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ORTHOGONALLY SPHERICAL OBJECTS AND SPHERICAL FIBRATIONS

RINA ANNO AND TIMOTHY LOGVINENKO

ABSTRACT. We introduce a relative version of the spherical objects of Seidel and Thomas [ST01]. Define an object E in the derived category $D(Z \times X)$ to be spherical over Z if the corresponding functor from $D(Z)$ to $D(X)$ gives rise to autoequivalences of $D(Z)$ and $D(X)$ in a certain natural way. Most known examples come from subschemes of X fibred over Z . This categorifies to the notion of an object of $D(Z \times X)$ orthogonal over Z . We prove that such an object is spherical over Z if and only if it possesses certain cohomological properties similar to those in the original definition of a spherical object. We then interpret this geometrically in the case when our objects are actual flat fibrations in X over Z .

1. INTRODUCTION

Let X be a smooth projective variety over \mathbb{C} and $D(X)$ be the bounded derived category of coherent sheaves on X . Following certain developments in mirror symmetry Seidel and Thomas introduced in [ST01] the notion of a *spherical object*:

Definition 1.1 ([ST01]). An object E of $D(X)$ is *spherical* if:

- (1) $\mathrm{Hom}_{D(X)}^i(E, E) = \begin{cases} \mathbb{C}, & \text{if } i = 0 \text{ or } \dim X, \\ 0, & \text{otherwise} \end{cases}$
- (2) $E \simeq E \otimes \omega_X$ where ω_X is the canonical bundle of X .

The motivating idea came from considering Lagrangian spheres on a symplectic manifold. Given such a sphere one can associate to it a symplectic automorphism called the Dehn twist. Correspondingly:

Theorem ([ST01]). Let $E \in D(X)$. The twist functor T_E is a cone we can associate to the natural transformation $E \otimes_{\mathbb{C}} \mathbf{R}\mathrm{Hom}_X(E, -) \xrightarrow{\mathrm{eval}} \mathrm{Id}_{D(X)}$. If E is spherical, then T_E is an autoequivalence of $D(X)$.

Spherical twists can be used to construct braid group actions on $D(X)$, as was indeed the main concern of [ST01]. They also deserve to be studied in their own right as some of the simplest non-trivial autoequivalences of $D(X)$ which do not come from autoequivalences of the underlying abelian category $\mathrm{Coh}(X)$. In fact, on smooth toric surfaces or on surfaces of general type whose canonical model has at worst A_n -singularities the whole of $\mathrm{Aut} D(X)$ is generated by spherical twists, $\mathrm{Aut} \mathrm{Coh}(X)$ and the shift functor ([IU05], [BP10]). In more complicated cases spherical twists are still an essential tool in studying the autoequivalences of $D(X)$ and stability conditions on it ([Bri08], [Bri09], [Bri06]).

In this paper we study a relative version of the construction above where instead of a single object we have a family of objects in $D(X)$ parametrised by a base Z . A geometric example of this is a subvariety W of X flatly fibred over Z . It can be thought of as a family of subschemes of X parametrised by Z . Even when the structure sheaf of W is not itself spherical in sense of [ST01] one may still produce an autoequivalence of $D(X)$ by exploiting the extra fibration structure which W possesses. Moreover, one can do this completely abstractly, working with families of arbitrary objects of $D(X)$ and not just families of subschemes of X . We characterize those families of objects of $D(X)$ for which this is possible in terms of applicable cohomological criteria similar to Definition 1.1 above. Our study is a self-contained exercise in derived categories of coherent sheaves and doesn't involve mirror symmetry. One should mention though that the original examples of these family twists were inspired by Kontsevich's proposal that the autoequivalences of $D(X)$ should correspond to loops in the moduli space of complex structures on its mirror, cf. [Hor99, §4.1], [Hor05], [Sze01], [Sze04]. Maybe in future our results could be used to construct further, more general examples of this correspondence.

Consider an object E in the derived category $D(Z \times X)$ of the product of Z and X . We can view E as a *family of objects in $D(X)$ parametrised by Z* by considering the fibres of E over points of Z to be the derived

pullbacks of E to the corresponding fibres of $X \times Z$ over Z :

$$\begin{array}{ccc} X & \xrightarrow{\iota_{X_p}} & Z \times X \\ \downarrow & & \downarrow \pi_Z \\ \bullet & \xrightarrow{\iota_p} & Z \end{array} \quad \forall p \in Z \quad E_p = \iota_{X_p}^* E$$

On the other hand, each object $E \in D(Z \times X)$ defines naturally a functor $\Phi_E: D(Z) \rightarrow D(X)$ called the *Fourier–Mukai transform with kernel E* which sends point sheaves \mathcal{O}_p on Z to the fibres $E_p \in D(X)$ [Huy06]. The interplay between these two points of view, moduli-theoretic and functorial, led to a string of celebrated results by Mukai, Bondal and Orlov, Bridgeland and others.

When Z is the point scheme $\text{Spec } \mathbb{C}$ the above formalism tells us to view an object $E \in D(X)$ as a functor $\Phi_E = E \otimes_{\mathbb{C}} (-)$ from $D(\mathbf{Vect})$ to $D(X)$. Then the functor $E \otimes_{\mathbb{C}} \mathbf{R}\text{Hom}_X(E, -)$ is the composition of Φ_E with its right adjoint Φ_E^{radj} and the above definition of the twist functor T_E amounts to T_E being a cone of the adjunction co-unit

$$(1.1) \quad \Phi_E \Phi_E^{\text{radj}} \longrightarrow \text{Id}_{D(X)}.$$

There is a subtlety involved here: taking cones, famously, is not functorial in $D(X)$, so the cone of a morphism between two functors is not *a priori* well defined. However in [AL12] it is shown that in a very general context we can represent both functors in (1.1) by Fourier–Mukai kernels and then represent the adjunction co-unit (1.1) by a natural morphism μ between these kernels. We can therefore define the twist functor T_E as the Fourier–Mukai transform whose kernel is the cone of μ and consider the following:

Problem: Describe the objects E in $D(Z \times X)$ for which the twist T_E is an autoequivalence of $D(X)$.

A partial answer was provided by Horja in [Hor05] for smooth Z and X . He considers only those objects E of $D(Z \times X)$ which come from the derived category of a smooth subscheme of X flatly fibred over Z . For these he gives a cohomological criterion sufficient for the twist T_E to be an autoequivalence of $D(X)$. Another special case was treated by Toda in [Tod07] who studied infinitesimal deformations and so assumed X to be a smooth projective variety and Z to be the Spec of a local artinian \mathbb{C} -algebra. In [AL13] we took different approach and abstracted out the properties of the functors Φ_E defined by spherical objects of [ST01] and [Hor05] which are exploited in proving that the twists T_E are autoequivalences. In all these cases not only T_E is an autoequivalence, but this autoequivalence identifies naturally the left and right adjoints of Φ_E . Moreover, in all these cases the same is true of the *co-twist* F_E , defined as the cone of the adjunction unit $\text{Id}_{D(Z)} \rightarrow \Phi_E^{\text{radj}} \Phi_E$ shifted by 1 to the right. In [AL13, Theorem 5.1] we prove a general result which implies that, in fact, for any Fourier–Mukai transform $D(Z) \xrightarrow{\Phi_E} D(X)$ we have

$$(1.2) \quad \left\{ \begin{array}{l} F_E \text{ is an autoequivalence} \\ \Phi_E^{\text{radj}} \simeq F_E \Phi_E^{\text{ladj}}[1] \end{array} \right\} \quad \text{if and only if} \quad \left\{ \begin{array}{l} T_E \text{ is an autoequivalence} \\ \Phi_E^{\text{ladj}} T_E[-1] \simeq \Phi_E^{\text{radj}} \end{array} \right\}$$

The functors which possess these equivalent properties are called *spherical*, cf. [AL13]. We thereby define:

Definition (Definition 3.4). An object $E \in D(Z \times X)$ is *spherical over Z* if the corresponding Fourier–Mukai transform $\Phi_E: D(Z) \rightarrow D(X)$ is spherical. In other words, if:

- (1) The co-twist F_E is an autoequivalence of $D(Z)$.
- (2) The natural transformation $\Phi_E^{\text{radj}} \xrightarrow{(2.38)} F_E \Phi_E^{\text{ladj}}[1]$ is an isomorphism of functors.

When $Z = \text{Spec } \mathbb{C}$ this is equivalent to Definition 1.1 above (Example 3.5). It also explains why most of the examples over a non-trivial base Z came from subschemes of X fibred over Z . These are the cases when the autoequivalence F_E has a particularly nice form. Indeed, for such fibrations the Fourier–Mukai kernel of F_E must be supported on the diagonal Δ of $Z \times Z$ (Lemma 3.9), and an autoequivalence of $D(Z)$ is supported on Δ if and only if it is simply tensoring by some shifted line bundle \mathcal{L}_E in $D(Z)$ (Prop. 3.7). This makes the Fourier–Mukai kernel of $\Phi_E^{\text{radj}} \Phi_E$, a certain $\mathbf{R}\text{Hom}$ complex, into an extension of $\Delta_* \mathcal{L}_E$ by $\Delta_* \mathcal{O}_X$. Pointwise, this becomes a familiar condition that a certain $\mathbf{R}\text{Hom}$ complex is $\mathbb{C} \oplus \mathbb{C}[d]$ for some $d \in \mathbb{Z}$.

In Section 3 of the present paper we show that this argument can be made very general. Let Z and X be arbitrary schemes of finite type over an algebraically closed field k of characteristic 0. No assumptions of smoothness or projectivity are made. Instead we make two assumptions on the object $E \in D(Z \times X)$: E is perfect (locally quasi-isomorphic to a bounded complex of free sheaves) and the support of E is proper over Z and over X . These are necessary for Φ_E to have adjoints which are again Fourier–Mukai transforms. We then categorify the notion of “a subscheme of X fibred over Z ”. The graphs of such subschemes in $Z \times X$ are characterised by the property that their fibres over points of Z are mutually disjoint in X . In

derived categories the notion of disjointness is expressed by orthogonality - vanishing of all Hom 's between two objects. Thus the objects we want are the objects in $D(Z \times X)$ which are *orthogonal* over Z , i.e. their fibres over points of Z are pairwise orthogonal in $D(X)$. In Lemma 3.9 we show that E is orthogonal over Z if and only if the support of the Fourier–Mukai kernel of the co-twist F_E lies within the diagonal Δ of $Z \times X$. It follows that such F_E is an autoequivalence if and only if it is the functor of tensoring by some invertible (locally a shifted line bundle) object of $D(Z)$. On the other hand, define an object \mathcal{L}_E to be the cone of

$$(1.3) \quad \mathcal{O}_Z \xrightarrow{\text{Definition 3.6}} \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_{X*} \pi_X^* E, E) \quad \pi_Z, \pi_X \text{ are projections } Z \times X \rightarrow Z, X$$

We show in Prop. 3.7 that if \mathcal{L}_E is invertible then necessarily $F_E \simeq (-) \otimes \mathcal{L}_E$. To check whether \mathcal{L}_E is invertible we restrict (1.3) to points of Z , whence we obtain our main theorem.

Theorem (Theorem 3.1). *Let Z and X be two separable schemes of finite type over k . Let E be a perfect object of $D(Z \times X)$ orthogonal over Z and proper over Z and X . Then E is spherical over Z if and only if:*

- (1) *For every closed point $p \in Z$ such that the fibre E_p is not zero*

$$\mathbf{R} \text{Hom}_X(\pi_{X*} E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}$$

and the natural morphism $\pi_{X} E \xrightarrow{(3.12)} E_p$ is non-zero.*

- (2) *The canonical morphism α (see Definition 3.11) is an isomorphism:*

$$E^\vee \otimes \pi_X^! (\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^! (\mathcal{L}_E)$$

Interestingly, a similar statement can be made for kernels of Fourier–Mukai equivalences, cf. Example 3.3.

If E is orthogonally spherical, then d_p in (1) has to be constant on every connected component of Z . We show further in Prop. 3.12 that for any Gorenstein $(z, x) \in \text{Supp}_{Z \times X} E$ we have

$$(1.4) \quad d_z = -(\dim_x X - \dim_z Z).$$

The canonical morphism α in (2) is a morphism of Fourier–Mukai kernels which induces the natural transformation $\Phi^{adj} \rightarrow F_E \Phi^{radj}[1]$ in Definition 3.6. Due to this indirect definition it may be very difficult, even in simple cases, to write α down explicitly and check that it is an isomorphism. It may be similarly difficult to check that $\pi_{X*} E \xrightarrow{(3.12)} E_p$ is non-zero in (1). In §3.4 we show that when applying Theorem 3.1 in the ‘if’ direction we can omit both of these awkward checks whenever the integer d_p in condition (1) is always negative, cf. Theorem 3.2. For Z and X reasonably nice e.g. abstract varieties this corresponds by (1.4) to the case where $\dim Z < \dim X$.

Setting $Z = \text{Spec } \mathbb{C}$ in Theorem 3.1 turns conditions (1) and (2) into the original definition of a spherical object E in [ST01]. Similarly, setting $Z = \text{Spec } R$ for some local artinian \mathbb{C} -algebra R yields the definition of an R -spherical object E in [Tod07], §2. Note that we also obtain the converse implication - if T_E is an autoequivalence of $D(X)$ which identifies the left and right adjoints of Φ_E , then E has to satisfy the conditions (1) and (2) of Theorem 3.1.

In Section 4 we reconsider the case of flat fibrations. Let $\xi: W \hookrightarrow X$ be a subscheme with $\pi: W \rightarrow Z$ a flat and surjective map. We apply the results of Section 3 to \mathcal{O}_W in $D(Z \times X)$. One of our goals is to understand what geometric properties a spherical fibration must possess. The two technical assumptions on the object E in Section 3 translate to the assumptions of the fibres of W over Z being proper and of \mathcal{O}_W being a perfect object of $D(Z \times X)$. We first give the most general analogue of Theorem 3.1 which applies to any flat fibration W with the above properties (Theorem 4.1). We improve on it for the case when either the fibres of W are Gorenstein schemes or ξ is a Gorenstein map, noting that for any spherical W these two conditions are, in fact, equivalent (Prop. 4.8). Finally, we treat the case when the immersion ξ is *regular*, i.e. locally on X the ideal of W is generated by a regular sequence. In such case the cohomology sheaves of $\xi^* \xi_* \mathcal{O}_W$ are the vector bundles $\wedge^j \mathcal{N}^\vee$ where \mathcal{N} is the normal sheaf of W in X . The object $\xi^* \xi_* \mathcal{O}_W$ is the key to computing the Ext complex in the condition (1) of Theorem 3.1 and therefore (1) can be deduced via a spectral sequence argument from fibrewise vanishing of the cohomology of $\wedge^j \mathcal{N}$. In fact, the reverse implication can also be obtained if the complex $\xi^* \xi_* \mathcal{O}_W$ actually splits up as a direct sum of $\wedge^j \mathcal{N}^\vee[j]$. In [AC10] Arinkin and Caldararu had shown that for a smooth X this happens if and only if \mathcal{N} extends to the first infinitesimal neighborhood of W in X , e.g. when W is carved out by a section of a vector bundle, or when W is the fixed locus of a finite group action, or when ξ can be split. For any regular immersion ξ we say that it is *Arinkin–Caldararu* if $\xi^* \xi_* \mathcal{O}_W$ splits up as the direct sum of its cohomology sheaves. Then:

Theorem (Theorem 4.2). *Let W be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Let \mathcal{N} be the normal sheaf of W in X . Then W is spherical if for any closed point $p \in Z$ the fibre W_p is a connected Gorenstein scheme and*

- (1) $H_{W_p}^i(\wedge^j \mathcal{N}|_{W_p}) = 0$ unless $i = j = 0$ or $i = \dim W_p$, $j = \operatorname{codim}_X W$.
(2) $(\omega_{W/X})|_{W_p} \simeq \omega_{W_p}$ where ω_{W_p} is the dualizing sheaf of W_p and $\omega_{W/X} = \wedge^{\operatorname{codim}_X} W \mathcal{N}$.

Conversely, if W is spherical, then each fibre W_p is a connected Gorenstein scheme and (2) holds. If ξ is an Arinkin-Caldararu immersion, then (1) also holds.

The ‘If’ implication here generalises the result in [Hor05], where Z , W and X are assumed to be smooth. The same argument works for any object in $D(W)$ and not just \mathcal{O}_W . We also obtain the converse implication. Any spherical fibration W which is Arinkin-Caldararu must therefore satisfy $H_{W_p}^i(\mathcal{O}_{W_p}) = 0$ for all $i > 0$, which matches the fact that in the known examples the fibres of spherical fibrations are Fano varieties.

Section 2 contains the preliminaries necessary for all of the above. In §2.2 we work out explicitly the morphisms of kernels which underly the left and right adjunction units of a general Fourier–Mukai functor. We need this to compute F_E since co-twist functors need to be defined as the cones of adjunction units. We get this result for free from the similar result for adjunction co-units in [AL12] using the Grothendieck duality arguments summarized in §2.1. We then review the formalism of spherical functors in Section §2.3.

Finally, in the Appendix we give an example of an orthogonally spherical object which is not a spherical fibration and which is a genuine complex and not just a shifted sheaf. It arises naturally when constructing an affine braid group action on (n, n) -fibre of the Grothendieck-Springer resolution of the nilpotent cone of $\mathfrak{sl}_{2n}(\mathbb{C})$. The authors hope that the tools developed in this paper will allow to construct more examples of orthogonally spherical objects which aren’t sheaves and to study explicitly the derived autoequivalences which they induce.

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2. PRELIMINARIES

Notation: Throughout the paper we define our schemes over the base field k which is assumed to be an algebraically closed field of characteristic 0. We also denote by \mathbf{Vect} the category of finite-dimensional vector spaces over k . Given a fibre product $X_1 \times \cdots \times X_n$ we usually denote by π_i the projection $X_1 \times \cdots \times X_n \rightarrow X_i$ onto the i -th component.

Let X be a scheme. We denote by $D_{\text{qc}}(X)$, resp. $D(X)$, the full subcategory of the derived category of \mathcal{O}_X - \mathbf{Mod} consisting of complexes with quasi-coherent, resp. bounded and coherent, cohomology. Given an object E in $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ we denote by $\mathcal{H}^i(E)$ the i -th cohomology sheaf of E and by E^\vee its derived dual, the object $\mathbf{R}Hom_X(E, \mathcal{O}_X)$.

All the functors in this paper are assumed to be derived unless mentioned otherwise.

We therefore omit all the usual \mathbf{R} ’s and \mathbf{L} ’s. An exception is made for the derived bi-functor $\mathbf{R}Hom_X(-, -)$ of taking the space of morphisms between a pair of sheaves in $\text{Coh}(X)$. This is to distinguish for any $A, B \in D(X)$ the complex $\mathbf{R}Hom_X(A, B)$ in $D(\mathbf{Vect})$ from the vector space $\text{Hom}_{D(X)}(A, B)$ of morphisms from A to B in $D(X)$. Another exception was made for the derived bi-functor $\mathbf{R}Hom_X(-, -)$ of taking the sheaf of morphisms between a pair of sheaves. This is for it to still look like a curly version of $\mathbf{R}Hom_X(-, -)$.

