) \in L_2(\mathbb{R}^d)$.

Indeed, the analogue of the formula (10) relating the integrability indices p for $\Delta_T(\lambda)$ and q for $f(\lambda)$ becomes in this case of the following form: $dk\frac{1}{p}+dk\frac{1}{q} = dk$, or $\frac{1}{p}+\frac{1}{q} = 1$. We need already $f(\lambda)$ to be in $L_2(\mathbb{R}^d)$ for a proper behavior of the second order cumulant, therefore, choosing $q = 2$, we can take p to be equal to 2 as soon as $p_* < 2$ in the Assumption K. Note, that in this case we appeal to the homogeneous HYBL inequality.

Thus, we derived the known result (see, for example, [28]):

Proposition 1 *If a stationary Gaussian filed $X(t), t \in \mathbb{R}^d$, has the spectral density $f(\lambda) \in L_2(\mathbb{R}^d)$ and Assumption K with $p_* < 2$ holds, then, as $T \to \infty$*

$$\frac{S_T(H_2(X(t)))}{T^{d/2}} \xrightarrow{D} N(0, \sigma^2), \tag{29}$$

where

$$\sigma^2 = (2\pi)^d\,|K| \int_{\mathbb{R}^d} f^2(\lambda)d\lambda. \tag{30}$$

As we can see, when taking into consideration the spectral domain, the application of the Hölder-Young-Brascamp-Lieb inequality allows to provide a very simple proof. Note also that this kind of technique has been used for linear sequences (which generalize Gaussian fields) as well [5].

Moreover, requiring more regularity on spectral density $f(\lambda)$, we are able to evaluate the rate of convergence (29) in the following way.

Let us consider

$$\check{S}_T = \frac{S_T(H_2(X(t)))}{(2\pi)^d\,|K|\,T^{d/2}\int_{R^d} f^2(\lambda)d\lambda}.$$

We have for $f(\lambda) \in L_2(\mathbb{R}^d)$ the convergence as $T \to \infty$

$$\check{S}_T \xrightarrow{D} N \backsim N(0,1). \tag{31}$$

We can state stronger version for this approximation, namely, that the convergence (31) takes place with respect to the Kolmogorov distance:

$$d_{Kol}(\check{S}_T, N) = \sup_{z\in\mathbb{R}} |P(\check{S}_T < z) - P(N < z)| \to 0, \tag{32}$$

and also we can provide an upper bound for $d_{Kol}(\check{S}_T, N)$. For this we apply the results from [38]: since $\check{S}_T$ is representable as a double stochastic Wiener-Itô integral we can use the Proposition 3.8 of [38] which is concerned with normal approximation in second Wiener Chaos and gives upper bounds for the

Kolmogorov distance solely in terms of the fourth and second cumulants. This bound is of the form

$$d_{Kol}(\check{S}_T, N) \leq \sqrt{\frac{1}{6} cum_4(\check{S}_T) + (cum_2(\check{S}_T) - 1)^2}. \quad (33)$$

So, we need only to control the fourth cumulant of $\check{S}_T$ and this can be done with the use of the Hölder-Young-Brascamp-Lieb inequality. Due to this inequality, analogously to our previous derivations, for $f(\lambda) \in L_q(\mathbb{R}^d)$, $q > 2$, and $\Delta_T(\lambda) \in L_p(\mathbb{R}^d)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we can write

$$\left| cum_k(\check{S}_T) \right| \leq C T^{kd(1-\frac{1}{p}) - \frac{kd}{2}} C_p^k(K) ||f||_q^k = C T^{kd(\frac{1}{q} - \frac{1}{2})} C_p^k(K) ||f||_q^k,$$

therefore,

$$d_{Kol}(\check{S}_T, N) \leq Const\ T^{-\frac{q-2}{q} d},$$

where the constant depends on K and f. Thus, the rate of convergence to the normal law depends on the index of integralbility of $f(\lambda)$, in particular, for $f(\lambda) \in L_4(\mathbb{R}^d)$ we obtain

$$d_{Kol}(\check{S}_T, N) \leq Const\ \frac{1}{T^{d/2}}.$$

The above technique can be also used for deriving CLT for $S_T(H_m(X(t)))$ with $m > 2$.

4 An invariance principle

Let us return now to the case of a general random field X_t of Assumption A.

In this section we discuss the invariance principle and demonstrate the applicability of the technique based on higher-order spectral densities and HYBL inequality in this situation. With this purpose in mind, we introduce some additional assumptions.

Assumption A*. Let the stationary random field X_t of Assumption A be isotropic of the second order, that is, its covariance function $cov(X_t, X_s)$ depends only on the distance between the points t and s:

$$cov(X_t, X_s) = EX_t X_s = c_2(t - s, 0) = B(|t - s|).$$

We also suppose that the field X_t is mean square continuous.

Note that in this case covariance function $B(u)$, $u \in R_+^1$, has the following representation:

$$B(u) = \int_0^\infty Y_d(\lambda u) G(d\lambda),$$

where $G(\cdot)$ is a finite measure on $[0, \infty)$ and the function $Y_d(\cdot)$ is defined by

$$Y_d(u) = 2^{(d-2)/2} \Gamma\left(\frac{d}{2}\right) J_{(d-2)/2}(u)\, u^{(2-d)/2}, \quad u \geq 0,$$

$J_{(d-2)/2}(\cdot)$ is the Bessel function of the first kind of order $(d-2)/2$.

In the case under consideration, when the spectral density of the second order $f_2(\lambda)$ exists, this function is such that $f_2(\lambda) = f(|\lambda|)$, and the measure $G(\cdot)$ has the density

$$g(\lambda) = |s(1)| f(\lambda) \lambda^{d-1}, \quad \lambda \in R_+,$$

where $|s(1)| = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere in R^d (for more detail, see, e.g., [28]).

Consider the space $C[0,1]$ of continuous functions with the uniform topology. We will state below the invariance principle (or functional central limit theorem) for the measures P_T, induced in the space $C[0,1]$ by the stochastic processes

$$Y_T(u) = \frac{1}{T^{d/2}} S_{u^{1/d}T} = \frac{1}{T^{d/2}} \int_{t \in u^{1/d} K_T} X_t dt, \quad u \in [0,1]. \tag{34}$$

We will need one more additional assumption which, together with Assumption A^*, will help to handle the asymptotic behavior of the variance of $Y_T(u)$.

Let us reconsider first the variance of the functional

$$S_T = S(K_T) = \int_{t \in K_T} X_t dt. \tag{35}$$

Taking into account Assumption A^*, the variance of the above integral can be expressed in terms of the covariance function $B(u)$ of the field X_t in the following way:

$$\begin{aligned} VarS_T &= \int_{K_T} \int_{K_T} B(|t-s|) dt ds = |K_T|^2 E B(\rho_{\alpha\beta}) \\ &= |K|^2 T^{2d} \int_0^{d(T)} B(u) dF_{K_T}(u), \end{aligned}$$

with $\rho_{\alpha\beta}$ being the distance between two points α and β chosen independently and randomly in the set K_T according to the uniform distribution, $F_{K_T}(u) = F_{\rho_{\alpha\beta}}(u)$ is the probability distribution of $\rho_{\alpha\beta}$, $d(T)$ is the diameter of K_T. (Here the so-called "method of randomization of the covariance function" is used, see, e.g. [28], Sections 1.4, 1.5.)

In the case of averaging in (35) over the balls $v(R)$ (centered at the origin of radius R), the density of the distribution function $F_{v(R)}(u)$ can be calculated in the closed form (see, [30], and also [28], Lemma 1.4.2), which gives the possibility to analyze the asymptotic behavior of $VarS(v(R))$.

In particular, the following asymptotic relation takes place:

$$VarS(v(R)) = c(d)\beta R^d (1 + o(1)) \text{ as } R \to \infty,$$

provided that $\int_0^\infty z^{d-1}|B(z)|dz < \infty$ and $\beta := \int_0^\infty z^{d-1} B(z) dz \neq 0$; here $c(d) = 4\pi^d d^{-1} \Gamma^{-2}(d/2)$ (see, [28], Lemma 1.5.1).

Moreover, if the density $f(\lambda)$ is continuous in the neighborhood of the point $\lambda = 0$, $f(0) \neq 0$ and $f(\lambda)$ is bounded on $[0, \infty)$, then

$$VarS(v(R)) = (2\pi)^d \, |s(1)| R^d f(0)(1 + o(1)) \text{ as } R \to \infty$$

(see, [28], Lemma 1.5.5).

Considering the covariance between S_{T_1} and S_{T_2}, we can derive:

$$\begin{aligned} \mathbb{E}S_{T_1}S_{T_2} &= E\int_{t\in K_{T_1}} X_t dt \int_{s\in K_{T_2}} X_s ds = \int_{K_{T_1}}\int_{K_{T_2}} B(|t-s|)dtds \\ &= |K|^2 \, (T_1T_2)^d \, EB(\rho_{\alpha\beta}), \end{aligned}$$

where now $\rho_{\alpha\beta}$ is the distance between two independent random points α and β chosen according to the uniform distribution in K_{T_1} and K_{T_2} correspondingly. Denoting by $F_{K_{T_1},K_{T_2}}(u)$ the probability distribution of $\rho_{\alpha\beta}$, we can write

$$\mathbb{E}S_{T_1}S_{T_2} = |K|^2 \, (T_1T_2)^d \int_0^{(d(T_1)+d(T_2))/2} B(u) dF_{K_{T_1},K_{T_2}}(u). \tag{36}$$

Assumption A**. Properties of the second-order characteristics of the field X_t of Assumption A^* and the properties of the set K allows for the asymptotic representation

$$\mathbb{E}S_{T_1}S_{T_2} = (\min\{T_1, T_2\})^d \, c(K, B \vee G)(1 + o(1)) \text{ as } T_1 + T_2 \to \infty,$$

where with notation $c(K, B \vee G)$ we mean a constant representable in terms of the set K and the covariance $B(u)$ or the spectral function $G(\lambda)$ (spectral density $f(\lambda)$). In what follows we will denote $c(K, B \vee G)$ simply as $c(K)$.