All the categories we consider are most certainly 1-categories. However given a morphism $A \rightarrow B$ in a category we can consider it as a (trivial) commutative diagram. For two commutative diagrams of the same shape there is a well defined notion of them being isomorphic, e.g. in our case $A \rightarrow B$ is isomorphic to another diagram $A' \rightarrow B'$ if and only if there exist isomorphisms which make the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \simeq \downarrow & & \downarrow \simeq \\ A' & \longrightarrow & B' \end{array}$$

commute. Sometimes as an abuse of notation we describe this by saying that morphism $A \rightarrow B$ is ‘isomorphic’ to morphism $A' \rightarrow B'$. Clearly this imposes an equivalence relation on the set of morphisms in a given category. This equivalence relation is important in the context of triangulated categories because all the morphisms in the same equivalence class have isomorphic cones.

2.1. On duality theories. The standard reference on Grothendieck-Verdier duality has for some time been [Har66]. There the duality theory is constructed by hand in a (comparatively) geometric and (comparatively) painful fashion. For a more modern and (comparatively) more elegant categorical approach which obtains the existence of the right adjoint to f_* by pure thought we can recommend the reader Lipman’s excellent

exposition in [Lip09] which expands greatly on the Deligne's elegant but brief note [Del66]. Below we give a brief overview of the results we intend to use. Our approach relies heavily on the notion of a perfect object in a derived category, both in an absolute sense and relative to a morphism. The reader may find this discussed at length in [Ill71b] and [Ill71a].

Let S be a Noetherian scheme. Let \mathcal{FT}_S be the category of separated schemes of finite type over S whose morphisms are separated S -scheme maps of finite type. We have the following (relative) duality theory $D_{\bullet/S}$ for schemes in \mathcal{FT}_S : for any $X \xrightarrow{f} S$ let $D_{X/S}$ denote the functor $\mathbf{R}\mathcal{H}om(-, f^! \mathcal{O}_X)$ from $D(\mathcal{O}_X\text{-Mod})$ to $D(\mathcal{O}_X\text{-Mod})^{op}$. Here $(-)^!$ is the twisted inverse image pseudo-functor, cf. [Lip09, Theorem 4.8.1]. It follows from [Ill71a, Cor. 4.9.2] that $D_{X/S}$ takes $D_{S\text{-perf}}(X)$, the full subcategory of $D(X)$ consisting of objects perfect over S , to itself in the opposite category and the restriction is a self-inverse equivalence

$$D_{X/S}: D_{S\text{-perf}}(X) \xrightarrow{\sim} D_{S\text{-perf}}(X)^{op}.$$

Now, given any two schemes X and Y in \mathcal{FT}_S and any exact functor $F: D_{S\text{-perf}}(X) \rightarrow D_{S\text{-perf}}(Y)$ we define its dual under $D_{\bullet/S}$ to be the functor $D_{Y/S} F D_{X/S}: D_{S\text{-perf}}(X) \rightarrow D_{S\text{-perf}}(Y)$. The double-dual of a functor is then the functor itself and we say that F and $D_{Y/S} F D_{X/S}$ are *dual under $D_{\bullet/S}$* . The (contravariant) notion of a dual of a morphism of functors is defined accordingly. One can then easily see that if a functor has a left (resp. right) adjoint then $D_{\bullet/S}$ sends it to the right (resp. left) adjoint of its dual and interchanges the adjunction units with the adjunction co-units.

Let X be a scheme in \mathcal{FT}_S and let E be a perfect (in an absolute sense) object of $D(\mathcal{O}_X\text{-Mod})$. Then the functor $E \otimes (-)$ takes $D_{S\text{-perf}}(X)$ to $D_{S\text{-perf}}(X)$, its adjoint, both left and right, is the functor $E^\vee \otimes (-)$ and for any $F \in D(\mathcal{O}_X\text{-Mod})$ we have by [AIL10, Lemma 1.4.6] a natural isomorphism

$$(2.1) \quad D_{X/S}(E \otimes F) \xrightarrow{\sim} E^\vee \otimes D_{X/S}F.$$

Therefore $E \otimes (-)$ and $E^\vee \otimes (-)$ are dual under $D_{\bullet/S}$. Consequently, $D_{\bullet/S}$ interchanges the adjunction unit $\text{Id} \rightarrow E^\vee \otimes E \otimes (-)$ and the adjunction co-unit $E^\vee \otimes E \otimes \text{Id} \rightarrow \text{Id}$.

Let $X \xrightarrow{f} Y$ be a proper map in \mathcal{FT}_S . Then f_* sends $D_{S\text{-perf}}(X)$ to $D_{S\text{-perf}}(Y)$. By the sheafified Grothendieck duality, cf. [Lip09, Cor. 4.4.2], for any $E \in D_{qc}(X)$ the natural map

$$(2.2) \quad D_{Y/S}(f_* E) \xrightarrow{\sim} f_*(D_{X/S}E).$$

is an isomorphism. It follows that f_* is self-dual under $D_{\bullet/S}$.

On the other hand, let $X \xrightarrow{f} Y$ be any map in \mathcal{FT}_S . By [Lip09, Prop. 4.10.1] there is for any $E \in D(Y)$ a natural isomorphism

$$(2.3) \quad D_{X/S}(f^* E) \xrightarrow{\sim} f^!(D_{Y/S}E).$$

If f^* takes $D_{S\text{-perf}}(Y)$ to $D_{S\text{-perf}}(X)$, e.g. f is perfect, it follows that f^* and $f^!$ are dual under $D_{\bullet/S}$.

Note that f^* is the left adjoint of f_* and, if f is proper, $f^!$ is its right adjoint. So for f proper and perfect f^* and $f^!$ being dual under $D_{\bullet/S}$ is precisely equivalent to f_* being self-dual.

If f is proper, then even if f^* doesn't take S -perfect objects to S -perfect objects, it still follows from the definitions of maps (2.2) and (2.3) in [Lip09] that for any $E \in D(Y)$ the following diagram commutes

$$(2.4) \quad \begin{array}{ccc} D_{Y/S}(f_* f^* E) & \xrightarrow{(\text{Id} \rightarrow f_* f^*)^{opp}} & D_{Y/S}E \\ \downarrow \simeq & \nearrow f_* f^! \rightarrow \text{Id} & \\ f_* f^! D_{Y/S}E & & \end{array}$$

i.e. $D_{\bullet/S}$ still send the adjunction unit $\text{Id} \rightarrow f_* f^*$ to the adjunction co-unit $f_* f^! \rightarrow \text{Id}$. We then also have:

Lemma 2.1. *Let $X \xrightarrow{f} Y$ be any map in \mathcal{FT}_S , let E be a perfect object in $D(\mathbf{Mod}\text{-}Y)$ and let F be an S -perfect object in $D(\mathbf{Mod}\text{-}Y)$. Then the natural map*

$$(2.5) \quad f^* E \otimes f^! F \rightarrow f^!(E \otimes F)$$

is an isomorphism.

Proof. By compactification [Nag62] such f decomposes into an open immersion followed by a proper map.

If f is an open immersion, then the map (2.5) is by definition the isomorphism $f^* E \otimes f^* F \rightarrow f^*(E \otimes F)$.

It remains to consider the case of f being a proper map. Then, by definition, the map (2.5) is the right adjoint with respect to f_* of the composition

$$(2.6) \quad f_*(f^*E \otimes f^!F) \xrightarrow{\text{inverse of projection formula map}} E \otimes f_*f^!F \xrightarrow{f_*f^! \rightarrow \text{Id}} E \otimes F.$$

Using the duality isomorphism $D_{\bullet/S}D_{\bullet/S}F \simeq F$, isomorphisms (2.1)-(2.3) and (2.4), we can re-write (2.6) as

$$(2.7) \quad D_{\bullet/S} \left(E^\vee \otimes D_{\bullet/S}F \xrightarrow{\text{Id} \rightarrow f_*f^*} E^\vee \otimes f_*f^*D_{\bullet/S}F \xrightarrow{\text{projection formula map}} f_*(f^*E^\vee \otimes f^*D_{\bullet/S}F) \right)^{\text{opp}}$$

which by [AL12], Lemma 2.1 is the same map as

$$(2.8) \quad D_{\bullet/S} \left(E^\vee \otimes D_{\bullet/S}F \xrightarrow{\text{Id} \rightarrow f_*f^*} f_*f^*(E^\vee \otimes D_{\bullet/S}F) \xrightarrow{f^*(-\otimes-) \rightarrow f^* \otimes f^*} f_*(f^*E^\vee \otimes f^*D_{\bullet/S}F) \right)^{\text{opp}}.$$

Using (2.1)-(2.3), (2.4) and $D_{\bullet/S}D_{\bullet/S}F \simeq F$ again, we deduce that (2.6) is the same map as

$$(2.9) \quad f_*(f^*E \otimes f^!F) \xrightarrow{f_*\alpha} f_*f^!(E \otimes F) \xrightarrow{f_*f^! \rightarrow \text{Id}} E \otimes F$$

where the map α is isomorphic to

$$(2.10) \quad D_{\bullet/S} \left(f^*(E^\vee \otimes D_{\bullet/S}F) \xrightarrow{f^*(-\otimes-) \rightarrow f^* \otimes f^*} f^*E^\vee \otimes f^*D_{\bullet/S}F \right)^{\text{opp}}.$$

Since the right adjoint of (2.9) with respect to f_* is clearly α , we conclude that the natural map (2.5) is precisely the map α which is evidently an isomorphism. \square

In the special case of $S = \text{Spec } k$ the category \mathcal{FT}_k is simply the category of all schemes of finite type over k . For any such scheme X we have $D_{S\text{-perf}}(X) = D_{\text{coh}}^b(X)$. The resulting duality theory $D_{\bullet/k}$ is the usual duality theory of [Har66] with $D_{X/k}(\mathcal{O}_X)$ being dualizing complexes in sense of [Har66], Chapter V.

On the other hand we have the perfect duality theory which exists in the category of arbitrary schemes. Let X be a scheme and let DP_X denote the functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ from $D(\mathcal{O}_X\text{-Mod})$ to $D(\mathcal{O}_X\text{-Mod})^{\text{opp}}$, i.e. $\text{DP}_X(E) = E^\vee$. It is shown in [Ill71b], §7 that DP_X takes $D_{\text{perf}}(X)$, the full subcategory of $D(\mathbf{Mod}\text{-}\mathcal{O}_X)$ consisting of perfect objects, to itself in the opposite category and the restriction is a self-inverse equivalence

$$\text{DP}_X: D_{\text{perf}}(X) \xrightarrow{\sim} D_{\text{perf}}(X)^{\text{opp}}.$$

Then, given any two schemes X and Y , we define just as above the notions of a dual under DP of any functor $F: D_{\text{perf}}(X) \rightarrow D_{\text{perf}}(Y)$ and of any natural transformation between two such functors. Once again, the duality interchanges left adjoints with right adjoints and the adjunction units with the adjunction counits.

Let $X \xrightarrow{f} Y$ be any scheme map. Then f^* sends $D_{\text{perf}}(Y)$ to $D_{\text{perf}}(X)$ and we have ([Ill71b], Prps. 7.1.2) for any $E \in D_{\text{perf}}(Y)$ a natural isomorphism

$$(2.11) \quad \text{DP}_X f^*E \xrightarrow{\sim} f^* \text{DP}_Y E.$$

It follows that f^* is self-dual under DP.

Now let $X \xrightarrow{f} Y$ be any scheme map such that f_* sends $D_{\text{perf}}(X)$ to $D_{\text{perf}}(Y)$, e.g. a quasi-perfect map of concentrated schemes ([Lip09], §4.7). Then, since f^* is self-dual, the dual of f_* under DP is the left adjoint f_{\dagger} of f^* . And when f is a separated finite-type perfect map of Noetherian schemes, we know ([AIL10], Lemma 2.1.10) that $f_{\dagger}(-)$ is naturally isomorphic to $f_*(f^!(\mathcal{O}_Y) \otimes -)$ in a way which makes the composition

$$f_{\dagger}f^* \xrightarrow{\sim} f_*(f^!(\mathcal{O}_Y) \otimes f^*(-)) \xrightarrow{\sim} f_*f^! \xrightarrow{\text{the adjunction co-unit}} \text{Id}$$

be precisely the adjunction co-unit $f_{\dagger}f^* \rightarrow \text{Id}$.

2.2. Adjunction units and Fourier–Mukai transforms. The definition of a spherical functor S in [AL13] demands for S to be a Fourier-Mukai transform whose left and right adjoints L and R are also Fourier Mukai transforms. Moreover, to compute the twist T_S and the co-twist F_S of S we need to write down the units and the co-units of these adjunctions on the level of Fourier-Mukai kernels.

Partly this was achieved in §3.1 of [AL12]. We give a brief summary here. Quite generally, let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object in $D(X_1 \times X_2)$. We have

a commutative diagram of projection morphisms:

$$(2.12) \quad \begin{array}{ccccc} & & X_1 \times X_2 \times X_1 & & \\ & \swarrow \pi_{12} & \downarrow \pi_{13} & \searrow \pi_{23} & \\ & X_1 \times X_2 & X_1 \times X_1 & X_2 \times X_1 & \\ \swarrow \pi_1 & & \swarrow \pi_2 & & \searrow \pi_1 \\ X_1 & & X_2 & & X_1 \\ \nwarrow \tilde{\pi}_1 & & \nwarrow \pi_2 & & \nwarrow \tilde{\pi}_2 \end{array}$$

Let $\Phi_E: D(X_1) \rightarrow D(X_2)$ be the Fourier–Mukai transform $\pi_{2*}(E \otimes \pi_1^*(-))$ with kernel E , then:

- (1) A left adjoint Φ_E^{ladj} to Φ_E exists and is isomorphic to the Fourier–Mukai transform $\Psi_{E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})}$ from $D(X_2)$ to $D(X_1)$.
- (2) A right adjoint Φ_E^{radj} to Φ_E exists and is isomorphic to the Fourier–Mukai transform $\Psi_{E^\vee \otimes \pi_2^!(\mathcal{O}_{X_2})}$ from $D(X_2)$ to $D(X_1)$.
- (3) The adjunction co-unit $\Phi_E^{\text{ladj}} \Phi_E \rightarrow \text{Id}_{D(X_1)}$ is isomorphic to the morphism $\Theta_Q \rightarrow \Theta_{\mathcal{O}_\Delta}$ of Fourier–Mukai transforms $D(X_1) \rightarrow D(X_1)$ induced by the morphism $Q \rightarrow \mathcal{O}_\Delta$ of objects of $D(X_1 \times X_1)$ written down explicitly in [AL12, Theorem 3.1] to which we refer the reader for all the details. An analogous statement holds for the adjunction co-unit $\Phi_E \Phi_E^{\text{radj}} \rightarrow \text{Id}_{D(X_2)}$, cf. [AL12, Theorem 3.2].
- (4) The condition of X_1 and X_2 being proper can be replaced by the condition of the support of E being proper over X_1 and over X_2 , cf. §2.2 of [AL12]. If E is a pushforward of an object in the derived category of a closed subscheme $X_1 \times X_2$ proper over X_1 and X_2 , then there is an alternative description of the morphisms of Fourier–Mukai kernels which produce the adjunction co-units $\Phi_E^{\text{ladj}} \Phi_E \rightarrow \text{Id}_{D(X_1)}$ and $\Phi_E \Phi_E^{\text{radj}} \rightarrow \text{Id}_{D(X_2)}$, cf. [AL12, Theorems 4.1 and 4.2].

What remains to be done is to obtain a similar result for the adjunction units $\text{Id}_{D(X_1)} \rightarrow \Phi_E^{\text{radj}} \Phi_E$ and $\text{Id}_{D(X_2)} \rightarrow \Phi_E \Phi_E^{\text{ladj}}$. Fortunately this can be obtained directly from the above results in [AL12] via the Grothendieck–Verdier duality in the following way.

The dual of the Fourier–Mukai transform

$$\Phi_E(-) = \pi_{2*}(E \otimes \pi_1^*(-))$$

under the duality theory $D_{\bullet/k}$ (see Section 2.1) is the functor

$$(2.13) \quad \pi_{2*} \mathbf{R} \mathcal{H}om(E, \pi_1^!(-)).$$

There are two ways to view this functor. Firstly, via natural isomorphisms

$$(2.14) \quad \mathbf{R} \mathcal{H}om(E, \pi_1^! \mathcal{O}_{X_1}) \otimes \pi_1^*(-) \xrightarrow{\sim} \mathbf{R} \mathcal{H}om(E, \pi_1^! \mathcal{O}_{X_1} \otimes \pi_1^*(-)) \xrightarrow{\sim} \mathbf{R} \mathcal{H}om(E, \pi_1^!(-))$$

we can identify (2.13) with the Fourier–Mukai transform $\Phi_{\mathbf{R} \mathcal{H}om(E, \pi_1^! \mathcal{O}_{X_1})}$ from $D(X_1)$ to $D(X_2)$. Secondly, observe that (2.13) is the right adjoint Ψ_E^{radj} of the Fourier–Mukai transform Ψ_E from $D(X_2)$ to $D(X_1)$.

Taking this second point of view, it immediately follows that the dual of Φ_E^{radj} is Ψ_E and the dual of the adjunction unit

$$(2.15) \quad \text{Id}_{D(X_1)} \rightarrow \Phi_E^{\text{radj}} \Phi_E$$

is the adjunction co-unit

$$(2.16) \quad \Psi_E \Psi_E^{\text{radj}} \rightarrow \text{Id}_{D(X_1)}.$$

By [AL12, Theorem 3.2], the adjunction co-unit (2.16) is isomorphic to the natural transformation

$$(2.17) \quad \Theta_{\tilde{Q}} \rightarrow \Theta_{\mathcal{O}_\Delta}$$

of Fourier–Mukai transforms $D(X_1) \rightarrow D(X_1)$ induced by the following morphism of objects of $D(X_1 \times X_1)$:

$$(2.18) \quad \tilde{Q} = \pi_{13*}(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_1^!(\mathcal{O}_{X_1})) \xrightarrow{\text{Id} \rightarrow \Delta_* \Delta^*} \pi_{13*} \Delta_* \Delta^*(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_1^!(\mathcal{O}_{X_1}))$$

$$(2.19) \quad \pi_{13*} \Delta_* \Delta^*(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_1^!(\mathcal{O}_{X_1})) \simeq \Delta_* \pi_{1*}(E^\vee \otimes E \otimes \pi_1^!(\mathcal{O}_{X_1}))$$

$$(2.20) \quad \Delta_* \pi_{1*}(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})) \xrightarrow{E^\vee \otimes E \otimes \text{Id} \rightarrow \text{Id}} \Delta_* \pi_{1*}(\pi_1^!(\mathcal{O}_{X_1}))$$

$$(2.21) \quad \Delta_* \pi_{1*}(\pi_1^!(\mathcal{O}_{X_1})) \xrightarrow{\pi_{1*} \pi_1^! \rightarrow \text{Id}} \Delta_* \mathcal{O}_{X_1}.$$

Identifying¹ the duals of $\Theta_{\tilde{Q}}$ and $\Theta_{\mathcal{O}_\Delta}$ under $D(\bullet/k)$ with $\Theta_{\mathbf{R}\mathcal{H}om(\tilde{Q}, \tilde{\pi}_1^! \mathcal{O}_{X_1})}$ and $\Theta_{\mathbf{R}\mathcal{H}om(\mathcal{O}_\Delta, \tilde{\pi}_1^! \mathcal{O}_{X_1})}$, we see that the dual of (2.17) under $D_{\bullet/k}$ is the morphism of Fourier–Mukai transforms induced by the morphism

$$(2.22) \quad \mathbf{R}\mathcal{H}om(\mathcal{O}_\Delta, \tilde{\pi}_1^! \mathcal{O}_{X_1}) \rightarrow \mathbf{R}\mathcal{H}om(\tilde{Q}, \tilde{\pi}_1^! \mathcal{O}_{X_1})$$

obtained by applying the relative dualizing functor $D_{X_1 \times X_1 / X_1} = \mathbf{R}\mathcal{H}om(-, \tilde{\pi}_1^! \mathcal{O}_{X_1})$ to (2.18) – (2.21).