Remark 7 Conditions for validity of Assumption A** for the case of balls $v(R)$ can be deduced in the spirit of the results of Section 1.5 of [28], basing on (36) and the exact formula for the density of the distribution function $F_{v(R_1),v(R_2)}(u)$ available for this case (see also [34]). Namely, this density is of the following form (see Lemma 1.4.3 in [28]):

$$\Psi_{v(R_1),v(R_2)}(u) = 2^{(d-2)/2}\Gamma(d/2) d^2 u^{d/2} (R_1R_2)^{d/2} \tag{37}$$

$$\times \int_0^\infty \rho^{-d/2} J_{(d-2)/2}(u\rho) J_{d/2}(R_1\rho) J_{d/2}(R_2\rho) \, d\rho, \quad u \in [0, R_1 + R_2],$$

where $J_\nu(u)$ is the Bessel function of the first kind of order ν. This formula allows to write down the expression for the covariance $\mathbb{E}S(v(R_1))S(v(R_2))$ in the closed form and evaluate its asymptotic behavior. In particular, if the spectral density $f(\lambda)$ is continuous in the neighborhood of the point $\lambda = 0$, $f(0) \neq 0$ and $f(\lambda)$ is bounded on $[0, \infty)$, then

$$\mathbb{E}S(v(R_1))S(v(R_2)) = (\min\{R_1, R_2\})^d \, |v(1)| (2\pi)^d f(0)(1+o(1)) \text{ as } R_1+R_2 \to \infty.$$

For other sets the formula for the density analogous to (37) is not known. This issue belongs to an interesting area for further research in geometric probability but go beyond the scope of the present paper.

Assumption A**, which is introduced here as a technical assumption, allows to deduce the asymptotic expression for covariances when checking the convergence of finite dimensional distributions in our proof of the invariance principle below. Asymptotics for higher-order cumulants in our approach are stated via the integrability assumptions on the spectral densities of the field. Note that in the existing literature on this topic another approach is exploited, namely, assumptions of weak dependence are introduced in the form of some kind of mixing or association for random fields (see, e.g. [31], [7] and references therein).

Consider the Brownian motion, that is the zero mean Gaussian random process $b(t), t \in [0,1]$, such that $b(0) = 0$ a.s., $\mathbb{E}b(t_1)b(t_2) = c(K)|K| \min\{t_1, t_2\}$.

Note that for the ball $K = v(1)$

$$\mathbb{E}b(t_1)b(t_2) = (2\pi)^d f_2(0)\, |v(1)| \min\{t_1, t_2\}, \quad t_1, t_2 \in [0,1].$$

In what follows we will use some definitions and terminology from the book of Billingsley (1968).

It is known that the stochastic process $b(t), t \in [0,1]$, induces the probability measure P in the space $C[0,1]$ with uniform topology.

We will prove now that the measures P_T, induced by the process (34) converge weakly ($\Longrightarrow$) to the Gaussian measure P in the space $C[0,1]$ as $T \to \infty$.

According to Billingsley (1968), we have to prove that:

(i) the finite dimensional distributions of (34) converge to those of the Gaussian process $b(t), t \in [0,1]$;

(ii) the family of probability measures $\{P_T\}_{T>0}$ is relatively compact in $C[0,1]$.

To prove (ii) let us check that Kolmogorov's criterion in the form

$$\mathbb{E}\left|Y_T(u_2) - Y_T(u_1)\right|^4 \le const\left|u_2 - u_1\right|^2, \ 0 \le u_1 \le u_2 \le 1, \tag{38}$$

for weak relative compactness of probability measures $\{P_T\}$ is satisfied (see again Billingsley (1968)).

Consider

$$\begin{aligned}
\mathbb{E}\left|Y_T(v)-Y_T(u)\right|^4 &= \frac{1}{T^{2d}}\mathbb{E}\left[\int_{t\in v^{1/d}K_T\setminus u^{1/d}K_T} X_t dt\right]^4 \\
&= \frac{1}{T^{2d}}\int_{\widetilde{K}_T^4} \mathbb{E}\left[X_{t_1}X_{t_2}X_{t_3}X_{t_4}\right] dt_1dt_2dt_3dt_4 \\
&= \frac{1}{T^{2d}}\int_{\widetilde{K}_T^4} [c_4(t_1-t_4,t_2-t_4,t_3-t_4,0) \\
&\quad +c_2(t_1-t_2,0)c_2(t_3-t_4,0)+c_2(t_1-t_3,0)c_2(t_2-t_4,0) \\
&\quad +c_2(t_1-t_4,0)c_2(t_2-t_3,0)]dt_1dt_2dt_3dt_4 \\
&= I_1+I_2+I_3+I_4. \qquad (39)
\end{aligned}$$

(We have denoted here $\widetilde{K}_T = v^{1/d}K_T\setminus u^{1/d}K_T$.)

We can write

$$\begin{aligned}
I_1 &= \frac{1}{T^{2d}}\int_{R^{3d}} f_4(\lambda_1,\lambda_2,\lambda_3)\prod_{j=1}^{3}\left[\int_{\widetilde{K}_T} e^{it_j\lambda_j}dt_j\right]\int_{\widetilde{K}_T} e^{-it_4\sum_{j=1}^3\lambda_j}dt_4d\lambda_1d\lambda_2d\lambda_3 \\
&= \frac{1}{T^{2d}}\int_{R^{3d}} f_4(\lambda_1,\lambda_2,\lambda_3)\prod_{j=1}^{3}\left[\Delta_{\widetilde{K}_T}(\lambda_j)\right]\Delta_{\widetilde{K}_T}(\sum_{j=1}^{3}\lambda_j)d\lambda_1d\lambda_2d\lambda_3. \qquad (40)
\end{aligned}$$

Now we can repeat here the chain of inequalities (23) with $k=4$, $p=3$, $q=3/2$ to obtain the following bound (supposing $f_4\in L_3$):

$$|I_1| \le K_{_H}\frac{1}{T^{2d}}\|f_4\|_3\left\{\left\|\Delta_{\widetilde{K}_T}\right\|_2\right\}^4. \qquad (41)$$

Consider

$$\begin{aligned}
\left\{\left\|\Delta_{\widetilde{K}_T}\right\|_2\right\}^4 &= \left\{\left\{\int_{R^d}\left|\int_{\widetilde{K}_T} e^{it\lambda}dt\right|^2 d\lambda\right\}^{1/2}\right\}^4 = \left\{\int_{R^d}\left|\int_{\widetilde{K}_T} e^{it\lambda}dt\right|^2 d\lambda\right\}^2 \\
&= \left\{(2\pi)^d|K|((Tv^{1/d})^d-(Tu^{1/d})^d)\right\}^2,
\end{aligned}$$

by Parseval's identity.

Therefore, under the assumption $f_4\in L_3(\mathbb{R}^{3d})$ (which is covered by the assumptions of Theorem 2)

$$I_1 \le const\ (v-u)^2.$$

Next, consider

$$\int_{\widetilde{K}_T^2} c_2(t_1-t_2,0)dt_1dt_2 = \int_{R^d} f_2(\lambda)\int_{\widetilde{K}_T^2} e^{i(t_1-t_2)\lambda}dt_1dt_2d\lambda = \int_{R^d} f_2(\lambda)\left|\Delta_{\widetilde{K}_T}(\lambda)\right|^2 d\lambda.$$

Supposing $f_2(\lambda)$ to be bounded we get

$$\int_{\widetilde{K}_T^2} c_2(t_1-t_2,0)dt_1dt_2 \le const \int_{R^d} \left|\Delta_{\widetilde{K}_T}(\lambda)\right|^2 d\lambda = const\ (2\pi)^d |K| T^d (v-u) \tag{42}$$

which implies that each term I_j, $j=2,3,4$ in (39) is bounded by

$$const(2\pi)^d|K|(v-u)^2.$$

Hence, (38) holds if we suppose that the second order spectral density f_2 is bounded and $f_4 \in L_3$.

We can summarize the above arguments in the next lemma.

Lemma 3 *Suppose that Assumptions A, K and B hold, and* $f_4(\lambda_1,\lambda_2,\lambda_3) \in L_3(\mathbb{R}^{3d})$. *Then the familly of measures* P_T, *induced by the stochastic processes (34) is relatively compact in the space* $C[0,1]$.

Let us show now the convergence of the finite-dimensional distributions, that is, the above statement (i). Let $c_1,...,c_m$ be fixed constants and consider the random variable $Z_T = T^{-d/2}\sum_{j=1}^m c_j Y_T(u_j)$. We have $\mathbb{E}Z_T = 0$ and convergence of the variance will hold under the Assumption A**. Consider now the cumulant of the general order $k \ge 3$:

$$\begin{aligned} cum_k\{Z_T,...,Z_T\} &= \frac{1}{T^{dk/2}} cum_k \Big\{ \sum_{j=1}^m c_j \int_{t\in u_j^{1/d}K_T} X(t)dt, ..., \\ &\quad \sum_{j=1}^m c_j \int_{t\in u_j^{1/d}K_T} X(t)dt \Big\}. \end{aligned} \tag{43}$$

Due to the multilinearity property of cumulants, the last expression can be represented as the sum of k-th order cumulants of the following form

$$\frac{1}{T^{dk/2}} cum_k \Big\{ \int_{t\in u_{j_1}^{1/d}K_T} X(t)dt, ..., \int_{t\in u_{j_k}^{1/d}K_T} X(t)dt \Big\}, \tag{44}$$

where $u_{j_l} \in \{1,...,m\}, l=1,...,k$. To evaluate the asymptotic behavior of the above cumulant we write it in the following form:

$$\begin{aligned} &\frac{1}{T^{dk/2}} \int_{t_k\in u_{j_k}^{1/d}K_T} ... \int_{t_1\in u_{j_1}^{1/d}K_T} c_k(t_1-t_k,..,t_{k-1}-t_k,0)dt_1...dt_k \\ = &\frac{1}{T^{dk/2}} \int_{(\lambda_1,...,\lambda_{k-1})\in\mathbb{R}^{d(k-1)}} f_k(\lambda_1,...,\lambda_{k-1}) \\ &\times \Delta_{u_{j_1}^{1/d}T}(\lambda_1)...\Delta_{u_{j_{k-1}}^{1/d}T}(\lambda_{k-1})\Delta_{u_{j_k}^{1/d}T}\left(-\sum_{i=1}^{k-1}\lambda_i\right) d\lambda_1...d\lambda_{k-1}, \end{aligned}$$

where $\Delta_{u^{1/d}T}(\lambda) = \int_{t\in u^{1/d}K_T} e^{it\lambda}dt$ and $\|\Delta_{u^{1/d}T}(\lambda)\|_p = T^{d(1-1/p)}C_p(u,K)$.

Now we can apply exactly the same reasonings as those in the proof of Theorem 2 to get the convergence of each term (44) to zero as $T \to \infty$, under the conditions of integrability of the spectral densities (21) and, therefore, convergence to zero of the cumulants (43) of all orders $k \geq 3$.

Summarizing all the above we come to the following result.