Treating (2.18) – (2.21) as morphisms of functors in \mathcal{O}_{X_1} and applying the results of Section 2.1, we see that $D_{X_1 \times X_1 / X_1}$ applied to (2.18) – (2.21) yields:

$$(2.23) \quad \Delta_* D_{X_1 / X_1}(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow \pi_{1*} \pi_1^*} \Delta_* \pi_{1*} \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1})$$

$$(2.24) \quad \Delta_* \pi_{1*} \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow E \otimes E^\vee \otimes} \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1}))$$

$$(2.25) \quad \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1})) \simeq \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^! \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1}))$$

$$(2.26) \quad \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^! \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1})) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{13*} (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^! \pi_1^* D_{X_1 / X_1}(\mathcal{O}_{X_1}))$$

By the above (2.23)–(2.26) induces a natural transformation of Fourier–Mukai transforms isomorphic to the dual of (2.17). Since (2.17) is itself isomorphic to the dual of $\text{Id}_{X_1} \rightarrow \Phi_E^{\text{radj}} \Phi_E$, we conclude that the natural transformation induced by (2.23)–(2.26) is isomorphic to $\text{Id}_{X_1} \rightarrow \Phi_E^{\text{radj}} \Phi_E$. Finally, since $D_{X_1 / X_1}(\mathcal{O}_{X_1}) \simeq \mathcal{O}_{X_1}$ and $\pi_{12}^! \pi_1^*(\mathcal{O}_{X_1}) \simeq \pi_{23}^! \pi_2^*(\mathcal{O}_{X_2})$, we obtain:

Proposition 2.2. *Let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object of $D(X_1 \times X_2)$. Then the adjunction unit $\text{Id}_{X_1} \rightarrow \Phi_E^{\text{radj}} \Phi_E$ is isomorphic to the morphism of Fourier–Mukai transforms induced by the following morphism of objects of $D(X_1 \times X_1)$:*

$$(2.27) \quad \Delta_*(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow \pi_{1*} \pi_1^*} \Delta_* \pi_{1*} \pi_1^*(\mathcal{O}_{X_1})$$

$$(2.28) \quad \Delta_* \pi_{1*} \pi_1^*(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow E \otimes E^\vee \otimes} \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1}))$$

$$(2.29) \quad \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})) \simeq \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^! \pi_2^*(\mathcal{O}_{X_2}))$$

$$(2.30) \quad \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^! \pi_2^*(\mathcal{O}_{X_2})) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{13*} (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^! \pi_2^*(\mathcal{O}_{X_2}))$$

In a similar fashion we also obtain:

Proposition 2.3. *Let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object of $D(X_1 \times X_2)$. Then the adjunction unit $\text{Id}_{X_1} \rightarrow \Psi_E \Psi_E^{\text{ladj}}$ is isomorphic to the morphism of Fourier–Mukai transforms induced by the following morphism of objects of $D(X_1 \times X_1)$:*

$$(2.31) \quad \Delta_*(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow \pi_{1*} \pi_1^*} \Delta_* \pi_{1*} \pi_1^*(\mathcal{O}_{X_1})$$

$$(2.32) \quad \Delta_* \pi_{1*} \pi_1^*(\mathcal{O}_{X_1}) \xrightarrow{\text{Id} \rightarrow E \otimes E^\vee \otimes} \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1}))$$

$$(2.33) \quad \Delta_* \pi_{1*} (E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})) \simeq \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^! \pi_2^*(\mathcal{O}_{X_2}))$$

$$(2.34) \quad \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^! \pi_2^*(\mathcal{O}_{X_2})) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{13*} (\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^! \pi_2^*(\mathcal{O}_{X_2}))$$

If X_1 and X_2 are not proper, but the support of E is proper over X_1 and X_2 , one can still apply the above results via compactification as described in §3.2 of [AL12]. If E is a pushforward of an object in the derived category of a closed subscheme $W \hookrightarrow X_1 \times X_2$ proper over both X_1 and X_2 one can also dualize Theorems 4.1 and 4.2 of [AL12] to obtain an alternative description of morphisms of kernels which induce both adjunction units. We leave this as an exercise for the reader.

2.3. Twists, co-twists and spherical functors. Let X_1 and X_2 be, as before, two separated proper schemes of finite type over k . Let E be a perfect object in $D(X_1 \times X_2)$ and let Φ_E be the Fourier–Mukai transform from $D(X_1)$ to $D(X_2)$ with kernel E . In Section 2.2 we’ve produced morphisms of Fourier–Mukai kernels which induce the adjunction units and co-units of Φ_E , Φ_E^{ladj} and Φ_E^{radj} . Taking cones of these morphisms allows us to construct the functorial exact triangles in the following definition:

¹One has to be a little careful here since \mathcal{O}_Δ , unlike \tilde{Q} , is not a perfect object of $D(X_1 \times X_1)$. However, both natural maps in the analogue of (2.14) are still isomorphisms so we can still make the same identification.

Definition 2.4. We define the twist T_E , the dual twist T'_E , the co-twist F_E , and the dual co-twist F'_E of Φ_E by the functorial exact triangles

$$(2.35) \quad \Phi_E \Phi_E^{\text{radj}} \rightarrow \text{Id}_{D(X_2)} \rightarrow T_E,$$

$$T'_E \rightarrow \text{Id}_{D(X_2)} \rightarrow \Phi_E \Phi_E^{\text{ladj}},$$

$$(2.36) \quad F_E \rightarrow \text{Id}_{D(X_1)} \rightarrow \Phi_E^{\text{radj}} \Phi_E,$$

$$\Phi_E^{\text{ladj}} \Phi_E \rightarrow \text{Id}_{D(X_1)} \rightarrow F'_E$$

constructed via the morphisms of Fourier-Mukai kernels produced in Section 2.2.

In [AL13, Prop. 5.3] we proved that T'_E and F'_E are the left adjoints of T_E and F_E , respectively. Consider now the following two natural transformations

$$(2.37) \quad \Phi_E^{\text{ladj}} T_E[-1] \xrightarrow{T_E[-1] \rightarrow \Phi_E \Phi_E^{\text{radj}} \text{ in (2.35)}} \Phi_E^{\text{ladj}} \Phi_E \Phi_E^{\text{radj}} \xrightarrow{\Phi_E^{\text{ladj}} \Phi_E \rightarrow \text{Id}} \Phi_E^{\text{radj}},$$

$$(2.38) \quad \Phi_E^{\text{radj}} \xrightarrow{\text{Id} \rightarrow \Phi_E \Phi_E^{\text{ladj}}} \Phi_E^{\text{radj}} \Phi_E \Phi_E^{\text{ladj}} \xrightarrow{\Phi_E^{\text{radj}} \Phi_E \rightarrow F_E[1] \text{ in (2.36)}} F_E[1] \Phi_E^{\text{ladj}}.$$

The following key notion was introduced in [AL13]:

Definition 2.5. We say that the Fourier–Mukai transform Φ_E is a *spherical functor* if:

- (1) T_E is an autoequivalence of $D(X_2)$,
- (2) F_E is an autoequivalence of $D(X_1)$,
- (3) $\Phi_E^{\text{ladj}} T_E[-1] \xrightarrow{(2.37)} \Phi_E^{\text{radj}}$ is an isomorphism of functors (“the twist identifies the adjoints”),
- (4) $\Phi_E^{\text{radj}} \xrightarrow{(2.38)} F_E[1] \Phi_E^{\text{ladj}}$ is an isomorphism of functors (“the co-twist identifies the adjoints”).

The following is the main result of [AL13]:

Theorem 2.1 ([AL13], Theorem 5.1). *Any two of the conditions in Definition 2.5 imply all four.*

Corollary 2.6. *If F_E is an autoequivalence of $D(X_1)$ and if $\Phi_E^{\text{radj}} \xrightarrow{(2.38)} F_E[1] \Phi_E^{\text{ladj}}$ is an isomorphism, then Φ_E is a spherical functor.*

Lemma 2.7. *The composition (2.38) is the unique morphism $\Phi_E^{\text{radj}} \xrightarrow{\alpha} F_E[1] \Phi_E^{\text{ladj}}$ which makes the following diagram commute:*

$$(2.39) \quad \begin{array}{ccc} \Phi_E^{\text{radj}} \Phi_E & \xrightarrow{(2.36)} & F_E[1] \\ \alpha \downarrow & \nearrow \text{adj. co-unit} & \\ F_E[1] \Phi_E^{\text{ladj}} \Phi_E & & \end{array}$$

Proof. We first show that the composition (2.38) makes (2.39) commute. Indeed, composing each term with Φ_E and composing the whole isomorphism with the adjunction co-unit $\Phi_E^{\text{ladj}} \Phi_E \rightarrow \text{Id}_{D(X_1)}$ we obtain

$$(2.40) \quad \Phi_E^{\text{radj}} \Phi_E \xrightarrow{\text{adj. unit}} \Phi_E^{\text{radj}} \Phi_E \Phi_E^{\text{ladj}} \Phi_E \xrightarrow{(2.36)} F_E[1] \Phi_E^{\text{ladj}} \Phi_E \xrightarrow{\text{adj. co-unit}} F_E[1].$$

Since clearly the following square commutes

$$(2.41) \quad \begin{array}{ccc} \Phi_E^{\text{radj}} \Phi_E \Phi_E^{\text{ladj}} \Phi_E & \xrightarrow{(2.36)} & F_E[1] \Phi_E^{\text{ladj}} \Phi_E \\ \text{adj. co-unit} \downarrow & & \downarrow \text{adj. co-unit} \\ \Phi_E^{\text{radj}} \Phi_E & \xrightarrow{(2.36)} & F_E[1] \end{array}$$

the composition (2.40) equals to

$$(2.42) \quad \Phi_E^{\text{radj}} \Phi_E \xrightarrow{\text{adj. unit}} \Phi_E^{\text{radj}} \Phi_E \Phi_E^{\text{ladj}} \Phi_E \xrightarrow{\text{adj. co-unit}} \Phi_E^{\text{radj}} \Phi_E \xrightarrow{(2.36)} F_E[1]$$

and is therefore simply $\Phi_E^{\text{radj}} \Phi_E \xrightarrow{(2.36)} F_E[1]$, as required.

Conversely, let $\alpha: \Phi_E^{radj} \rightarrow F_E \Phi_E^{ladj}[1]$ be a morphism which makes (2.39) commute. We then have a commutative diagram

$$(2.43) \quad \begin{array}{ccccc} \Phi_E^{radj} & \xrightarrow{\text{adj. unit}} & \Phi_E^{radj} \Phi_E \Phi_E^{ladj} & \xrightarrow{(2.36)} & F_E[1] \Phi_E^{ladj} \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow = \\ F_E[1] \Phi_E^{ladj} & \xrightarrow{\text{adj. unit}} & F_E[1] \Phi_E^{ladj} \Phi_E \Phi_E^{ladj} & \xrightarrow{\text{adj. co-unit}} & F_E[1] \Phi_E^{ladj} \end{array}$$

Since the bottom row is the identity morphism, we conclude that α equals to the morphism given by the top row, i.e. to the composition (2.38). \square

2.4. Miscellaneous. In this section we give two technical lemmas we make use of throughout the paper.

Recall that the *support* $\text{Supp } E$ of an object E in the derived category $D(\mathcal{O}_X\text{-Mod})$ of a scheme X is the union of the supports of its cohomology sheaves $\mathcal{H}^i(E)$. The *support* of a coherent sheaf \mathcal{F} on a scheme X is defined, as per [Har77], to be the set of all $x \in X$ such that the stalk \mathcal{F}_x is not zero. By [Har77, Ex. 5.6(c)] the support of a coherent sheaf on a noetherian scheme is closed. These are the definitions employed in e.g. [AIL10] whose results we make use of.

Lemma 2.8. *Let X be a noetherian scheme and let $E \in D(X)$. A point $x \in X$ lies in $\text{Supp } E$ if and only if $\iota_x^* E \neq 0$.*

Proof. First, we claim that given a coherent sheaf \mathcal{F} on X a point $x \in X$ lies in $\text{Supp } \mathcal{F}$ if and only if the ordinary, non-derived pullback $\mathbf{L}^0 \iota_x^* \mathcal{F} \neq 0$. This is because the stalk \mathcal{F}_x is a finite $\mathcal{O}_{X,x}$ -module and by [Mat86] any finite module for a Noetherian local ring has a minimal free resolution

$$\cdots \rightarrow L_2 \rightarrow L_1 \rightarrow L_0$$

whose differentials die under ι_x^* , i.e. $\dim \mathbf{L}^i \iota_x^* \mathcal{F} = \text{rk } L_i$. By definition $x \in \text{Supp } \mathcal{F}$ if and only if $\mathcal{F}_x \neq 0$. On the other hand, $\mathcal{F}_x \neq 0$ if and only if $L_0 \neq 0$, which by above is equivalent to $\mathbf{L}^0 \iota_x^* \mathcal{F} \neq 0$.

Now let E be an object of $D_{coh}^b(X)$. Consider the standard spectral sequence

$$\mathbf{L}^p \iota_x^* \mathcal{H}^q E \Rightarrow \mathbf{L}^{p+q} \iota_x^* E.$$

Suppose $\mathbf{L}^0 \iota_x^* \mathcal{H}^q E \neq 0$ for some $q \in \mathbb{Z}$. Take minimal q for which this holds — we can do that since $\mathcal{H}^j(E) \neq 0$ for only finite number of $j \in \mathbb{Z}$. Then $\mathbf{L}^0 \iota_x^* \mathcal{H}^q E$ is the lower-left corner of the non-zero terms of the spectral sequence and hence survives yielding $\mathbf{L}^q \iota_x^* E \neq 0$. On the other hand, if $\mathbf{L}^0 \iota_x^* \mathcal{H}^q E = 0$ for all $q \in \mathbb{Z}$, all the higher pullbacks $\mathbf{L}^p \iota_x^* \mathcal{H}^q E = 0$ also vanish by the minimal free resolution argument above. Thus all terms of the spectral sequence are zero and thus $\iota_x^* E = 0$.

We have thus shown that $\iota_x^* E \neq 0$ if and only if $\mathbf{L}^0 \iota_x^* \mathcal{H}^q E \neq 0$ for some $q \in \mathbb{Z}$. By the first claim, this is equivalent to $x \in \text{Supp } \mathcal{H}^q E$ for some $q \in \mathbb{Z}$ and that is the definition of x lying in $\text{Supp } E$. \square

Lemma 2.9. *Let X_1 and X_2 be two noetherian schemes. Let E_1 and E_2 be two objects of $D(X_1 \times X_2)$ and let α be a morphism from E_1 to E_2 . Then α is an isomorphism if and only if the induced morphism of functors $\Phi_{E_1} \rightarrow \Phi_{E_2}$ is an isomorphism.*

Proof. The ‘only if’ statement is obvious. For the ‘if’ statement we use the fact that for any closed point $p \in X_1$ and any A in $D(X_1)$ we have a natural isomorphism $\Phi_A(\mathcal{O}_p) \xrightarrow{\sim} \iota_p^*(A)$ which is functorial in A . So if $\Phi_{E_1} \rightarrow \Phi_{E_2}$ is an isomorphism then the pullback of α to any closed point of X_1 is an isomorphism. This implies that the pullback of the cone α to any closed point of X_1 is 0. By Lemma 2.8 the cone of α is itself 0, and thus α is an isomorphism. \square

3. ORTHOGONALLY SPHERICAL OBJECTS

Let Z and X be two separable schemes of finite type over k . Given a closed point p in Z we denote by ι_p the closed immersion $\text{Spec } k \hookrightarrow Z$ and by ι_{Xp} the corresponding immersion $X \hookrightarrow Z \times X$:

$$(3.1) \quad \begin{array}{ccc} X & \xrightarrow{\iota_{Xp}} & Z \times X \\ \pi_k \downarrow & & \downarrow \pi_Z \searrow \pi_X \\ \text{Spec } k & \xrightarrow{\iota_p} & Z \end{array}$$

Given a perfect object E in $D(Z \times X)$ we define *the fibre E_p of E at p* to be the object $\iota_{Xp}^* E$ in $D(X)$. In this way we can think of any perfect object in $D(Z \times X)$ as a family of objects of $D(X)$ parametrised

by Z . We assume throughout this section that either Z and X are both proper or that the support of the object E in $Z \times X$ is proper over both Z and X . This ensures that all of our Fourier–Mukai transforms take complexes with bounded coherent cohomologies to complexes with bounded coherent cohomologies. It also makes applicable the results in Section 2.2 on the adjunctions units/co-units for Fourier–Mukai transforms.

3.1. Orthogonal objects. Our first goal is to come up with a categorification of the notion of a subscheme W of X fibred over Z . Our motivation is the following geometric example:

Example 3.1. Let W be a flat fibration in X over Z with proper fibres. By this we mean a scheme W equipped with a morphism $\xi: W \hookrightarrow X$ which is a closed immersion and a morphism $\pi: W \rightarrow Z$ which is flat and proper. Denote by ι_W the map $W \hookrightarrow Z \times X$ given by the product of π and ξ . We set E to be the structure sheaf of the graph of W in $Z \times X$, that is - the object $\iota_{W*}\mathcal{O}_W$ in $D(Z \times X)$.

An arbitrary subscheme W' of $Z \times X$ is a graph of some subscheme W of X fibred over Z if and only if the fibres of W' over closed points of Z are disjoint as subschemes of X . In derived categories the notion of disjointness corresponds to the notion of orthogonality, that is, to the vanishing of all the Ext's between them. This suggests the following as a categorification of the notion of a subscheme of X fibred over Z :

Definition 3.2. Let E be a perfect object of $D(Z \times X)$. We say that E is *orthogonal over Z* if for any two distinct points p and q in Z the fibres E_p and E_q are orthogonal in $D(X)$. Or in other words

$$(3.2) \quad \mathrm{Hom}_{D(X)}^i(E_p, E_q) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

Since E is a perfect object we have $(E^\vee)_p = (E_p)^\vee$. So if E is orthogonal over Z , then its dual E^\vee is also orthogonal over Z .