Theorem 3 *Suppose that conditions of Theorem 2 and, in addition, Assumptions A^*, A^{**} hold. Then $P_T \Longrightarrow P$ in $C[0,1]$, where the measures P_T and P are induced by the stochastic processes (34) and $b(t), t \in [0,1]$, respectively.*

Remark 8 To summarize, the invariance principle is stated here under some conditions on the integrability of spectral densities and some technical assumptions introduced to manage the behavior of the second order cumulants (Assumptions A^*, A^{**}), as we intended here to stress on the points where the HYBL inequality works. However, we would like to note once more that these additional assumptions hold for the balls ($K = v(1)$) under appropriate assumptions on the covariance or spectrum (which are available in the literature); therefore, here we have presented a new proof of the invariance principle for this case, which do not rely on some forms of mixing conditions or association, but uses conditions in terms of higher order spectral densities instead. Note that the essential element for establishing the validity of Assumption A^{**} is the possibility to deduce the exact formula for the density of the distribution function for the distance between two points chosen independently and randomly in the sets K_{T_1} and K_{T_2}. Such kind of problems are of ongoing research interest in geometric probability and related applied areas (see, e.g., [18], [21], [44] and references therein). This problem deserves a separate study. As long as the new progress in this area is achieved, Assumption A^{**} can be stated for new classes of sets and, therefore, new classes of sets will be covered by Theorem 3.

5 Non-homogeneous random fields

We discuss now the central limit theorem for non-homogeneous random fields of special form.

Assumption C: For a given $\beta > 0$, assume that a real (weight) function $w(t), t \in \mathbb{R}^d$, is (positively) homogeneous of degree β, that is, for any $a > 0$, it holds that

$$w(at) = w(at_1, ..., at_d) = a^\beta w(t), \quad t \in \mathbb{R}^d.$$

Assumption D: Assume that there exists

$$w_1(\lambda) = \int_{t \in K} w(t) e^{it\lambda} dt, \ \lambda \in \mathbb{R}^d.$$

Under Assumptions C and D

$$w_T(\lambda) = \int_{t \in K_T} w(t) e^{it\lambda} dt = T^{d+\beta} w_1(\lambda T), \ \lambda \in \mathbb{R}^d.$$

Example 1. The function $w_1(t) = \|t\|^{\nu}, \nu \geq 0$, is homogeneous of degree $\beta = \nu$, if $\nu > 0$. For example if $d = 1$, $K = [0,1]$ and $\nu \geq 0$ is an integer,we obtain

$$w_1(\lambda) = \int_{t \in [0,1]} t^{\nu} e^{it\lambda} dt = \frac{1}{i^{\nu}} \frac{\partial^{\nu}}{\partial \lambda^{\nu}} \int_{t \in [0,1]} e^{it\lambda} dt = \frac{1}{i^{\nu}} \frac{\partial^{\nu}}{\partial \lambda^{\nu}} \frac{e^{i\lambda} - 1}{i\lambda}, \ \lambda \in \mathbb{R}^1.$$

Example 2. Another example of the homogeneous function of degree $\beta > 0$, is $w_2(t) = |t_1 + ... + t_d|^{\nu}$, where again $\beta = \nu$, if $\nu > 0$.

Example 3. The function $w_3(t) = ||t_1|^{\gamma} + ... + |t_d|^{\gamma}|^{\nu}$ is homogeneous of degree $\beta = \nu\gamma$, if $\nu > 0, \gamma > 0$.

Example 4. All arithmetic, geometric and harmonic averages of $|t_1|, ..., |t_d|$ are homogeneous functions of degree one.

Under Assumption C we investigate below the asymptotic normality of integrals

$$S_T^w = \int_{t \in K_T} w(t) X_t dt$$

as $T \to \infty$.

We denote

$$W^2(T) = \int_{t \in K_T} w^2(t) dt = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |w_T(\lambda)|^2 d\lambda. \tag{45}$$

Assumption E: Let the finite measures

$$\mu_T(d\lambda) = \frac{|w_T(\lambda)|^2 d\lambda}{\int_{\mathbb{R}^d} |w_T(\lambda)|^2 d\lambda}, \lambda \in \mathbb{R}^d$$

converge weakly to some finite measure $\mu(d\lambda)$, and the spectral density $f_2(\lambda)$ is positive on set $B \subseteq \mathbb{R}^d$ of positive μ-measure ($\mu(B) > 0$).

We recall that the weak convergence of probability measures means that for any continuous and bounded function $f(\lambda)$ as $T \to \infty$

$$\lim \int_{\mathbb{R}^d} f(\lambda) \mu_T(d\lambda) = \int_{\mathbb{R}^d} f(\lambda) \mu(d\lambda).$$

Then we have that the variance

$$\mathbb{E}\left[\frac{S_T^w}{W(T)}\right]^2 = \frac{1}{W^2(T)} \int_{\mathbb{R}^d} f_2(\lambda) \left[\int_{t_1 \in K_T} w(t_1) e^{it_1\lambda} dt_1\right] \overline{\left[\int_{t_2 \in K_T} w(t_2) e^{it_2\lambda} dt_2\right]} d\lambda =$$

$$= (2\pi)^d \int_{\mathbb{R}^d} f_2(\lambda)\mu_T(d\lambda) \to (2\pi)^d \int_{\mathbb{R}^d} f_2(\lambda)\mu(d\lambda) = \sigma^2 > 0,$$

as $T \to \infty$, under Assumption B.