Any object whose support in $Z \times X$ is the graph of a subscheme of X fibred over Z is immediately orthogonal over Z — as all the Ext's between two objects with disjoint supports must vanish. Another class of examples comes from Fourier–Mukai equivalences:

Example 3.3. The kernel F of any fully faithful Fourier–Mukai transform $\Phi_F: D(Z) \xrightarrow{\sim} D(X)$ is orthogonal over Z , since for any $p \in Z$ the fibre F_p is the image under Φ_F of the skyscraper sheaf \mathcal{O}_p . Moreover, we have also $\Phi_F^{\mathrm{radj}} \Phi_F(\mathcal{O}_p) \simeq \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(F, \pi_X^!(F_p))$. The adjunction unit $\mathcal{O}_p \rightarrow \Phi_F^{\mathrm{radj}} \Phi_F(\mathcal{O}_p)$ is an isomorphism as Φ_F is fully faithful. Applying π_{k*} , where π_k is the structure morphism $Z \rightarrow \mathrm{Spec} k$, to this adjunction unit we obtain $\mathbf{R} \mathcal{H}om_X(\pi_{X*}F, F_p) = k$. It is possible using the same techniques as in the proof of Proposition 3.7 below to show that the converse is also true, i.e. Φ_F is fully faithful if and only if F is orthogonal over Z and $\mathbf{R} \mathcal{H}om_X(\pi_{X*}F, F_p) = k$ for all $p \in Z$. Suppose now that Φ_F is further an equivalence, then all its adjunction units and co-units are isomorphisms. By Lemma 2.9 the morphisms of Fourier–Mukai kernels which induce them are also isomorphisms. In particular, the isomorphism of functors

$$(3.3) \quad \Phi_F^{\mathrm{radj}} \xrightarrow{\text{adj. unit}} \Phi_F^{\mathrm{radj}} \Phi_F \Phi_F^{\mathrm{ladj}} \xrightarrow{\text{inverse of adj. co-unit}} \Phi_F^{\mathrm{ladj}}$$

must come from an isomorphism $F^\vee \otimes \pi_X^!(\mathcal{O}_X) \rightarrow F^\vee \otimes \pi_Z^!(\mathcal{O}_Z)$ of their Fourier–Mukai kernels. Conversely, any isomorphism $F^\vee \otimes \pi_X^!(\mathcal{O}_X) \rightarrow F^\vee \otimes \pi_Z^!(\mathcal{O}_Z)$ induces an isomorphism $\Phi_F^{\mathrm{radj}} \xrightarrow{\sim} \Phi_F^{\mathrm{ladj}}$. On the other hand, when X is connected by [Bri99, Theorem 3.3] Φ_F being fully faithful and Φ_F^{radj} being isomorphic to Φ_F^{ladj} imply together that Φ_F is an equivalence. We conclude that when X is connected the kernels of Fourier–Mukai equivalences are precisely the objects which are orthogonal over Z and for which

$$(3.4) \quad \mathbf{R} \mathcal{H}om_X(\pi_{X*}F, F_p) = k \text{ for all } p \in Z$$

$$(3.5) \quad F^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq F^\vee \otimes \pi_Z^!(\mathcal{O}_Z).$$

Our goal is to show that the orthogonal objects which are one step up from that, in the sense that $\mathbf{R} \mathcal{H}om_X(\pi_{X*}F, F_p) = k \oplus k[d]$ for some $d \in \mathbb{Z}$ and a similar condition to (3.5) holds, are kernels of spherical Fourier–Mukai transforms.

3.2. Spherical objects.

Definition 3.4. Let E be a perfect object of $D(Z \times X)$. We say that E is *spherical over Z* if the Fourier–Mukai transform $\Phi_E: D(Z) \rightarrow D(X)$ is a spherical functor, cf. Defn. 2.5 and Cor. 2.6. In other words, if:

- (1) The co-twist F_E is an autoequivalence of $D(Z)$,
- (2) The natural map $\Phi_E^{\mathrm{radj}} \xrightarrow{(2.38)} F_E \Phi_E^{\mathrm{ladj}}[1]$ is an isomorphism of functors.

If E is also orthogonal over Z we say further that E is *orthogonally spherical over Z* .

Example 3.5. The spherical objects introduced by Seidel and Thomas in [ST01] can be thought of as the objects spherical over $\mathrm{Spec} k$. Indeed let $Z = \mathrm{Spec} k$ and let X be a smooth variety over k . Then π_X is an isomorphism which identifies $\mathrm{Spec} k \times X$ with X . Under this identification $\pi_X^!(\mathcal{O}_X)$ becomes simply \mathcal{O}_X and $\pi_k^!(k)$ becomes the dualizing complex $D_{X/k}$ which is isomorphic to $\omega_X[\dim X]$ since X is smooth. Therefore the Fourier–Mukai kernel of the right adjoint Φ_E^{radj} is E^\vee and the Fourier–Mukai kernel of the left adjoint Φ_E^{ladj} is $E^\vee \otimes \omega_X[\dim X]$. The triple product $\mathrm{Spec} k \times X \times \mathrm{Spec} k$ is identified with X by the projection π_2 and under this identification the projection π_{13} becomes the map $\pi_k: X \rightarrow \mathrm{Spec} k$. Therefore the Fourier–Mukai kernel of the composition $\Phi_E^{\mathrm{radj}}\Phi_E$ is

$$\pi_{k*}(E^\vee \otimes E) \simeq \pi_{k*} \mathbf{R} \mathcal{H}om_X(E, E) \simeq \mathbf{R} \mathrm{Hom}_X(E, E)$$

and by the results of Section 2.2 the adjunction unit $\mathrm{Id}_{D(\mathbf{Vect})} \rightarrow \Phi_E^{\mathrm{radj}}\Phi_E$ comes from the natural morphism $k \rightarrow \mathbf{R} \mathrm{Hom}_X(E, E)$ of Fourier–Mukai kernels which sends 1 to the identity automorphism of E . Denote this morphism by γ .

The first condition for Φ_E to be a spherical functor is for the co-twist F_E to be an autoequivalence of $D(\mathbf{Vect})$. The only autoequivalences of $D(\mathbf{Vect})$ are the shifts $(-)[d]$ by some $d \in \mathbb{Z}$ and their Fourier–Mukai kernels are $k[d]$. The Fourier–Mukai kernel of F_E is the shift by 1 to the left of the cone of $k \xrightarrow{\gamma} \mathbf{R} \mathrm{Hom}_X(E, E)$. If E is non-zero the morphism γ is non-zero and then F_E is an autoequivalence if and only if $\mathbf{R} \mathrm{Hom}_X(E, E)$ is $k \oplus k[d]$ for some $d \in \mathbb{Z}$. If this does hold then $F_E = (-)[d-1]$. If E is 0, then the cone of γ is k and therefore F_E is the identity functor $\mathrm{Id}_{D(\mathbf{Vect})}$. Note that E is trivially isomorphic to its single fibre over the single closed point of $\mathrm{Spec} k$. Hence we’ve shown that F_E is an autoequivalence if and only if for every closed point $p \in Z$ such that the fibre E_p is non-zero we have $\mathbf{R} \mathrm{Hom}_X(\pi_{X*}E, E_p) = k \oplus k[d]$ for some $d \in \mathbb{Z}$.

By Lemma 2.7 the second condition for Φ_E to be spherical is an isomorphism $\alpha: E^\vee \xrightarrow{\sim} E^\vee \otimes \omega_X[\dim X + d]$ which makes the diagram (2.39) commute. If E is 0 then this condition is trivially satisfied, so assume otherwise. Since E^\vee and $E^\vee \otimes \omega_X$ are bounded complexes with non-zero cohomologies in exactly the same degrees, the isomorphism α is only possible when $d = -\dim X$. On the other hand, the diagram (2.39) on the level of the corresponding Fourier–Mukai kernels is just

$$(3.6) \quad \begin{array}{ccc} k \oplus k[d] & \xrightarrow{0 \oplus \mathrm{Id}} & k[d] \\ \alpha' \downarrow & \nearrow 0 \oplus \mathrm{Id} & \\ k \oplus k[d] & & \end{array}$$

where α' is the isomorphism induced by α . The diagram commutes if α' restricts to the identity morphism on the component $k[d]$ and we can achieve that by multiplying any given α by an appropriate scalar in k .

We conclude that E is spherical over $\mathrm{Spec} k$ if and only if either E is 0 or if $\mathbf{R} \mathrm{Hom}_X(E, E) = k \oplus k[-\dim X]$ and $E \simeq E \otimes \omega_X$, which is precisely the definition given in [ST01]. And since the base $\mathrm{Spec} k$ is a single point, any object spherical over $\mathrm{Spec} k$ is orthogonally spherical.

3.3. A cohomological criterion for sphericity. We now introduce the object in the derived category $D(Z)$ of the base Z which is relative case version of the cone of the natural morphism $k \rightarrow \mathbf{R} \mathrm{Hom}_X(E, E)$ of the Example 3.5 where the base Z is just the single point $\mathrm{Spec} k$:

Definition 3.6. For any perfect object E of $D(Z \times X)$ denote by \mathcal{L}_E the object of $D(Z)$ which is the cone of the following composition of morphisms:

$$(3.7) \quad \mathcal{O}_Z \rightarrow \pi_{Z*} \mathcal{O}_{Z \times X} \rightarrow \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(E, E) \rightarrow \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} E, E).$$

Here the first morphism is induced by the adjunction unit $\mathrm{Id}_{D(Z)} \rightarrow \pi_{Z*} \pi_Z^*$, the second by the adjunction unit $\mathrm{Id}_{D(Z \times X)} \rightarrow \mathbf{R} \mathcal{H}om(E, E \otimes -)$ and the third by the adjunction co-unit $\pi_X^* \pi_{X*} \rightarrow \mathrm{Id}_{D(Z \times X)}$.

Let p be any closed point of the base Z . Apply the pullback functor ι_p^* to the composition (3.7) to obtain a morphism $k \rightarrow \iota_p^* \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} E, E)$. We have a sequence of natural isomorphisms:

$$(3.8) \quad \iota_p^* \pi_{Z*} \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} E, E) \xrightarrow{\text{base change iso. around (3.1)}} \pi_{k*} \iota_{Xp}^* \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} E, E)$$

$$(3.9) \quad \pi_{k*} \iota_{Xp}^* \mathbf{R} \mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} E, E) \xrightarrow{[\text{M171b}], \text{Prps. 7.1.2}} \pi_{k*} \mathbf{R} \mathcal{H}om_X(\iota_{Xp}^* \pi_X^* \pi_{X*} E, \iota_{Xp}^* E)$$

$$(3.10) \quad \pi_{k*} \mathbf{R} \mathcal{H}om_X(\iota_{Xp}^* \pi_X^* \pi_{X*} E, \iota_{Xp}^* E) \xrightarrow{\pi_X \circ \iota_{Xp} = \mathrm{Id}_X} \pi_{k*} \mathbf{R} \mathcal{H}om_X(\pi_{X*} E, E_p) \simeq \mathbf{R} \mathrm{Hom}_X(\pi_{X*} E, E_p).$$

One can check that these natural isomorphisms identify ι_p^* (3.7) with the morphism

$$(3.11) \quad k \rightarrow \mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p)$$

which sends 1 to the natural composition

$$(3.12) \quad \iota_{X_p}^* \left(\pi_X^* \pi_{X^*} E \xrightarrow{\text{adj. co-unit}} E \right)$$

where we identify the LHS with $\pi_{X^*} E$ via the scheme map identity $\pi_X \circ \iota_{X_p} = \text{Id}_X$. Thus pointwise restriction of (3.7) gives a natural morphism $k \rightarrow \mathbf{R} \text{Hom}_{D(X)}(\pi_{X^*} E, E_p)$ for each closed point $p \in Z$. It turns out that for an orthogonal E the criterion for the co-twist F_E of E to be an autoequivalence of $D(Z)$ is for the cone of each of these morphisms to be $k[d]$ for some $d \in \mathbb{Z}$:

Proposition 3.7. *Let E be a perfect object of $D(Z \times X)$ orthogonal over Z . The following are equivalent:*

- (1) *For every closed point $p \in Z$ such that the fibre E_p is non-zero*

$$\mathbf{R} \text{Hom}_{D(X)}(\pi_{X^*} E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}$$

and the natural morphism $\pi_{X^} E \xrightarrow{(3.12)} E_p$ is not zero.*

- (2) *The object \mathcal{L}_E is an invertible object of $D(Z)$. That is - on every connected component of Z it is isomorphic to a shift of a line bundle, cf. [AIL10, §1.5].*
 (3) *The co-twist F_E of the transform $\Phi_E: D(Z) \rightarrow D(X)$ is an autoequivalence of $D(Z)$.*

When the conditions above are satisfied:

- *Locally around any closed point $p \in Z$ we have $\mathcal{L}_E \simeq \mathcal{O}_Z[d_p]$ where*

$$d_p = \begin{cases} \text{the same integer as in (1)} & \text{if } E_p \neq 0 \\ 1 & \text{if } E_p = 0 \end{cases}$$

- *F_E is isomorphic the functor $\mathcal{L}_E \otimes (-)[-1]$*

We see therefore that for the orthogonally spherical objects the geometric meaning of the object \mathcal{L}_E defined above is that its restriction to each connected component of Z is a (shifted) line bundle which induces the co-twist autoequivalence F_E of $D(Z)$.

To prove Proposition 3.7 we need two technical lemmas. Recall that by Proposition 2.2 the adjunction unit $\text{Id}_{D(Z)} \rightarrow \Phi_E^{\text{radj}} \Phi_E$ is isomorphic to the morphism of Fourier–Mukai transforms induced by the morphism

$$(3.13) \quad \Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q = \pi_{13*} (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_X^! \mathcal{O}_X)$$

in $D(Z \times Z)$. Here π_{ij} are the natural projection morphisms in the following commutative diagram:

$$(3.14) \quad \begin{array}{ccccc} & & Z \times X \times Z & & \\ & & \swarrow \pi_{12} & \downarrow \pi_{13} & \searrow \pi_{23} \\ & Z \times X & & Z \times Z & & X \times Z \\ & \swarrow \pi_Z & \searrow \tilde{\pi}_1 & \swarrow \pi_X & \searrow \tilde{\pi}_2 & \swarrow \pi_Z \\ Z & & & X & & Z \end{array}$$

Lemma 3.8. *Let E be a perfect object of $D(Z \times X)$ and let $p \in Z$ be a closed point. Then the following two morphisms in $D(Z)$ are isomorphic:*

$$(3.15) \quad \tilde{\pi}_{1*} \left(\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \right)$$

and

$$(3.16) \quad \mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*} \mathbf{R} \text{Hom}(\pi_X^* \pi_{X^*} E, E).$$

Consequently, for every closed point $p \in Z$ the natural morphism $k \xrightarrow{(3.11)} \mathbf{R} \text{Hom}_X(\pi_{X^*} E, E_p)$ is isomorphic to $\pi_{k*} \left(\mathcal{O}_p \xrightarrow{\text{adj. unit}} \Phi_E^{\text{radj}} \Phi_E(\mathcal{O}_p) \right)$ and therefore $\pi_{k*} F_E(\mathcal{O}_p) \simeq \iota_p^* \mathcal{L}_E[-1]$.

Proof. For the first claim we need to show that $\mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*} \mathbf{R} \mathcal{H}om(\pi_X^* \pi_{X*} E, E)$ is isomorphic to:

$$(3.17) \quad \tilde{\pi}_{1*} \Delta_*(\mathcal{O}_Z) \xrightarrow{\text{Id} \rightarrow \pi_{Z*} \pi_Z^*} \tilde{\pi}_{1*} \Delta_* \pi_{Z*} \pi_Z^*(\mathcal{O}_Z)$$

$$(3.18) \quad \tilde{\pi}_{1*} \Delta_* \pi_{Z*} \pi_Z^*(\mathcal{O}_Z) \xrightarrow{\text{Id} \rightarrow E \otimes E^\vee \otimes} \tilde{\pi}_{1*} \Delta_* \pi_{Z*} (E \otimes E^\vee \otimes \pi_Z^*(\mathcal{O}_Z))$$

$$(3.19) \quad \tilde{\pi}_{1*} \Delta_* \pi_{Z*} (E \otimes E^\vee \otimes \pi_Z^*(\mathcal{O}_Z)) \simeq \tilde{\pi}_{1*} \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_X^!(\mathcal{O}_X))$$

$$(3.20) \quad \tilde{\pi}_{1*} \pi_{13*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_X^!(\mathcal{O}_X)) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \tilde{\pi}_{1*} \pi_{13*} (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_X^!(\mathcal{O}_X))$$

By the scheme map identity $\tilde{\pi}_1 \circ \Delta = \text{Id}_Z$ we have $\tilde{\pi}_{1*} \Delta_* \simeq \text{Id}_{D(Z)}$ and this identifies (3.17) and (3.18) with the first and the second morphisms in the composition (3.7). It remains to show that

$$(3.21) \quad \pi_{Z*} \mathbf{R} \mathcal{H}om_X(E, E) \xrightarrow{\pi_X^* \pi_{X*} \rightarrow \text{Id}} \pi_{Z*} \mathbf{R} \mathcal{H}om(\pi_X^* \pi_{X*} E, E)$$

is isomorphic to (3.20).

By the scheme map identity $\tilde{\pi}_1 \circ \pi_{13} = \pi_Z \circ \pi_{12}$ from (3.14) we have $\tilde{\pi}_{1*} \pi_{13*} \simeq \pi_{Z*} \pi_{12*}$. By the independent fibre square

$$(3.22) \quad \begin{array}{ccc} Z \times X \times Z & \xrightarrow{\pi_{23}} & X \times Z \\ \pi_{12} \downarrow & & \downarrow \pi_X \\ Z \times X & \xrightarrow{\pi_X} & X \end{array}$$

we also have $\pi_{23}^* \pi_X^! \simeq \pi_{12}^* \pi_X^!$, cf. [Lip09], §3.10. We can therefore rewrite (3.20) as

$$(3.23) \quad \pi_{Z*} \pi_{12*} \Delta_* \Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X}) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{Z*} \pi_{12*} (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X}).$$

Now observe that $\pi_{12}^* E$ is perfect, while $\pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X}$ is a tensor product of a perfect object and a π_{12} -perfect object and therefore itself π_{12} -perfect. Hence, even though Δ is not perfect, by Lemma 2.1 the natural map $\Delta^! (\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X}) \rightarrow \Delta^* (\pi_{12}^* E) \otimes \Delta^! (\pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X})$ is still an isomorphism. It therefore follows from [AL12, Lemma 2.2] that (3.23) is isomorphic to

$$(3.24) \quad \pi_{Z*} (E \otimes \pi_{12*} \Delta_* \Delta^! (\pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X})) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{Z*} (E \otimes \pi_{12*} (\pi_{23}^* E^\vee \otimes \pi_{12}^* \mathcal{O}_{Z \times X})).$$

It remains to show that

$$(3.25) \quad \mathbf{R} \mathcal{H}om_X(E, \mathcal{O}_{Z \times X}) \xrightarrow{\pi_X^* \pi_{X*} \rightarrow \text{Id}} \mathbf{R} \mathcal{H}om(\pi_X^* \pi_{X*} E, \mathcal{O}_{Z \times X})$$

is isomorphic to

$$(3.26) \quad \pi_{12*} \Delta_* \Delta^! \mathbf{R} \mathcal{H}om(\pi_{23}^* E, \pi_{12}^* \mathcal{O}_{Z \times X}) \xrightarrow{\Delta_* \Delta^! \rightarrow \text{Id}} \pi_{12*} \mathbf{R} \mathcal{H}om(\pi_{23}^* E, \pi_{12}^* \mathcal{O}_{Z \times X}).$$

Rewriting (3.25) and (3.26) in terms of the relative duality theory $D_{\bullet/Z \times X}$ (see Section 2.1) we obtain

$$D_{\bullet/Z \times X} \left(\pi_X^* \pi_{X*} E \xrightarrow{\pi_X^* \pi_{X*} \rightarrow \text{Id}} E \right)^{\text{opp}} \quad \text{and} \quad D_{\bullet/Z \times X} \left(\pi_{12*} \pi_{23}^* E \xrightarrow{\text{Id} \rightarrow \Delta_* \Delta^*} \pi_{12*} \Delta_* \Delta^* \pi_{23}^* E \right)^{\text{opp}}$$

respectively and these are isomorphic by [AL12], Lemma 3.2. This settles the first claim of this lemma.

For the second claim, we have an independent fibre square

$$(3.27) \quad \begin{array}{ccc} Z & \xrightarrow{\iota_{p,Z}} & Z \times Z \\ \pi_k \downarrow & & \downarrow \tilde{\pi}_1 \\ \text{Spec } k & \xrightarrow{\iota_p} & Z \end{array}$$

and for any $A \in D(Z \times Z)$ we have a standard isomorphism

$$(3.28) \quad \Phi_A(\mathcal{O}_p) \xrightarrow{\sim} \iota_{p,Z}^* A$$

which is functorial in A . The adjunction unit morphism $\mathcal{O}_p \rightarrow \Phi_E^{\text{radj}} \Phi_E(\mathcal{O}_p)$ is isomorphic to the morphism $\Phi_{\Delta_* \mathcal{O}_Z}(\mathcal{O}_p) \rightarrow \Phi_Q(\mathcal{O}_p)$ induced by $\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q$ and is therefore isomorphic to $\iota_{p,Z}^* \left(\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \right)$.

By the base change around (3.27) we have $\pi_{k*} \iota_{p,Z}^* \simeq \iota_p^* \tilde{\pi}_{1*}$ and therefore $\pi_{k*} \left(\mathcal{O}_p \rightarrow \Phi_E^{\text{radj}} \Phi_E(\mathcal{O}_p) \right)$ is isomorphic to $\iota_p^* \tilde{\pi}_{1*} \left(\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \right)$ and hence, by the first claim, to $\iota_p^* \left(\mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*} \mathbf{R} \mathcal{H}om(\pi_X^* \pi_{X*} E, E) \right)$,

which is precisely the natural morphism $k \xrightarrow{(3.11)} \mathbf{R}\mathrm{Hom}_X(\pi_{X^*}E, E_p)$. This settles the second claim of the lemma and the last claim follows immediately by taking cones. \square

Lemma 3.9. *Let E be a perfect object of $D(Z \times X)$. Then E is orthogonal over Z if and only if the support of the object $Q = \pi_{13^*}(\pi_{12^*}E \otimes \pi_{23^*}E^\vee \otimes \pi_{23^*}\pi_X^!\mathcal{O}_X)$ is contained within the diagonal $\Delta \subset Z \times Z$. Consequently, if E is orthogonal over Z then for any closed point $p \in Z$ the object $F_E(\mathcal{O}_p)$, if non-zero, is supported at precisely the point p .*

Proof. Let q_1 and q_2 be closed points of Z , let $\bar{q} = (q_1, q_2)$ be the corresponding point of $Z \times Z$ and denote by $\iota_{\bar{q}}$ the closed embedding $\bar{q} \hookrightarrow Z \times Z$. Since $Q \in D(Z \times Z)$ its cohomology sheaves are coherent and only finite number of them are non-zero. It follows from the standard spectral sequence $\mathbf{L}^i \iota_{\bar{q}}^* \mathcal{H}^j Q \Rightarrow \mathbf{L}^{i+j} \iota_{\bar{q}}^* Q$ that $\bar{q} \in \mathrm{Supp}_{Z \times Z} Q$ if and only if $\iota_{\bar{q}}^* Q \neq 0$.

We have an independent fibre square

$$(3.29) \quad \begin{array}{ccc} X & \xrightarrow{\iota_{X\bar{q}}} & Z \times X \times Z \\ \pi_k \downarrow & & \downarrow \pi_{13} \\ \mathrm{Spec} k & \xrightarrow{\iota_{\bar{q}}} & Z \times Z \end{array}$$

and thus

$$(3.30) \quad \begin{aligned} \iota_{\bar{q}}^* Q &= \iota_{\bar{q}}^* \pi_{13^*}(\pi_{12^*}E \otimes \pi_{23^*}E^\vee \otimes \pi_{23^*}\pi_X^!\mathcal{O}_X) \simeq \\ &\simeq \pi_{k^*} \iota_{X\bar{q}}^*(\pi_{12^*}E \otimes \pi_{23^*}E^\vee \otimes \pi_{23^*}\pi_X^!\mathcal{O}_X) \simeq \\ &\simeq \pi_{k^*}(\mathbf{R}\mathrm{Hom}(E_{q_2}, E_{q_1}) \otimes \iota_{Xq_2}^* \pi_X^!\mathcal{O}_X) \end{aligned}$$

We have a pair of independent fibre squares

$$(3.31) \quad \begin{array}{ccccc} X & \xrightarrow{\iota_{Xq_2}} & Z \times X & \xrightarrow{\pi_X} & X \\ \pi_k \downarrow & & \downarrow \pi_Z & & \downarrow \pi_k \\ \mathrm{Spec} k & \xrightarrow{\iota_{q_2}} & Z & \xrightarrow{\pi_k} & \mathrm{Spec} k \end{array}$$

and thus $\pi_X^!\mathcal{O}_X \simeq \pi_Z^* D_{Z/k}$, where $D_{Z/k}$ is the dualizing complex $\pi_k^!(k)$ on Z . Therefore

$$\iota_{Xq_2}^* \pi_X^!\mathcal{O}_X \simeq \iota_{Xq_2}^* \pi_Z^* D_{Z/k} \simeq \pi_k^* \iota_{q_2}^* D_{Z/k},$$

and so finally:

$$(3.32) \quad \begin{aligned} \iota_{\bar{q}}^* Q &= \pi_{k^*}(\mathbf{R}\mathrm{Hom}(E_{q_2}, E_{q_1}) \otimes \iota_{Xq_2}^* \pi_X^!\mathcal{O}_X) \simeq \\ &\simeq \pi_{k^*}(\mathbf{R}\mathrm{Hom}(E_{q_2}, E_{q_1}) \otimes \pi_k^* \iota_{q_2}^* D_{Z/k}) \simeq \\ &\simeq \pi_{k^*} \mathbf{R}\mathrm{Hom}(E_{q_2}, E_{q_1}) \otimes \iota_{q_2}^* D_{Z/k} \simeq \mathbf{R}\mathrm{Hom}_{D(X)}(E_{q_2}, E_{q_1}) \otimes \iota_{q_2}^* D_{Z/k}. \end{aligned}$$

By [AIL10, Lemma 1.3.7] the support of any semi-dualizing (and, in particular, of any dualizing) complex on a noetherian scheme is the whole of the scheme. Therefore $\iota_{q_2}^* D_{Z/k}$ is non-zero for any $q_2 \in Z$. It then follows from (3.32) that $\iota_{\bar{q}}^* Q \neq 0$ if and only if $\mathrm{Hom}_{D(X)}^i(E_{q_2}, E_{q_1}) \neq 0$ for some $i \in \mathbb{Z}$. Therefore the support of Q in $Z \times Z$ consists precisely of all points (q_1, q_2) for which $\mathrm{Hom}_{D(X)}^i(E_{q_2}, E_{q_1}) \neq 0$ for some $i \in \mathbb{Z}$. Whence the assertion that E is orthogonal over Z if and only if the support of Q lies within the diagonal of $Z \times Z$.

For the second assertion, recall that $\Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p) \simeq \iota_{p,Z}^* Q$ and therefore $\iota_{p,Z}^* Q$ fits into an exact triangle

$$\mathcal{O}_p \rightarrow \iota_{p,Z}^* Q \rightarrow F_E(\mathcal{O}_p)[1]$$

in $D(Z)$. Since the support of \mathcal{O}_p is p and the support of $\iota_{p,Z}^* Q$ lies within $\iota_{p,Z}^{-1} \mathrm{Supp}_{Z \times Z} Q \subseteq \iota_{p,Z}^{-1} \Delta = p$, it follows that the support of $F_E(\mathcal{O}_p)$ also lies within the point p . If the object $F_E(\mathcal{O}_p)$ is non-zero its support is closed and non-empty and must therefore be precisely p . \square

Proof of Proposition 3.7. (1) \Leftrightarrow (2): By [AIL10], Theorem 1.5.2 the object \mathcal{L}_E is invertible if and only if for every closed point $p \in Z$ it is isomorphic in the neighborhood of p to $\mathcal{O}_Z[d_p]$ for some $d_p \in \mathbb{Z}$. This is equivalent to having $\iota_p^* \mathcal{L}_E = k[d_p]$. We have an exact triangle

$$(3.33) \quad k \xrightarrow{(3.11)} \mathbf{R}\mathrm{Hom}_{D(X)}(\pi_{X^*}E, E_p) \rightarrow \iota_p^* \mathcal{L}_E$$

in $D(\mathbf{Vect})$. Hence $\iota_p^* \mathcal{L}_E = k[d_p]$ for some $d_p \in \mathbb{Z}$ is equivalent to either $\mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p) = 0$ and $d_p = 1$ or to $\mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p) = k \oplus k[d_p]$ and (3.11), and hence (3.12), not being the zero morphism. Therefore to establish (1) \Leftrightarrow (2) and the first of the two assertions in the end it remains only to show that if $E_p \neq 0$ then $\mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p) \neq 0$.

By Lemma 3.8 the morphism (3.11) is isomorphic to π_{k*} applied to the adjunction unit $\mathcal{O}_p \rightarrow \Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$. If $E_p = \Phi_E(\mathcal{O}_p) \neq 0$, then this adjunction unit is non-zero and hence $\Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p) \neq 0$. In Lemma 3.9 we've shown that both \mathcal{O}_p and $\Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$ are supported at $p \in Z$. It suffices therefore to show that the functor π_{k*} is injective on objects of the full subcategory $D_p(Z)$ of $D(Z)$ consisting of the complexes whose cohomology is supported at $p \in Z$. Indeed, let U be any open affine subset of Z containing p , let ι_U be the corresponding open immersion and observe that ι_{U*} restricts to an equivalence $\iota_{U*}: D_p(U) \xrightarrow{\sim} D_p(X)$ whose inverse is ι_U^* . On the other hand, $D_p(U) \xrightarrow{\pi_{k*}} D(\mathbf{Vect})$ decomposes as

$$(3.34) \quad D_p(U) \xrightarrow{\Gamma^*} D_p(\mathcal{O}_X(U)\text{-Mod}) \xrightarrow{\text{forgetful}} D(\mathbf{Vect})$$

Here Γ^* is the derived global sections functor and it is an equivalence since U is affine. The functor of forgetting the $\mathcal{O}_X(U)$ -module structure is injective on objects. The claim now follows.

(2) \Leftrightarrow (3): The object \mathcal{L}_E is invertible if and only if for every closed point $p \in Z$ we have $\iota_p^*(\mathcal{L}_E) = k[d_p]$ for some $d_p \in \mathbb{Z}$. By Lemma 3.8 we have $\iota_p^*(\mathcal{L}_E) = \pi_{k*} F_E(\mathcal{O}_p)[1]$. By Lemma 3.9 the object $F_E(\mathcal{O}_p)$ lies in the full subcategory $D_p(Z)$ of $D(Z)$ consisting of the complexes whose cohomology is supported at p . Finally, the decomposition (3.34) makes it clear that the only object of $D_p(Z)$ whose image in $D(\mathbf{Vect})$ under π_{k*} is precisely k is the point sheaf \mathcal{O}_p . We conclude that \mathcal{L}_E is invertible if and only if

$$(3.35) \quad \forall p \in Z, \quad F_E(\mathcal{O}_p) = \mathcal{O}_p[d] \quad \text{for some } d \in \mathbb{Z}.$$

Suppose (3.35) holds. Let Q' be the Fourier–Mukai kernel of the co-twist F_E . Since $\iota_{p,Z}^* Q' \simeq F_E(\mathcal{O}_p)$ by the semicontinuity theorem ([GD63], *Théorème 7.6.9*) the shift d in (3.35) is constant on every connected component of Z . Let U be such a connected component, then the spectral sequence argument of [Bri99], Lemma 4.3 shows that the restriction of Q' to $U \times Z$ is the shift by d of a coherent sheaf flat over U , whose restriction to the fibre $\{p\} \times Z$ over every point $p \in U$ is precisely \mathcal{O}_p . Any such sheaf is necessarily a line bundle supported on the diagonal $U \hookrightarrow U \times Z$. We conclude that globally $Q' = \Delta_* \mathcal{L}'$ for some invertible object \mathcal{L}' of $D(Z)$. This immediately implies that the corresponding Fourier–Mukai transform F_E is an equivalence.

Conversely, suppose F_E is an equivalence. Let p be any closed point of Z . As F_E is an equivalence we have $\mathrm{Hom}_{D(Z)}^{\leq 0}(F_E(\mathcal{O}_p), F_E(\mathcal{O}_p)) = 0$ and $\mathrm{Hom}_{D(Z)}^0(F_E(\mathcal{O}_p), F_E(\mathcal{O}_p)) = k$. By Lemma 3.9 the support of $F_E(\mathcal{O}_p)$ is precisely p . Now the same spectral sequence argument as in Proposition 2.2 of [BO01] shows that $F_E(\mathcal{O}_p) = \mathcal{O}_p[d]$ for some $d \in \mathbb{Z}$.

For the second of the two assertions in the end: it follows from the definition of F_E that Q' is the object

$$\mathrm{Cone} \left(\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \right) [-1]$$

of $D(Z \times Z)$. Therefore by Lemma 3.8 we have $\tilde{\pi}_{1*} Q' \simeq \mathcal{L}_E[-1]$. Above we've shown that $Q' = \Delta_* \mathcal{L}'$ for some invertible object $\mathcal{L}' \in D(Z)$ and since $\tilde{\pi}_{1*} \Delta_* \simeq \mathrm{Id}_{D(Z)}$ it follows that $\mathcal{L}' \simeq \mathcal{L}_E[-1]$. \square

The following lemma shows that when verifying the condition (1) of Prop. 3.7 one doesn't have to check that $\pi_{X*} E \xrightarrow{(3.12)} E_p$ is non-zero provided the integer d_p is non-positive:

Lemma 3.10. *Let E be a perfect object of $D(Z \times X)$ orthogonal over Z . Let $p \in Z$ be such that*

$$\mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}.$$

If $d_p \leq 0$ then the natural morphism $\pi_{X} E \xrightarrow{(3.12)} E_p$ is non-zero.*

Proof. Recall that $k \xrightarrow{(3.11)} \mathbf{R} \mathrm{Hom}_{D(X)}(\pi_{X*} E, E_p)$ sends 1 to the morphism (3.12). By Lemma 3.8 the morphism (3.11) is isomorphic to π_{k*} applied to the adjunction unit $\mathcal{O}_p \rightarrow \Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$. By adjunction

$$\pi_{k*} \left(\mathcal{O}_p \rightarrow \Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p) \right)$$

being non-zero is equivalent to

$$\pi_k^* \pi_{k*} \mathcal{O}_p = \mathcal{O}_Z \rightarrow \mathcal{O}_p \rightarrow \Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$$

being non-zero. It suffices therefore to show that the induced morphism $\mathcal{O}_p \rightarrow \mathcal{H}^0(\Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p))$ is non-zero.

By Lemma 3.9 the support of $\Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p)$ is p . Hence $\pi_{k*}\Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p) = k \oplus k[d_p]$ implies that the only non-zero cohomology sheaves of $\Phi_E^{\text{radj}}\Phi_E$ are \mathcal{O}_p in degree 0 and $-d_p$. The standard spectral sequence

$$\text{Ext}_Z^i\left(\mathcal{O}_p, \mathcal{H}^j(\Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p))\right) \Rightarrow \text{Hom}_{D(Z)}^{i+j}\left(\mathcal{O}_p, \Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p)\right)$$

and the fact that $d_p \leq 0$ imply that

$$\text{Hom}_{D(Z)}\left(\mathcal{O}_p, \Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p)\right) \simeq \text{Hom}_Z\left(\mathcal{O}_p, \mathcal{H}^0(\Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p))\right),$$

the isomorphism being given by restriction to the 0th cohomology sheaves. Therefore it suffices to show that the adjunction unit $\mathcal{O}_p \rightarrow \Phi_E^{\text{radj}}\Phi_E(\mathcal{O}_p)$ itself is a non-zero morphism. But since $\mathbf{R}\text{Hom}_{D(X)}(\pi_{X*}E, E_p) \neq 0$, we must also have $E_p \neq 0$ and hence the adjunction unit is non-zero. \square

Suppose now that E satisfies the equivalent conditions of Proposition 3.7. Then the co-twist F_E is an autoequivalence of $D(Z)$ whose Fourier–Mukai kernel is $\Delta_*\mathcal{L}_E[-1]$. Recall the definition of Φ_E being spherical given in Defn. 2.5. The four functorial exact triangles in it are constructed on the level of Fourier–Mukai kernels. In other words, we have implicitly fixed once and for all a completion to exact triangles of the four morphisms given in Section 2.2 which underlie the adjunction units and co-units of Φ_E .

Let κ be the morphism in the chosen completion of (2.27)–(2.30) to an exact triangle

$$(3.36) \quad \Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} \pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!\mathcal{O}_X)) \xrightarrow{\kappa} \Delta_*\mathcal{L}_E.$$

Denote by κ_{FM} the corresponding natural transformation in the corresponding functorial exact triangle (2.36).