It turns out that we need the following

Assumption F:

$$C_{p,w}(K) := ||w_1(\lambda)||_p = \left(\int_{\mathbb{R}^d} \left| \int_{t \in K} w(t) e^{it\lambda} dt \right|^p d\lambda \right)^{1/p} < \infty, \quad \forall p > p_* \geq 1.$$

Then by scaling property we obtain the following formula:

$$\|w_T(\lambda)\|_p = T^{d(1-\frac{1}{p})+\beta} C_{p,w}(K),$$

and in particular

$$W^2(T) = \int_{t \in K_T} w^2(t) dt = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |w_T(\lambda)|^2 d\lambda = \frac{1}{(2\pi)^d} \left[T^{\frac{d}{2}+\beta} C_{2,w}(K) \right]^2. \tag{46}$$

Similarly to the proof of Theorem 1 we obtain that the cumulant of order $k \geq 3$ is of the form

$$I_T^{(k)} = cum_k \left\{ \frac{S_T^w}{W(T)}, ..., \frac{S_T^w}{W(T)} \right\} =$$

$$= \frac{1}{W(T)^k} \int_{t_1 \in K_T} ... \int_{t_k \in K_T} w(t_1)...w(t_k) c_k(t_1 - t_k, .., t_{k-1} - t_k, 0) dt_1 ... dt_k =$$

$$= \frac{1}{W(T)^k} \int_{(\lambda_1, ..., \lambda_{k-1}) \in \mathbb{R}^{d(k-1)}} w_T(\lambda_1) w_T(\lambda_2) ... w_T(\lambda_{k-1}) w_T(-\lambda_1 - ... - \lambda_{k-1}) \times$$

$$\times f_k(\lambda_1, ..., \lambda_{k-1})\, d\lambda_1 ... d\lambda_{k-1}, \tag{47}$$

and then with the same reasonings as those used for derivation of the formula (15) we obtain for some $C > 0$ the bound

$$\left| I_T^{(k)} \right| \leq C \frac{T^{kd(1-\frac{1}{p_1})}}{T^{\frac{kd}{2}} C_{2,w}^k(K)} C_{p_1,w}^k(K) \|f_k\|_{p_{k+1}} = C T^{-\nu} C_{p_1,w}^k(K) C_{2,w}^{-k}(K) \|f_k\|_{p_{k+1}},$$

where

$$\nu = kd \left(\frac{1}{2} - \left(1 - \frac{1}{p_1} \right) \right).$$

Similarly to the proof of the Theorem 2, we come to the restrictions on p_1 and p_{k+1}, and, therefore, derive the following

Theorem 4 *If Assumptions A, B, C, D, E and F hold, and for $k \geq 3$*

$$f_k(\lambda_1, ..., \lambda_{k-1}) \in L_{p_k}(\mathbb{R}^{d(k-1)}),$$

where $p_k = \frac{2(k-1)}{k-2}$. *Then, as* $T \to \infty$

$$\frac{S_T^w}{W(T)} \xrightarrow{D} N(0, \sigma^2),$$

where $\sigma^2 = (2\pi)^d \int_{\mathbb{R}^d} f_2(\lambda)\mu(d\lambda)$, *and the finite measure* μ *is defined in assumption E.*

This theorem can be applied to the statistical problem of estimation of unknown coefficient of linear regression observed on the increasing convex sets.

Remark 9 Analogously to Section 4, the invariance principle for the above situation can be considered and Theorem 3 can be extended to the analog of Theorem 2. We just point out the key steps here.

First, we note that for the monotonically increasing function $V(T) := W^2(T)$ (with $W^2(T)$ given by (45)) there exists the unique inverse function which we will denote $V^{(-1)}(T)$. Then we make the modification in the definition of the process (34) and introduce the corresponding limiting process.

Instead of (34) we consider the process

$$Y_T^w(u) = \frac{1}{V(T)^{1/2}} \int_{t \in V^{(-1)}(u)K_T} w(t) X_t dt, \quad u \in [0,1]. \tag{48}$$

Consider the multiparameter Brownian motion of Chentsov's type (see [42] for example), that is the zero mean Gaussian random field $\tilde{b}(t), t \in \mathbb{R}^d$, such that

(i) $\tilde{b}(t) = 0$, if $t_j = 0$ for at least one $j \in \{1, ..., d\}$;

(ii) $\mathbb{E}\tilde{b}(t_1)\tilde{b}(t_2) = \prod_{j=1}^{d} \min\left\{t_1^{(j)}, t_2^{(j)}\right\}, t_l = \left(t_l^{(j)}, j = 1, ..., d\right), l \in \{1, 2\}$.

Define now the process

$$L_K^w(u) = (\tilde{c}(K))^{1/2} \int_{t \in V^{(-1)}(u)K} d\tilde{b}(t), \quad u \in [0,1], \tag{49}$$

where $\tilde{c}(K)$ is coming from the asymptotics for the second order cumulants of the process (48) (under appropriate assumptions which can be formulated analogously to Assumption A^{**} of the previous section).

Note that $L_K^w(u)$ is the Gaussian process with zero mean and the covariance function

$$\begin{aligned}
\rho(u_1, u_2) &= \mathbb{E}L_K^w(u_1)L_K^w(u_2) = \tilde{c}(K)\left|V^{(-1)}(u_1)K \cap V^{(-1)}(u_2)K\right| \\
&= \tilde{c}(K)\,|K| \min\{(V^{(-1)}(u_1))^d, (V^{(-1)}(u_2))^d\} \\
&= C(\min\{u_1, u_2\})^{2H},\ u_1, u_1 \in [0,1],
\end{aligned} \tag{50}$$

where $H = \frac{d}{2(d+2\beta)}$ and

$$C = \tilde{c}(K)\,|K| \left(\int_K w^2(t)\,dt\right)^{-\frac{d}{d+2\beta}}.$$

From the above expression for the covariance function $\rho(u_1, u_2)$ we have

$$\rho(au_1, au_2) = a^{2H}\rho(u_1, u_2),$$

which means that the process $L_K^w(u)$ given by (49) is a self-similar process with the self-similarity parameter $H = \frac{d}{2(d+2\beta)} < 1$.

From (50) we also obtain

$$E\left[(L_K^w(u_2) - L_K^w(u_1))^2\right] = C(u_2^{2H} - u_1^{2H}),$$

for $u_1 < u_2$, which entails the following equality in distribution:

$$L_K^w(u_2) - L_K^w(u_1) \overset{d}{=} \sqrt{C(u_2^{2H} - u_1^{2H})}\, Z,$$

where $Z \sim N(0,1)$, and thus we can write

$$E|L_K^w(u_2) - L_K^w(u_1)|^{2k} = E|Z|^{2k} C^k |u_2^{2H} - u_1^{2H}|^k$$

with $E|Z|^{2k} = \pi^{-1/2}\, 2^k \Gamma(k+1/2)$. Since the function $f(u) = u^\gamma$ $(\gamma \leq 1)$, $u \in [0,1]$, is Hölder continuous with the Hölder exponent $0 < \alpha \leq \gamma$, then $|u_2^{2H} - u_1^{2H}| \leq const|u_2 - u_1|^{2H}$ and for any $k \geq 1$ we obtain:

$$E|L_K^w(u_2) - L_K^w(u_1)|^{2k} \leq const|u_2 - u_1|^{kd/(d+2\beta)}.$$

Taking $k > 1 + \frac{2\beta}{d}$, we have the exponent in the right hand side $\frac{kd}{d+2\beta} > 1$, that is, the process $L_K^w(u)$ satisfies Kolmogorov's criterion. Thus, the stochastic process (49) induces the probability measure P in the space $C[0,1]$ of continuous functions with the uniform topology.

Analogously to the derivations of the previous section, basing the proof of weak compactness of measures P_T induced by the stochastic processes (48) on Kolmogorov's criterion in the form (38), we must check now that

$$\frac{1}{V(T)^2}\mathbb{E}\left[\int_{t\in V^{(-1)}(v)K_T \setminus V^{(-1)}(u)K_T} w(t)X_t dt\right]^4 \leq const\,|v-u|^2,\ 0 \leq u \leq v \leq 1. \tag{51}$$

The same derivations as those in Section 4 will lead to the expression for the right hand side of (51) in the form of the sum $I_1 + I_2 + I_3 + I_4$, where now the function $w(t)$ will be involved and correspondingly in the formulas (40), (41) and (42) $\Delta_{\widetilde{K}_T}(\lambda)$ will be changed for $\Delta^w_{\widetilde{K}_T}(\lambda) = \int_{t\in\widetilde{K}_T} w^2(t)dt$ with $\widetilde{K}_T$ being now of the form $\widetilde{K}_T = V^{(-1)}(v)K_T \setminus V^{(-1)}(u)K_T$.

Therefore, supposing f_2 to be bounded and $f_4 \in L_3$, we come to the following bound

$$
\begin{aligned}
& \mathbb{E}\left[\int_{t\in V^{(-1)}(v)K_T\setminus V^{(-1)}(u)K_T} w(t)X_t dt\right]^4 \\
\leq\ & const\left\{\int_{R^d}\left|\Delta^w_{\widetilde{K}_T}(\lambda)\right|^2 d\lambda\right\}^2 \\
=\ & const\left\{\int_{R^d}\left|\int_{t\in\widetilde{K}_T} w(t)e^{it\lambda}dt\right|^2 d\lambda\right\}^2 \\
=\ & const(2\pi)^{2d}\left\{\int_{t\in\widetilde{K}_T} w^2(t)dt\right\}^2 . \qquad (52)
\end{aligned}
$$

(Note that (52) can be compared with the formula (1.8.11) in [28], which gives more general result, namely, the bounds for odd order higher moments).

Using (46) we can derive

$$
\begin{aligned}
\int_{t\in\widetilde{K}_T} w^2(t)dt &= \int_{t\in V^{(-1)}(v)K_T\setminus V^{(-1)}(u)K_T} w^2(t)dt \\
&= V(TV^{(-1)}(v)) - V(TV^{(-1)}(u)) \\
&= T^{d+2\beta}(V(V^{(-1)}(v)) - V(V^{(-1)}(u))) \\
&= T^{d+2\beta}(v-u). \qquad (53)
\end{aligned}
$$

From (46) we know also that $V(T) = (2\pi)^{-2d}T^{d+2\beta}const$, which combined with (53) and (52) gives (51). Therefore, weak compactness of measures P_T induced by the stochastic processes (48) takes place under the conditions that the second order spectral density f_2 is bounded and the fourth order spectral density f_4 is in L_3.

To prove the convergence of finite-dimensional distributions of (48) we can consider the sum $Z_T = V(T)^{-1/2}\sum_{j=1}^m c_j Y_T^w(u_j)$ and argue analogously to the corresponding derivations in Section 4, introducing also some additional conditions to guarantee the convergence of variance. However, in this general case, the conditions for appropriate asymptotic behavior of variance, if formulated analogously to those in the previous section, will look rather artificial. We believe that this point can be investigated further elsewhere for particular sets K_T.

Appendix A. The homogeneous and nonhomogeneous Hölder-Young-Brascamp-Lieb inequalities

We have mentioned already in the introduction that the Hölder-Young-Brascamp-Lieb inequality gives the possibility to evaluate the integrals of the form (3) under conditions on integrability indices of functions f_i.