By Cor. 2.6 the functor Φ_E is spherical, and thus the object E is spherical over Z , if and only if F_E is an autoequivalence and the composition

$$(3.37) \quad \Phi_E^{\text{radj}} \xrightarrow{\Phi_E^{\text{radj}}(\text{Id} \rightarrow \Phi_E\Phi_E^{\text{ladj}})} \Phi_E^{\text{radj}}\Phi_E\Phi_E^{\text{ladj}} \xrightarrow{\kappa_{FM}} F_E[1]\Phi_E^{\text{ladj}}$$

is an isomorphism of functors. The Fourier–Mukai kernel of Φ_E^{radj} is $E^\vee \otimes \pi_X^!(\mathcal{O}_X)$ and the Fourier–Mukai kernel of $F_E[1]\Phi_E^{\text{ladj}}$ is

$$\pi_Z^*(\mathcal{L}_E) \otimes (E^\vee \otimes \pi_Z^!(\mathcal{O}_Z)) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E).$$

We can therefore define:

Definition 3.11. Define α to be the morphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

of Fourier–Mukai kernels which underlies the natural morphism (3.37) if we choose (2.31)–(2.34) and κ as underlying morphisms of $\text{Id} \rightarrow \Phi_E\Phi_E^{\text{ladj}}$ and κ_{FM} , respectively.

By Lemma 2.9 the composition (3.37) is an isomorphism if and only if the underlying morphism α is. We therefore obtain immediately the main result of this section:

Theorem 3.1. *Let X and Z be two separable schemes of finite type over k . Let E be a perfect object of $D(Z \times X)$ orthogonal over Z . Then E is spherical over Z if and only if*

- (1) *For every closed point $p \in Z$ such that the fibre E_p is non-zero*

$$\mathbf{R}\text{Hom}_{D(X)}(\pi_{X*}E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}$$

and the natural morphism $\pi_{X}E \xrightarrow{(3.12)} E_p$ is not zero.*

- (2) *The canonical morphism*

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E) \quad (\text{see Definition 3.11})$$

is an isomorphism.

Whenever E is orthogonally spherical over Z , the object \mathcal{L}_E is invertible in $D(Z)$ and so locally around any closed point $p \in Z$ we have $\mathcal{L}_E \simeq \mathcal{O}_Z[d_p]$ for some $d_p \in \mathbb{Z}$. Over the locus where Z and X are not too degenerate this integer is the difference in dimensions between X and Z :

Proposition 3.12. *Let E be an object of $D(Z \times X)$ orthogonally spherical over Z . Let $(p, q) \in Z \times X$ be a Gorenstein point in the support of E if such exists. Then*

$$d_p = -(\dim_q X - \dim_p Z).$$

Proof. If E is spherical over Z the canonical map

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

is an isomorphism. Let us restrict it to $\text{Spec } \mathcal{O}_{Z \times X, (p, q)}$. Since $\mathcal{O}_{Z \times X, (p, q)} = \mathcal{O}_{Z, p} \otimes \mathcal{O}_{X, q}$ is Gorenstein, the structure map $\text{Spec } \mathcal{O}_{Z \times X, (p, q)} \rightarrow \text{Spec } k$ is Gorenstein. Therefore the projections $\pi_{Z, p}$ and $\pi_{X, q}$ are Gorenstein, since we can filter $\text{Spec } \mathcal{O}_{Z \times X, (p, q)} \rightarrow \text{Spec } k$ through them

$$(3.38) \quad \begin{array}{ccc} \text{Spec } \mathcal{O}_{Z \times X, (p, q)} & \xrightarrow{\pi_{X, q}} & \text{Spec } \mathcal{O}_{X, q} \\ \pi_{Z, q} \downarrow & & \downarrow \\ \text{Spec } \mathcal{O}_{Z, p} & \longrightarrow & k \end{array}$$

and for perfect maps (and therefore for flat maps such as these) the composition of two maps is Gorenstein if and only if both components are, [AF90, Prop. 2.3]. Therefore

$$\begin{aligned} \pi_{Z, p}^!(\mathcal{O}_{Z, p}) &= \mathcal{O}_{Z \times X, (p, q)}[\dim \mathcal{O}_{Z \times X, (p, q)} - \dim \mathcal{O}_{Z, p}] = \mathcal{O}_{Z \times X, (p, q)}[\dim \mathcal{O}_{X, q}] \\ \pi_{X, q}^!(\mathcal{O}_{X, q}) &= \mathcal{O}_{Z \times X, (p, q)}[\dim \mathcal{O}_{Z \times X, (p, q)} - \dim \mathcal{O}_{X, q}] = \mathcal{O}_{Z \times X, (p, q)}[\dim \mathcal{O}_{Z, p}] \end{aligned}$$

and so α restricts to $\text{Spec } \mathcal{O}_{Z \times X, (p, q)}$ as

$$E^\vee|_{\mathcal{O}_{Z \times X, (p, q)}}[\dim \mathcal{O}_{Z, p}] \xrightarrow{\sim} E^\vee|_{\mathcal{O}_{Z \times X, (p, q)}}[\dim \mathcal{O}_{X, q} + d_p]$$

Since (p, q) lies in the support of E , the restriction $E^\vee|_{\mathcal{O}_{Z \times X, (p, q)}}$ is a non-zero bounded complex. So

$$\dim \mathcal{O}_{Z, p} = \dim \mathcal{O}_{X, q} + d_p$$

whence the claim. \square

3.4. Concerning the canonical morphism α . A reader who wasn't at all disturbed by the words "the canonical morphism α is an isomorphism" in Theorem 3.1 probably doesn't need to read this section. However to apply Theorem 3.1 to show that an object is spherical one needs to compute the canonical morphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

described in Definition 3.11 and show it to be an isomorphism. In all but few very simple examples computing this morphism directly, by computing the morphisms of the kernels underlying both terms of (3.37) and then composing them, is not a pleasant endeavour.

Fortunately Lemma 2.7 gives us a different characterisation of α by telling us that α induces the unique natural transformation α_{FM} which makes the diagram

$$(3.39) \quad \begin{array}{ccc} \Phi_E^{radj} \Phi_E & \xrightarrow{\kappa_{FM}} & F_E[1] \\ \alpha_{FM} \downarrow & \nearrow & \\ F_E[1] \Phi_E^{ladj} \Phi_E & & F_E[1](\Phi_E^{ladj} \Phi_E \rightarrow \text{Id}) \end{array}$$

commute. It follows by Lemma 2.9 that showing α to be an isomorphism is equivalent to exhibiting some isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ which makes diagram (3.39) commute.

Proposition 3.13. *Let E be a perfect object of $D(Z \times X)$ orthogonal over Z and suppose that \mathcal{L}_E is an invertible object of $D(Z)$. Assume (for simplicity) that Z is connected, then $\mathcal{L}_E = L[d]$ for some line bundle $L \in \text{Pic } Z$ and $d \in \mathbb{Z}$. Assume further that $d < 0$ or, more generally, that $d \neq 0, 1$ and*

$$\text{Ext}_{Z \times Z}^d(\Delta_* \mathcal{O}_Z, \Delta_* L) = 0.$$

If $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ then the canonical map α is an isomorphism.

Proof. Let α' denote some isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$. By [AL12, Theorem 3.1] the natural transformation $F_E[1] \Phi_E^{ladj} \Phi_E \rightarrow F_E[1]$ is induced by the following morphism of Fourier-Mukai kernels:

$$(3.40) \quad \pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E \otimes \pi_Z^!(\mathcal{L}_E))) \xrightarrow{\text{Id} \rightarrow \Delta_* \Delta^*} \Delta_* \Delta^* \pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E)))$$

$$(3.41) \quad \Delta_* \Delta^* \pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E))) \xrightarrow{\Delta^* \pi_{13*}(\pi_{12}^* \otimes \pi_{13}^*) \simeq \pi_{Z*}(- \otimes -)} \Delta_* \pi_{Z*}(E \otimes (E^\vee \otimes \pi_Z^!(\mathcal{L}_E)))$$

$$(3.42) \quad \Delta_* \pi_{Z*}(E \otimes (E^\vee \otimes \pi_Z^!(\mathcal{L}_E))) \xrightarrow{E \otimes E^\vee \otimes \text{Id}} \Delta_* \pi_{Z*} \pi_Z^!(\mathcal{L}_E)$$

$$(3.43) \quad \Delta_* \pi_{Z*} \pi_Z^!(\mathcal{L}_E) \xrightarrow{\pi_{1*} \pi_1^! \rightarrow \text{Id}} \Delta_* \mathcal{L}_E.$$

Thus the analogue of (3.39) for α' is induced by the following diagram of Fourier–Mukai kernel morphisms

$$(3.44) \quad \begin{array}{ccc} \pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!\mathcal{O}_X)) & \xrightarrow{\kappa} & \Delta_*\mathcal{L}_E \\ \pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(\alpha')) \downarrow & \nearrow_{(3.40)-(3.43)} & \\ \pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!\mathcal{L}_E)) & & \end{array}$$

It is enough to show that α' can be chosen so that (3.44) commutes.

Denote by Q the object $\pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!(\mathcal{O}_X)))$. We have an exact triangle in $D(Z \times Z)$:

$$\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \xrightarrow{\kappa} \Delta_*L[d]$$

Denote by \mathcal{H}^i the functor of taking i -th cohomology of a complex. Since $d \neq 0, 1$, the associated long exact sequence of cohomologies shows that the complex Q has exactly two non-zero cohomologies: $\Delta_*\mathcal{O}_Z$ in degree 0 and Δ_*L in degree $-d$. More precisely, it shows that the morphisms

$$\Delta_*\mathcal{O}_Z \xrightarrow{\mathcal{H}^0((2.27)-(2.30))} \mathcal{H}^0(Q) \quad \text{and} \quad \mathcal{H}^{-d}(Q) \xrightarrow{\mathcal{H}^{-d}(\kappa)} \Delta_*L$$

are isomorphisms. Use them from now on to identify the spaces involved. Tautologically, the map

$$(3.45) \quad \text{Hom}_{D(Z \times Z)}(Q, \Delta_*L[d]) \xrightarrow{\mathcal{H}^{-d}(-)} \text{Hom}_{Z \times Z}(\Delta_*L, \Delta_*L) \xrightarrow{\sim} \text{Hom}_Z(L, L) \xrightarrow{\sim} \Gamma(\mathcal{O}_Z)$$

sends κ to the element 1 of $\Gamma(\mathcal{O}_Z)$.

Claim: The map (3.45) is injective.

Proof. Clearly it suffices to show that the map \mathcal{H}^{-d} in (3.45) is an isomorphism. Choose an injective resolution I^\bullet of Δ_* . There is a standard spectral sequence associated to the filtration by columns of the total complex of the bicomplex $\text{Hom}(Q^\bullet, I^\bullet)$:

$$E_2^{p,-q} = \text{Ext}_{Z \times Z}^p(H^q(Q), \Delta_*L) \quad \Rightarrow \quad E_\infty^{p,-q} = \text{Hom}_{D(Z \times Z)}^{p,-q}(Q, \Delta_*L).$$

We are interested in the space $\text{Hom}_{D(Z \times X)}(Q, \Delta_*L[d])$ which is the limit E_∞^d of the above spectral sequence. Since the complex Q has cohomology only in two degrees, there are only two potentially non-zero terms $E_2^{p,-q}$ with $p - q = d$. These are $E_2^{0,-d} = \text{Hom}_{Z \times Z}(\Delta_*L, \Delta_*L)$ and $E_2^{d,0} = \text{Ext}_{Z \times Z}^d(\Delta_*\mathcal{O}_Z, \Delta_*L)$. The space $E_2^{d,0}$ is 0 by assumption, so we have $E_\infty^d = E_\infty^{0,-d}$. But observe that there are no non-zero elements $E_2^{p,-q}$ with $p < 0$, and therefore at every page of the spectral sequence the incoming differential $E_r^{-r,-d+r-1} \rightarrow E_r^{0,-d}$ will be zero. Therefore we have natural inclusions $E_{r+1}^{0,-d} \hookrightarrow E_r^{0,-d}$ and the spectral sequence technology ensures that the natural map

$$\text{Hom}_{D(Z \times Z)}(Q, \Delta_*L[d]) \xrightarrow{\mathcal{H}^{-d}(-)} E_2^{0,-d}$$

lifts along each of these inclusions. Let β denote the map we obtain at the limit:

$$(3.46) \quad \begin{array}{ccc} \text{Hom}_{D(Z \times Z)}(Q, \Delta_*L[d]) & \xrightarrow{\mathcal{H}^{-d}(-)} & E_2^{0,-d} \\ & \searrow \beta & \uparrow \\ & & E_\infty^{0,-d} \end{array}$$

Since $E_\infty^{0,-d}$ is the only surviving component of E_∞^d the map β is an isomorphism, and the claim follows. \square

Let Q' denote $\pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E)))$, let λ denote the composition $Q \xrightarrow{\alpha'} Q' \xrightarrow{(3.40)-(3.43)} \Delta_*L[d]$ in diagram (3.44) and let $f \in \Gamma(\mathcal{O}_Z)$ be the image of λ under map (3.45). Since (3.45) is injective, λ is then necessarily the composition $Q \xrightarrow{\kappa} \Delta_*L[d] \xrightarrow{\pi_1^*f} \Delta_*L[d]$. Thus showing that $\lambda = \kappa$, i.e. (3.44) commutes, is equivalent to showing $f = 1$. In fact, it is enough to show that f is invertible, as then scaling α by $\pi_Z^*\frac{1}{f}$ would scale f to 1.

Suppose f isn't invertible, then there exists some closed point $p \in Z$ such that $f(p) = 0$. Let ι_p be the inclusion $p \hookrightarrow Z$ and ι_{Z_p} to be the corresponding inclusion $Z \xrightarrow{(p,-)} Z \times Z$. It is enough to show that

$$(3.47) \quad \iota_{Z_p}^* \left(Q' \xrightarrow{(3.40)-(3.43)} \Delta_*L[d] \right)$$

is the zero map, as then $\Phi_E^{ladj} \Phi_E(\mathcal{O}_p) \xrightarrow{\text{adj. co-unit}} \mathcal{O}_p$ would also be a zero map, implying $E_p = \Phi_E(\mathcal{O}_p) = 0$. By Proposition 3.7 we would then have $d = 1$ which contradicts our assumptions.

Since $Q \xrightarrow{\lambda} \Delta_* L[d]$ is a composition $Q' \xrightarrow{(3.40)-(3.43)} \Delta_* L[d]$ and an isomorphism, it suffices to show that $\iota_{Z_p}^*(Q \xrightarrow{\lambda} \Delta_* L[d])$ is the zero map. By adjunction this is equivalent to the following composition vanishing:

$$Q \xrightarrow{\lambda} \Delta_* L[d] \xrightarrow{\text{adj. unit}} \iota_{Z_p}^* \iota_{Z_p}^* \Delta_* L[d] = \mathcal{O}_{p,p}[d]$$

This holds since λ decomposes as $Q \xrightarrow{\kappa} \Delta_* L[d] \xrightarrow{\pi_1^* f} \Delta_* L[d]$ and $\pi_1^* f(p, p) = f(p) = 0$. \square

Together with Lemma 3.10 this allows us to significantly strengthen the “if” direction of Theorem 3.1 when the integer d_p is everywhere negative. Note that by Lemma 3.12 all objects spherical over Z necessarily have $d_p < 0$ if $\dim Z < \dim X$ and Z and X are not too degenerate.

Theorem 3.2. *Let X and Z be two separable schemes of finite type over k . Let E be a perfect object of $D(Z \times X)$ orthogonal over Z . Then E is spherical over Z if*

- (1) *For every closed point $p \in Z$ such that the fibre E_p is non-zero*

$$\mathbf{R} \text{Hom}_{D(X)}(\pi_{X*} E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}_{<0}.$$

- (2) $E^\vee \otimes \pi_X^! (\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^! (\mathcal{L}_E)$.

4. SPHERICAL FIBRATIONS

The results of Section 3 are rather general and category-theoretic owing to a rather general and category-theoretic nature of the objects it considers: arbitrary complexes in the derived category of the fibre product $Z \times X$. We now choose to restrict ourselves to a more geometric setup and study what our results imply in that context.

Firstly, we assume Z and X to be abstract varieties. Previously we have assumed them to only be separated schemes of finite type over k , now we assume them to also be reduced and irreducible. Strictly speaking, neither assumption is essential for what we prove below. Omitting them, however, would make the arguments more technically involved and the results less clear.

Secondly, and more importantly, we restrict the range of objects we consider from arbitrary complexes in $D(Z \times X)$ to subschemes of X flatly fibred over Z .

4.1. Flat and perfect fibrations with proper fibres.

Definition 4.1. A *flat fibration* W in X over Z is a scheme W equipped with a closed immersion $\xi: W \hookrightarrow X$ and a flat surjective map $\pi: W \rightarrow Z$. For any closed point $p \in Z$ we denote by W_p the set-theoretic fibre of W over p :

$$(4.1) \quad \begin{array}{ccccc} W_p & \hookrightarrow & W & \xrightarrow{\xi} & X \\ \pi_k \downarrow & & \downarrow \pi & & \\ \text{Spec } k & \xrightarrow{\iota_p} & Z & & \end{array}$$

Denote by ι_W the map $W \hookrightarrow Z \times X$ given by the product of π and ι . We have $\xi = \pi_X \circ \iota_W$ and $\pi = \pi_Z \circ \iota_W$. Denote by ξ_p the composition $\xi \circ \iota_{W_p}$, it is the inclusion of the fibre W_p into X . Let E denote the object $\iota_{W*} \mathcal{O}_W$ of $D(Z \times X)$. We think of this object as representing W in the derived category $D(Z \times X)$.

The flatness of W over Z ensures that the category-theoretic notion of a fibre considered in the Section 3 coincides for W with the usual set-theoretic one:

Lemma 4.2. *Let W be a flat fibration in X over Z , let \mathcal{E} be an object of $D(W)$ and let $E = \iota_{W*} \mathcal{E}$ be the corresponding object in $D(Z \times X)$. For any closed point $p \in Z$ denote by \mathcal{E}_p the fibre $\iota_{W_p}^* \mathcal{E}$. We have*

$$E_p \simeq \xi_{p*} \mathcal{E}_p$$

as objects of $D(X)$. In particular, when $\mathcal{E} = \mathcal{O}_W$ we have

$$E_p \simeq \xi_{p*} \mathcal{O}_{W_p}.$$

Proof. The fibre square in the diagram (4.1) decomposes into two fibre squares:

$$(4.2) \quad \begin{array}{ccccc} W_p & \xrightarrow{\iota_{W_p}} & W & & \\ \xi_p \downarrow & & \downarrow \iota_W & \searrow \xi & \\ X & \xrightarrow{\iota_{X_p}} & Z \times X & \xrightarrow{\pi_X} & X \\ \pi_k \downarrow & & \downarrow \pi_Z & & \\ \text{Spec } k & \xrightarrow{\iota_p} & Z & & \end{array}$$

The fibre E_p was defined to be the object $\iota_{X_p}^* E$ of $D(X)$. We have therefore $E_p = \iota_{X_p}^* \iota_{W*} \mathcal{E}$. Consider the base change map

$$(4.3) \quad \iota_{X_p}^* \iota_{W*} \rightarrow \xi_{p*} \iota_{W_p}^*$$

for the top fibre square in the diagram (4.2). Applying it to \mathcal{E} yields a morphism

$$E_p \rightarrow \xi_{p*} \mathcal{E}_p.$$

To show (4.3) to be an isomorphism it suffices to prove that the top fibre square in (4.2) is independent in the sense of [Lip09], §3.10. This follows via [Wie94, Prop. 3.2.9] from π_Z and $\pi = \pi_Z \circ \iota_W$ both being flat. \square

In particular, this makes it clear that $\iota_{W*} \mathcal{O}_W$ is an object of $D(Z \times X)$ which is orthogonal over Z . For any two distinct points p and q of Z the fibres W_p and W_q are disjoint in X and therefore all Hom's between $\xi_{p*} \mathcal{O}_{W_p}$ and $\xi_{q*} \mathcal{O}_{W_q}$ vanish.