The Hölder-Young-Brascamp-Lieb inequality was clarified and considerably generalized recently by Ball [8], Barthe [9], Carlen, Loss and Lieb [14], and Bennett, Carbery, Christ and Tao [11], [10], the end result being of replacing the linear functionals with surjective linear operators: $l_j(x) : S^m \rightarrow S^{n_j}, j = 1, ..., k$, with $\cap_1^k ker(l_j) = \{0\}$.

Following the remarkable exposition of [10], [11], we give the formulation of this inequality in the way the most relevant to the context of the present paper (see Theorem 2.1 of [10]).

Let H, H_1, ..., H_m be Hilbert spaces of finite positive dimensions, each being equipped with the corresponding Lebesgue measure; functions $f_j : H_j \rightarrow R$, $j = 1, ..., m$, satisfy the integrability conditions $f_j \in L_{p_j}$, $j = 1, ..., m$.

Theorem A.1 below specifies, in terms of certain linear inequalities on

$$z_j = \frac{1}{p_j}, \; j = 1, \ldots, m,$$

the "power counting polytope" PCP within which the Hölder inequality is valid.

Theorem A.1 *(Nonhomogeneous Hölder-Young-Brascamp-Lieb inequality). Let $l_j(x), j = 1, ..., m$ be surjective linear transformations $l_j : H \rightarrow H_j$, $j = 1, ..., m$. Let f_j, $j = 1, \ldots, m$ be functions $f_j \in L_{p_j}(\mu(dx))$, $1 \leq p_j \leq \infty$ defined on H_j, where $\mu(dx)$ is Lebesgue measure.*

Then, the Hölder-Young-Brascamp-Lieb inequality

$$(GH) \qquad \left| \int_H \prod_{j=1}^m f_j(l_j(\mathbf{x}))\mu(d\mathbf{x}) \right| \leq K_H \prod_{j=1}^m \|f_j\|_{p_j}$$

holds if and only if

(1) $\quad \dim(H) = \sum_j z_j \dim(H_j),$

and

(2) $\quad \dim(V) \leq \sum_j z_j \dim(l_j(V))$, *for every subspace $V \subset H$.*

Given that (1) holds, (2) is equivalent to

(3) $\quad \mathrm{codim}_H(V) \geq \sum_j z_j \mathrm{codim}_{H_j}(l_j(V))$, *for every subspace $V \subset H$.*

Here $\dim(V)$ *denotes the dimension of the vector space V and* $\mathrm{codim}_H(V)$ *denotes the codimension of a subspace $V \subset H$.*

Note also that any two of conditions (1), (2), (3) imply the third.

We recall now Theorem C.1 of [6] in which the homogeneous HYBL inequality is stated simultaneously in the three cases:

(C1) $\mu(dx_j)$ is normalized Lebesgue measure on the torus $[-\pi, \pi]^d$, and M has all its coefficients integers.
(C2) $\mu(dx_j)$ is counting measure on $\mathbb{Z}^d$, M has all its coefficients integers, and is unimodular, i.e. all its non-singular minors of dimension $m \times m$ have determinant ± 1.
(C3) $\mu(dx_j)$ is Lebesgue measure on R^d.

The domain of validity of the generalized Hölder's inequality is specified in terms of linear inequalities on $z_j = {p_j}^{-1},\ j = 1, \ldots, m$, which involve the rank $r(A)$ of arbitrary subsets A of columns of the matrix M or "dual rank" $r^*(A)$ defined by a dual matrix M^* whose lines are orthogonal to those of M.

Theorem C.1 *(Homogeneous Hölder-Young-Brascamp-Lieb inequality).*

Let $l_j(x) = x^t \alpha_j, j = 1, ..., k$, be linear functionals $l_j : S^m \to S$, where the space S is either the torus $[-\pi, \pi]^d$, $\mathbb{Z}^d$, or $\mathbb{R}^d$. Let M denote the matrix with columns $\alpha_j, j = 1, ..., k$.

Let $f_j,\ j = 1, \ldots, k$ be functions $f_j \in L_{p_j}(\mu(dx)),\ 1 \le p_j \le \infty$, defined on S, where $\mu(dx)$ is respectively normalized Lebesgue measure, counting measure and Lebesgue measure, $z_j = {p_j}^{-1},\ j = 1, \ldots, k$, and $z = (z_1, ..., z_k)$.

The Hölder-Young-Brascamp-Lieb inequality (GH) will hold (with $K_{_H} = K_{_H}(z) < \infty$) throughout the "power counting polytopes" PCP defined respectively by:

(c1) $\displaystyle\sum_{j \in A} z_j \le r(A),\ \forall A$

(c2) $\displaystyle\sum_{j \in A} z_j \ge r(M) - r(A^c), \quad \Leftrightarrow \quad \sum_{j \in A} (1 - z_j) \le r^*(A)\ \forall A$

(c3) $\displaystyle\sum_{j=1}^{k} z_j = m$, *and one of the conditions (c1) or (c2) is satisfied.*

Notes: 1) The domain of convergence (for fixed $(l_1, ..., l_k)$) is called "power counting polytope" PCP, cf. the terminology in the physics literature, where this polytope was already known (at least as integrability conditions for power functions), in the case $n_j = 1, \forall j$. Note that a general explicit form of the facets of PCP when $n_j > 1$ for some j, has not been found yet.

2) Some related and interesting inequalities and an application to an analysis of integrals involving cyclic products of kernels can be found in [17].

Acknowledgments

We thank the referee for constructive remarks and suggestions which helped to improve the presentation significantly.

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