In Section 3 we had to make two technical assumptions on the object E of $D(Z \times X)$. These were necessary for the adjoints of the Fourier–Mukai transform Φ_E to exist and to behave in a sensible way. The first assumption was that the support of E is proper over Z . The support of $\iota_{W*} \mathcal{O}_W$ in $Z \times X$ is the image of W under ι_W , so this assumption is equivalent to saying that the fibration morphism $\pi: W \rightarrow Z$ is proper. And π being proper is equivalent to all the fibres of W over closed points of Z being schemes proper over k , cf. [GD61, Corollaire 5.4.5].

The second assumption was that E is a perfect object of $D(Z \times X)$. This corresponds to $\iota_{W*} \mathcal{O}_W$ being perfect. We call a fibration W *perfect* if this holds. Since π is flat this condition can also be checked fibrewise:

Lemma 4.3. *Let W be a flat fibration in X over Z . Then it is perfect if and only if for every closed $p \in Z$ the object $\xi_{p*} \mathcal{O}_{W_p}$ is perfect in $D(X)$.*

Proof. We first claim that $\iota_{W*} \mathcal{O}_W$ is perfect relative to $\pi_Z: Z \times X \rightarrow Z$. There is a commutative diagram

$$(4.4) \quad \begin{array}{ccc} W & \xrightarrow{\iota_W} & Z \times X \\ \pi \downarrow & \swarrow \pi_Z & \\ Z & & \end{array}$$

and since ι_W is a closed immersion, and therefore proper, it takes π -perfect object to π_Z -perfect objects, cf. [Ill71a, Prop. 4.8]. Since π is flat the structure sheaf \mathcal{O}_W is π -flat and therefore most certainly π -perfect. We conclude that $\iota_{W*} \mathcal{O}_W$ is π_Z -perfect.

By the fibrewise criterion for perfection [Ill71a, Corollaire 4.6.1] an object of $D(Z \times X)$ is globally perfect if and only if it is π_Z -perfect and its fibre over every closed point of Z is globally perfect. By Lemma 4.2 the fibre of $\iota_{W*} \mathcal{O}_W$ over any closed $p \in Z$ is $\xi_{p*} \mathcal{O}_{W_p}$. The claim now follows. \square

The following are the two typical situations in which $\iota_{W*} \mathcal{O}_W$ would be perfect in $D(Z \times X)$:

Corollary 4.4. *Let W be a flat fibration in X over Z . Any one of the following conditions is sufficient for W to be perfect:*

- (1) X is smooth.
- (2) Z is smooth and $\xi: W \hookrightarrow X$ is a regular immersion.

Proof. By Lemma 4.3 it suffices to prove that for every closed $p \in Z$ the object $\xi_{p*} \mathcal{O}_{W_p}$ is perfect in $D(X)$.

If X is smooth, then every object of $D(X)$ is perfect and the claim follows trivially.

Suppose that Z is smooth and ξ is a regular immersion. To prove that $\xi_{p*} \mathcal{O}_{W_p}$ is perfect in $D(X)$ it suffices to prove that ξ_p is a regular immersion. A regular immersion is both proper and perfect and hence takes perfect objects to perfect objects, cf. [Ill71a, Corollaire 4.8.1].

Recall the commutative diagram (4.1). Smoothness of Z is equivalent to ι_p being a regular immersion for every closed point p of Z . Since π is faithfully flat, ι_p is regular if and only if ι_{W_p} is regular. Since ξ_p is the composition

$$W_p \xrightarrow{\iota_{W_p}} W \xrightarrow{\xi} X$$

and since a composition of two regular immersions is again a regular immersion, we conclude that ξ_p is regular for every closed $p \in Z$. \square

Thus we arrive at the class of objects we want to work with: flat and perfect fibrations in X over Z with proper fibres. For such fibrations the results of Section 3 can be restated in a more natural way and improved upon. Our goal is to give a satisfying description of the following:

Definition 4.5. Let W be a flat and perfect fibration in X over Z with proper fibres. We say that W is a *spherical fibration* if the object $E = \iota_{W*}\mathcal{O}_W$ is spherical over Z in $D(Z \times X)$.

So let W be a flat and perfect fibration in X with proper fibres and let $E = \iota_{W*}\mathcal{O}_W$ be the corresponding object in $D(Z \times X)$. Recall that the co-twist F_E of the Fourier–Mukai transform Φ_E was defined as the cone of the adjunction unit $\mathrm{Id}_{D(Z)} \rightarrow \Phi_E \Phi_E^{\mathrm{radj}}$ and that the first of the two conditions for E to be spherical was for F_E to be an autoequivalence of $D(Z)$.

Denote by \mathcal{L}_W the object \mathcal{L}_E of $D(Z)$. It was defined in Defn. 3.6 as the cone of a certain composition (3.7) of morphisms in $D(Z)$. This composition was later shown in Lemma 3.8 to be the pushdown from $Z \times X$ to Z via $\tilde{\pi}_{1*}$ of the composition (2.27)–(2.30). Recall that (2.27)–(2.30) is the morphism of the Fourier–Mukai kernels which induces the adjunction unit $\mathrm{Id}_{D(Z)} \rightarrow \Phi_E \Phi_E^{\mathrm{radj}}$. In §4 of [AL12] we have demonstrated that whenever the object E of $D(Z \times X)$ is a pushforward of an object from some closed subscheme of $Z \times X$, as is the case here, there exists a better, more economical decomposition of this morphism of Fourier–Mukai kernels than (2.27)–(2.30). A pushdown of this more economical decomposition to Z via $\tilde{\pi}_1$ could be expected to produce a better description of the defining morphism of \mathcal{L}_W than the composition (3.7). For the sake of simplicity we choose to state this better description directly and prove directly that it is isomorphic to the composition (3.7). An interested reader could check that dualising the composition in [AL12, Theorem 4.2] as described in Section 2.2 of this paper and applying $\tilde{\pi}_{1*}$ would yield the following:

Proposition 4.6. *Let W be a flat and perfect fibration in X with proper fibres and let $E = \iota_{W*}\mathcal{O}_W$ be the corresponding object in $D(Z \times X)$.*

Then the composition (3.7)

$$\mathcal{O}_Z \xrightarrow{\mathrm{Id} \rightarrow \pi_{Z*}\pi_Z^*} \pi_{Z*}\mathcal{O}_{Z \times X} \xrightarrow{\mathrm{Id} \rightarrow \mathbf{R}\mathcal{H}om(E, E \otimes -)} \pi_{Z*}\mathbf{R}\mathcal{H}om_{Z \times X}(E, E) \xrightarrow{(\pi_X^*\pi_{X*} \rightarrow \mathrm{Id})^{\mathrm{opp}}} \pi_{Z*}\mathbf{R}\mathcal{H}om_{Z \times X}(\pi_X^*\pi_{X*}E, E)$$

is isomorphic to

$$(4.5) \quad \mathcal{O}_Z \xrightarrow{\mathrm{Id} \rightarrow \pi_*\pi^*} \pi_*\mathcal{O}_W \xrightarrow{(\xi^*\xi_* \rightarrow \mathrm{Id})^{\mathrm{opp}}} \pi_*(\xi^*\xi_*\mathcal{O}_W)^\vee.$$

In particular, the object \mathcal{L}_W is isomorphic to the cone of (4.5).

Proof. We have $\pi = \pi_Z \circ \iota_W$ and $\xi = \pi_X \circ \iota_W$. Decomposing $\mathrm{Id} \rightarrow \pi_*\pi^*$ as $\mathrm{Id} \rightarrow \pi_{Z*}\pi_Z^* \rightarrow \pi_{Z*}\iota_{W*}\iota_W^*\pi_Z^*$ we see that (4.5) is the composition of $\mathrm{Id} \rightarrow \pi_{Z*}\pi_Z^*$ with the image under π_{Z*} of

$$\mathcal{O}_{Z \times X} \xrightarrow{\mathrm{Id} \rightarrow \iota_{W*}\iota_W^*} \iota_{W*}\mathcal{O}_W \xrightarrow{\gamma(\mathcal{O}_W)} \iota_{W*}\mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) \xrightarrow{(\xi^*\xi_* \rightarrow \mathrm{Id})^{\mathrm{opp}}} \iota_{W*}\mathbf{R}\mathcal{H}om_W(\iota_W^*\pi_X^*\pi_{X*}\iota_{W*}\mathcal{O}_W, \mathcal{O}_W)$$

where given an object A we denote by $\gamma(A)$ the adjunction unit $\mathrm{Id} \rightarrow \mathbf{R}\mathcal{H}om(A, A \otimes -)$. On the other hand (3.7) is the composition of $\mathrm{Id} \rightarrow \pi_{Z*}\pi_Z^*$ with the image under π_{Z*} of

$$\mathcal{O}_{Z \times X} \xrightarrow{\gamma(\iota_{W*}\mathcal{O}_W)} \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*}\mathcal{O}_W, \iota_{W*}\mathcal{O}_W) \xrightarrow{(\pi_X^*\pi_{X*} \rightarrow \mathrm{Id})^{\mathrm{opp}}} \mathbf{R}\mathcal{H}om_{Z \times X}(\pi_X^*\pi_{X*}\iota_{W*}\mathcal{O}_W, \iota_{W*}\mathcal{O}_W).$$

We claim that these two compositions are identified by

$$\iota_{W*}\mathbf{R}\mathcal{H}om_W(\iota_W^*\pi_X^*\pi_{X*}\iota_{W*}\mathcal{O}_W, \mathcal{O}_W) \xrightarrow{\alpha(\iota_W)} \mathbf{R}\mathcal{H}om_{Z \times X}(\pi_X^*\pi_{X*}\iota_{W*}\mathcal{O}_W, \iota_{W*}\mathcal{O}_W)$$

where $\alpha(\iota_W)$ is the natural bifunctorial isomorphism $\iota_{W*}\mathbf{R}\mathcal{H}om(\iota_W^*-, -) \rightarrow \mathbf{R}\mathcal{H}om(-, \iota_{W*}-)$.

Dualizing [AL12, Prop. 4.1] under the relative duality $D_{\bullet/X}$ (where X is in the notation of loc. cit.) we see that the morphism

$$\mathcal{O}_{Z \times X} \xrightarrow{\gamma(\iota_{W*}\mathcal{O}_W)} \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*}\mathcal{O}_W, \iota_{W*}\mathcal{O}_W)$$

equals the morphism

$$\mathcal{O}_{Z \times X} \xrightarrow{\mathrm{Id} \rightarrow \iota_{W*}\iota_W^*} \iota_{W*}\mathcal{O}_W \xrightarrow{\gamma(\mathcal{O}_W)} \iota_{W*}\mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) \xrightarrow{\beta(\iota_W)} \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*}\mathcal{O}_W, \iota_{W*}\mathcal{O}_W)$$

where given a scheme map f we denote by $\beta(f)$ the natural morphism $f_* \mathbf{R}\mathcal{H}om(\cdot, \cdot) \rightarrow \mathbf{R}\mathcal{H}om(f_*, f_*)$. It remains to establish the commutativity of the diagram

$$\begin{array}{ccc} \iota_{W*} \mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) & \xrightarrow{(\iota_W^* \iota_{W*} \rightarrow \text{Id})^{\text{opp}}} & \iota_{W*} \mathbf{R}\mathcal{H}om_W(\iota_W^* \iota_{W*} \mathcal{O}_W, \mathcal{O}_W) & \xrightarrow{(\pi_X^* \pi_{X*} \rightarrow \text{Id})^{\text{opp}}} & \iota_{W*} \mathbf{R}\mathcal{H}om_W(\iota_W^* \pi_X^* \pi_{X*} \iota_{W*} \mathcal{O}_W, \mathcal{O}_W) \\ \downarrow \beta(\iota_W) & & & & \downarrow \alpha(\iota_W) \\ \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) & \xrightarrow{(\pi_X^* \pi_{X*} \rightarrow \text{Id})^{\text{opp}}} & \mathbf{R}\mathcal{H}om_{Z \times X}(\pi_X^* \pi_{X*} \iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) & & \end{array}$$

By the functoriality of $\alpha(\iota_W)$ it suffices to show that the diagram

$$(4.6) \quad \begin{array}{ccc} \iota_{W*} \mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) & \xrightarrow{(\iota_W^* \iota_{W*} \rightarrow \text{Id})^{\text{opp}}} & \iota_{W*} \mathbf{R}\mathcal{H}om_W(\iota_W^* \iota_{W*} \mathcal{O}_W, \mathcal{O}_W) \\ \downarrow \beta(\iota_W) & \swarrow \alpha(\iota_W) & \\ \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) & & \end{array}$$

commutes. But the isomorphism $\alpha(\iota_W)$ was defined as the composition

$$\iota_{W*} \mathbf{R}\mathcal{H}om(\iota_W^* \cdot, -) \xrightarrow{\beta(\iota_W)} \mathbf{R}\mathcal{H}om(\iota_{W*} \iota_W^* \cdot, \iota_{W*} \cdot) \xrightarrow{(\text{Id} \rightarrow \iota_{W*} \iota_W^*)^{\text{opp}}} \mathbf{R}\mathcal{H}om(\cdot, \iota_{W*} \cdot)$$

and therefore we can re-write the diagram (4.6) as

$$\begin{array}{ccc} \iota_{W*} \mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) & \xrightarrow{(\iota_W^* \iota_{W*} \rightarrow \text{Id})^{\text{opp}}} & \iota_{W*} \mathbf{R}\mathcal{H}om_W(\iota_W^* \iota_{W*} \mathcal{O}_W, \mathcal{O}_W) \\ \downarrow \beta(\iota_W) & & \downarrow \beta(\iota_W) \\ \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) & \xleftarrow{(\text{Id} \rightarrow \iota_{W*} \iota_W^*)^{\text{opp}}} & \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \iota_W^* \iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) \end{array}$$

By the functoriality of $\beta(\iota_W)$ it remains only to check that the diagram

$$\begin{array}{ccc} \iota_{W*} \mathbf{R}\mathcal{H}om_W(\mathcal{O}_W, \mathcal{O}_W) & \xrightarrow{\beta(\iota_W)} & \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) \\ \downarrow \beta(\iota_W) & & \downarrow (\iota_W^* \iota_{W*} \rightarrow \text{Id})^{\text{opp}} \\ \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) & \xleftarrow{(\text{Id} \rightarrow \iota_{W*} \iota_W^*)^{\text{opp}}} & \mathbf{R}\mathcal{H}om_{Z \times X}(\iota_{W*} \iota_W^* \iota_{W*} \mathcal{O}_W, \iota_{W*} \mathcal{O}_W) \end{array}$$

commutes, which follows from

$$\iota_{W*} \mathcal{O}_W \xrightarrow{\text{Id} \rightarrow \iota_{W*} \iota_W^*} \iota_{W*} \iota_W^* \iota_{W*} \mathcal{O}_W \xrightarrow{\iota_W^* \iota_{W*} \rightarrow \text{Id}} \iota_{W*} \mathcal{O}_W$$

being an identity morphism. \square

Next we give an analogue of Proposition 3.7:

Proposition 4.7. *Let W be a flat and perfect fibration in X over Z with proper fibres. The following are equivalent:*

- (1) *There exists $d \in \mathbb{Z}$ such that for every closed point $p \in Z$ we have*

$$\mathbf{R}\text{Hom}_{D(X)}(\xi_* \mathcal{O}_W, \xi_{p*} \mathcal{O}_{W_p}) = k \oplus k[d].$$

- (2) *We have $\mathcal{L}_W \simeq L[d]$ for some $L \in \text{Pic } Z$ and $d \in \mathbb{Z}$.*
- (3) *The co-twist F_E is an autoequivalence of $D(Z)$.*

When the conditions above are satisfied $F_E \simeq (-) \otimes \mathcal{L}_W[-1]$ and the integers d in (1) are (2) are equal.

Proof. Since Z is connected any invertible object of $D(Z)$ is a shift of line bundle [AIL10, Theorem 1.5.2]. Thus our conditions (2) and (3) are equivalent to conditions (2) and (3) of Proposition 3.7.

As $\xi = \pi_X \circ \iota_W$ we have $\pi_{X*} E \simeq \pi_{X*} \iota_{W*} \mathcal{O}_W = \xi_* \mathcal{O}_W$. By Lemma 4.2 the categorical fibre E_p is $\xi_{p*} \mathcal{O}_{W_p}$. Under these identifications the morphism $\pi_{X*} E \xrightarrow{(3.12)} E_p$ is readily seen to be the sheaf restriction map $\xi_* \mathcal{O}_W \rightarrow \xi_* \mathcal{O}_{W_p}$ and thus non-zero for every $p \in Z$. Therefore our condition (1) is equivalent to condition (1) of Proposition 3.7 with an extra assumption that the integer d_p is the same for all $p \in Z$.

Now the assertion of this Proposition can be seen to follow directly from those of Proposition 3.7. \square

We could similarly re-state Theorem 3.1. However under a mild non-degeneracy assumption on W we can apply the results of Section 3.4 to make a stronger and more geometric statement. Since Z and X are abstract varieties they are generically non-singular. Hence the Gorenstein locus of $Z \times X$ is certainly dense in $Z \times X$. Our non-degeneracy assumption is that the graph of W doesn't lie completely outside this locus:

Theorem 4.1. *Let W be a flat and perfect fibration in X with proper fibres. Then W is spherical if:*

(1) *For any closed $p \in Z$ we have*

$$\mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) = k \oplus k[-(\dim X - \dim Z)].$$

(2) *There exists an isomorphism*

$$\iota_{W*}\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \iota_{W*}\pi^!(\mathcal{L}_W).$$

If the graph of W in $Z \times X$ doesn't lie outside the Gorenstein locus the reverse implication also holds.

Proof. We have the following natural isomorphisms:

$$\iota_{W*}\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \iota_{W*}\mathbf{R}\mathrm{Hom}_{Z \times X}(\mathcal{O}_W, \iota_W^!\pi_X^!(\mathcal{O}_X)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{Z \times X}(\iota_{W*}\mathcal{O}_W, \pi_X^!(\mathcal{O}_X)) \xrightarrow{\sim} E^\vee \otimes \pi_X^!(\mathcal{O}_X)$$

where the second isomorphism is due to the sheafified Grothendieck duality and the third is due to $E = \iota_{W*}\mathcal{O}_W$ being perfect. Similarly we obtain $\iota_{W*}\pi^!(\mathcal{L}_W) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_W)$. Therefore (2) is equivalent to there existing an isomorphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_W).$$

Suppose that (1) and (2) hold. By Proposition 3.7 the assumption (1) implies that the co-twist F_E is an autoequivalence and $\mathcal{L}_W \simeq L[-(\dim X - \dim Z)]$ for some $L \in \mathrm{Pic}(Z)$. Since $\dim X - \dim Z > 0$ it follows from Proposition 3.13 that the existence of any isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_W)$ implies that the canonical morphism α of Definition 3.11 is an isomorphism. Thus F_E is an autoequivalence and α is an isomorphism, and so W is a spherical fibration.

Conversely, suppose that W is a spherical fibration whose graph doesn't lie outside the Gorenstein locus of $Z \times X$. Then the co-twist F_E is an autoequivalence and so by Proposition 3.7 we have $\mathcal{L}_E \simeq L[d]$ for some $L \in \mathrm{Pic} Z$ and $d \in \mathbb{Z}$. By the non-degeneracy assumption there exists a point $p \in W$ such that $\xi(p)$ is Gorenstein in X and $\pi(p)$ is Gorenstein in Z . By Proposition 3.12 we then have $d = -(\dim X - \dim Z)$. Applying Proposition 3.7 again yields the assertion (1). On the other hand, since W is spherical the canonical morphism α is an isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_W)$ whence the assertion (2). \square

Recall the notion of a Gorenstein map, cf. §2.4 of [AIL10] or [AF90] for the local picture. A scheme map $f: S \rightarrow T$ is called Gorenstein if it is perfect and if $f^!(\mathcal{O}_T)$ is an invertible object of $D(S)$. If S is connected $f^!(\mathcal{O}_T)$ is a shift of some line bundle in $\mathrm{Pic} S$. We call this line bundle the *relative dualizing sheaf* and denote it by $\omega_{S/T}$. For any Gorenstein scheme S over k the (global) *dualizing sheaf* of S is the relative dualizing sheaf of $S \rightarrow \mathrm{Spec} k$ and we denote it by ω_S . If S is smooth then ω_S is the canonical bundle.

In our case the map $\pi: W \rightarrow Z$ is faithfully flat and thus Gorenstein if and only if its fibres are Gorenstein schemes [AIL10, Prop, 2.5.10]. On the other hand, the map ξ is the composition

$$W \xrightarrow{\iota_W} Z \times X \xrightarrow{\pi_X} X.$$

The closed immersion ι_W is perfect by the assumption that $\iota_{W*}\mathcal{O}_W$ is perfect [Ill71a, Prop. 4.4]. Hence ξ is perfect as it is a composition of two perfect maps. Thus ξ is Gorenstein if and only if $\xi^!(\mathcal{O}_X)$ is invertible.

If either π or ξ are Gorenstein we can re-state the second part of the Theorem 4.1 in terms of the line bundles involved and get rid of the non-degeneracy assumption on W :

Proposition 4.8. *Let W be a flat and perfect fibration in X over Z with proper fibres and assume that either the immersion $\xi: W \hookrightarrow X$ or the fibration $\pi: W \rightarrow Z$ is Gorenstein. Then W is spherical if and only if:*

(1) *For any closed $p \in Z$ we have*

$$\mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) = k \oplus k[-(\dim X - \dim Z)]$$

By Proposition 4.7 this implies that $\mathcal{L}_W = L[-(\dim X - \dim Z)]$ for some $L \in \mathrm{Pic} Z$.

(2) *Both ξ and π are Gorenstein and we have in $\mathrm{Pic}(W)$ an isomorphism*

$$\omega_{W/X} \simeq \pi^*L \otimes \omega_{W/Z}.$$

Proof. ‘If’: We have

$$\xi^!(\mathcal{O}_X) \simeq \omega_{W/X}[-(\dim X - \dim W)] \quad \pi^!(\mathcal{O}_X) \simeq \omega_{W/Z}[\dim W - \dim Z]$$

and therefore the condition (2) implies $\xi^!(\mathcal{O}_X) \simeq \pi^!(\mathcal{L}_W)$. Therefore W is spherical by Theorem 4.1.

‘Only if’: Suppose W is spherical. Arguing as in the proof of Theorem 4.1 shows that \mathcal{L}_W is invertible and we have an isomorphism

$$(4.7) \quad \iota_{W*}\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \iota_{W*}\pi^!(\mathcal{L}_W).$$

Our assumptions imply that one of $\xi^!(\mathcal{O}_X)$ or $\pi^!(\mathcal{L}_W)$ is invertible. Thus (4.7) is an isomorphism of (shifted) coherent sheaves. Since ι_{W*} is a closed immersion it restricts to a fully faithful functor $\text{Coh}(W) \rightarrow \text{Coh}(Z \times X)$. Hence isomorphism (4.7) lifts to an isomorphism

$$(4.8) \quad \xi^!(\mathcal{O}_X) \xrightarrow{\sim} \pi^!(\mathcal{L}_W).$$

Therefore $\xi^!(\mathcal{O}_X)$ and $\pi^!(\mathcal{L}_W)$ are both invertible, i.e. π and ξ are both Gorenstein.

Since \mathcal{L}_W is invertible it is of form $L[d]$ for some $L \in \text{Pic } Z$ and $d \in \mathbb{Z}$. We can re-write (4.8) as

$$\omega_{W/X}[-(\dim X - \dim W)] \simeq \pi^* L[d] \otimes \omega_{W/Z}[\dim W - \dim Z]$$

whence $d = -(\dim X - \dim Z)$ and the isomorphism (2). Finally, since $\mathcal{L}_W \simeq L[-(\dim X - \dim Z)]$ we can apply Proposition 4.7 to obtain the assertion (1). \square

Corollary 4.9. *Let W be a spherical fibration in X over Z . Then $\xi: W \hookrightarrow X$ is a Gorenstein immersion if and only if all the fibres of $\pi: W \rightarrow Z$ are Gorenstein schemes.*

4.2. Regularly immersed fibrations. One class of Gorenstein maps is that of regular immersions, cf. [GD67], §16 and §19 and [Ber71]. A closed immersion $\iota: Y \hookrightarrow X$ of two schemes is called *regular* if the ideal sheaf \mathcal{I}_Y of Y in X is locally generated by a regular sequence. It follows that locally on X the Koszul complex of Y is a resolution of the sheaf $\iota_* \mathcal{O}_Y$ by free sheaves. In particular, the co-normal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a locally free sheaf on Y whose rank c is the codimension of Y in X . We denote by $\mathcal{N}_{Y/X}$ its dual $(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$, the normal sheaf of Y in X .

It follows by §III.7 of [Har66] that

$$\iota^!(\mathcal{O}_X) = \wedge^c \mathcal{N}_{Y/X}[-c]$$

i.e. the relative dualizing sheaf $\omega_{Y/X}$ is the line bundle $\wedge^c \mathcal{N}_{Y/X}$. By [Ber71, Prop. 2.5] the cohomology sheaves of $\iota^* \iota_* \mathcal{O}_Y$ are

$$\mathcal{H}^{-i}(\iota^* \iota_* \mathcal{O}_Y) = \wedge^i \mathcal{N}_{Y/X}^\vee \quad \forall i \in \mathbb{Z}.$$

Let A be any object of $D(Y)$. By projection formula we have

$$\iota_*(\iota^* \iota_* A) \simeq \iota_* A \otimes \iota_* \mathcal{O}_Y \simeq \iota_*(A \otimes \iota^* \iota_* \mathcal{O}_Y).$$

As ι_* is exact and fully faithful on the level of abelian categories of coherent sheaves it follows that the cohomology sheaves of $\iota^* \iota_* A$ are isomorphic to those of $A \otimes \iota^* \iota_* \mathcal{O}_Y$.

One can ask when does $\iota^* \iota_* \mathcal{O}_Y$ split up as the direct sum of its cohomology sheaves:

$$(4.9) \quad \iota^* \iota_* \mathcal{O}_Y \xrightarrow{\sim} \bigoplus_i \wedge^i \mathcal{N}_{Y/X}^\vee[i]$$

This is true when a global Koszul resolution of Y in X exists, i.e. when Y is carved out in X by a section of a vector bundle. For smooth X a more general answer was provided by Arinkin and Caldararu in [AC10]: $\iota^* \iota_* \mathcal{O}_Y$ is isomorphic to $\bigoplus_i \wedge^i \mathcal{N}_{Y/X}^\vee[i]$ if and only if the normal sheaf $\mathcal{N}_{Y/X}$ extends to the first infinitesimal neighborhood of Y in X . The examples of when this holds include: when Y is carved out by a section of a vector bundle, when the immersion $\iota: Y \hookrightarrow X$ can be split and when Y is the fixed locus of a finite group action on X .

For arbitrary schemes we make the the following definition:

Definition 4.10. Let Y and X be a pair of schemes and let $\iota: Y \rightarrow X$ be a regular immersion. We say that ι is an *Arinkin-Caldararu immersion* if $\iota^* \iota_* \mathcal{O}_Y$ is isomorphic to $\bigoplus_i \wedge^i \mathcal{N}_{Y/X}^\vee[i]$ in $D(X)$.

Going back to our setup, we say that a fibration W in X over Z is *regularly immersed* if $\xi: W \hookrightarrow X$ is a regular immersion. Knowing the cohomology sheaves of $\xi^* \xi_* \mathcal{O}_W$ allows us to reduce the condition

$$\mathbf{R}\text{Hom}_X(\xi_* \mathcal{O}_W, \xi_{p*} \mathcal{O}_{W_p}) = k \oplus k[-(\dim X - \dim Z)]$$

via a spectral sequence argument to a statement on the vanishing of the sheaf cohomologies of $\wedge^i \mathcal{N}$ on W_p . If, moreover, $\xi^* \xi_* \mathcal{O}_W$ breaks up as a sum of its cohomologies then there is no need for the spectral sequence argument and we also obtain the converse implication:

Theorem 4.2. *Let W be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Let c be the codimension of W in X , let d be the dimension of the fibres of W and let $\mathcal{N} = \mathcal{N}_{W/X}$.*

Then W is spherical if for any closed point $p \in Z$ the fibre W_p is a connected Gorenstein scheme and

- (1) $H_{W_p}^i(\wedge^j \mathcal{N}|_{W_p}) = 0$ unless $i = j = 0$ or $i = d$, $j = c$.
- (2) $(\omega_{W/X})|_{W_p} \simeq \omega_{W_p}$.

Conversely, if W is spherical then each fibre W_p is a connected Gorenstein scheme and (2) holds. If, moreover, ξ is an Arinkin-Caldararu immersion then (1) also holds.

The following lemma is a global version of the fibrewise conditions of Theorem 4.2:

Lemma 4.11. *Let W be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Assume that for any closed point $p \in Z$ the fibre W_p is a connected Gorenstein scheme.*

Then having for every closed point $p \in Z$

$$(4.10) \quad H_{W_p}^i(\wedge^j \mathcal{N}|_{W_p}) = 0 \text{ unless } i = j = 0 \text{ or } i = d, j = c$$

$$\omega_{W/X}|_{W_p} = \omega_{W_p}$$

is equivalent to having

$$(4.11) \quad \pi_* \mathcal{O}_W = \mathcal{O}_Z, \quad \pi_* \wedge^j \mathcal{N} = 0 \text{ for all } 0 < j < c, \quad \pi_* \omega_{W/X} = L[d],$$

$$\omega_{W/X} = \pi^* L \otimes \omega_{W/Z}$$

for some $L \in \text{Pic } Z$. In particular, (4.10) implies that $H_{W_p}^0(\mathcal{O}_{W_p}) \simeq H_{W_p}^d(\omega_{W/X}|_{W_p}) \simeq k$.

Proof. By flat base change around the square

$$\begin{array}{ccc} W_p & \xrightarrow{\iota_{W_p}} & W \\ \pi_k \downarrow & & \downarrow \pi \\ \text{Spec } k & \xrightarrow{\iota_p} & Z \end{array}$$

we have a functorial isomorphism $\iota_p^* \pi_* \simeq \pi_{k*} \iota_{W_p}^*$. Since $H_{W_p}^i(\wedge^j \mathcal{N}|_{W_p})$ is the i -th cohomology of $\pi_{k*} \iota_{W_p}^*(\wedge^j \mathcal{N})$ restricting (4.11) to any closed $p \in Z$ by ι_p^* gives (4.10).

Conversely, assume that (4.10) holds for every closed $p \in Z$. By the Grothendieck duality for W_p we have

$$H_{W_p}^d(\omega_{W/X}|_{W_p}) \simeq H_{W_p}^d(\omega_{W_p}) \simeq H_{W_p}^0(\mathcal{O}_{W_p})$$

and since W_p is proper and connected we have $H_{W_p}^0(\mathcal{O}_{W_p}) \simeq k$. Thus by the above base change we have for every closed $p \in Z$

$$\begin{aligned} \iota_p^* \pi_* \mathcal{O}_Z &\simeq k \\ \iota_p^* \pi_* \wedge^j \mathcal{N}|_{W_p} &= 0 \quad \text{for all } 0 < j < c \\ \iota_p^* \pi_* \omega_{W/X} &\simeq k[-d]. \end{aligned}$$

Therefore $\pi_* \wedge^j \mathcal{N}$ vanishes for $0 < j < c$, while $\pi_* \mathcal{O}_W \simeq L'$ and $\pi_* \omega_{W/X}[d] \simeq L$ for some $L', L \in \text{Pic } Z$. But then $L' \simeq \mathcal{O}_Z$ since the adjunction unit $\mathcal{O}_Z \rightarrow \pi_* \pi^* \mathcal{O}_Z$ gives a nowhere vanishing morphism $\mathcal{O}_Z \rightarrow L'$ of line bundles. This is because the restriction of the adjunction unit $\mathcal{O}_Z \rightarrow \pi_* \pi^* \mathcal{O}_Z$ to any $p \in Z$ is the adjunction unit $k \rightarrow \pi_{k*} \pi_k^* k$ which certainly doesn't vanish.

Similarly, by the sheafified Grothendieck duality

$$L \simeq \pi_* \omega_{W/X}[d] \simeq \pi_* \mathbf{R} \mathcal{H} \text{om}(\omega_{W/X}^{-1} \otimes \omega_{W/Z}, \omega_{W/Z}[d]) \simeq \left(\pi_*(\omega_{W/X}^{-1} \otimes \omega_{W/Z}) \right)^\vee.$$

Therefore the adjunction co-unit $\pi^* \pi_*(\omega_{W/X}^{-1} \otimes \omega_{W/Z}) \rightarrow (\omega_{W/X}^{-1} \otimes \omega_{W/Z})$ gives a nowhere vanishing line bundle morphism $\pi^* L^\vee \rightarrow \omega_{W/X}^{-1} \otimes \omega_{W/Z}$, whence the final assertion that $\omega_{W/X} \simeq \pi^* L \otimes \omega_{W/Z}$. \square

Proof of Theorem 4.2. 'If' direction: Since ξ_p is the composition $W_p \xrightarrow{\iota_{W_p}} W \xrightarrow{\xi} X$ we have by adjunction

$$\mathbf{R} \text{Hom}_X(\xi_* \mathcal{O}_W, \xi_{p*} \mathcal{O}_{W_p}) \simeq \mathbf{R} \text{Hom}_W(\xi^* \xi_* \mathcal{O}_W, \iota_{W_p*} \mathcal{O}_{W_p}).$$

Consider the standard spectral sequence

$$E_2^{i,j} = \text{Ext}_W^i(\mathcal{H}^{-j}(\xi^* \xi_* \mathcal{O}_W), \iota_{W_p*} \mathcal{O}_{W_p}) \Rightarrow E_\infty^{i+j} = \text{Hom}_{D(W)}^{i+j}(\xi^* \xi_* \mathcal{O}_W, \iota_{W_p*} \mathcal{O}_{W_p}).$$

Since for any $j \in \mathbb{Z}$ we have $\mathcal{H}^{-j}(\xi^* \xi_* \mathcal{O}_W) = \wedge^j \mathcal{N}^\vee$ it follows by adjunction that

$$E_2^{i,j} \simeq \text{Ext}_W^i(\wedge^j \mathcal{N}^\vee, \iota_{W_p*} \mathcal{O}_{W_p}) \simeq \text{Ext}_{W_p}^i(\wedge^j \mathcal{N}^\vee|_{W_p}, \mathcal{O}_{W_p}) \simeq H_{W_p}^i(\wedge^j \mathcal{N}|_{W_p}).$$

Since the fibers of W are proper and connected $H_{W_p}^0(\mathcal{O}_{W_p}) \simeq k$. Moreover

$$H_{W_p}^d(\omega_{W/X}|_{W_p}) \simeq H_{W_p}^d(\omega_{W_p}) \simeq H_{W_p}^0(\mathcal{O}_{W_p}) \simeq k.$$

by the assumption (2) and the Grothendieck duality.

Thus by assumption (1) and by Lemma 4.11 all $E_2^{i,j}$ are zero except for

$$E_2^{0,0} = H_{W_p}^0(\mathcal{O}_{W_p}) \simeq k \text{ and } E_2^{d,c} = H_{W_p}^d(\omega_{W/X}|_{W_p}) \simeq k.$$

Since $d + c = \dim X - \dim Z \neq 0$ the convergence of the spectral sequence implies that

$$\mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) \simeq k \oplus k[-(\dim X - \dim Z)].$$

By Prop. 4.7 we have $\mathcal{L}_W \simeq L[c + d]$ for some $L \in \mathrm{Pic} Z$. By Prop. 4.6 we have an exact triangle

$$\mathcal{O}_Z \rightarrow \pi_* \mathbf{R}\mathrm{Hom}(\xi^*\xi_*\mathcal{O}_W, \mathcal{O}_W) \rightarrow L[-(c + d)].$$

Since $c + d > 0$ it follows that the $(c + d)$ -th cohomology sheaf of the complex $\pi_* \mathbf{R}\mathrm{Hom}(\xi^*\xi_*\mathcal{O}_W, \mathcal{O}_W)$ is isomorphic to L . On the other hand, computing this cohomology sheaf via a spectral sequence similar to the one above yields $\pi_*\omega_{W/X}[d]$. Thus $L \simeq \pi_*\omega_{W/X}[d]$ which implies by Lemma 4.11 that $\omega_{W/X} \simeq \pi^*L \otimes \omega_{W/Z}$. By Prop. 4.8 we conclude that W is spherical.

‘Only If’ direction:

Conversely, suppose W is spherical. By Proposition 4.8 the fibres of π are Gorenstein schemes and we have

$$\mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) \simeq k \oplus k[-(\dim X - \dim Z)]$$

for each fibre W_p . The same spectral sequence as before shows that the 0-th cohomology of the complex $\mathbf{R}\mathrm{Hom}_X^i(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p})$ is isomorphic to $H_{W_p}^0(\mathcal{O}_{W_p})$. Therefore $H_{W_p}^0(\mathcal{O}_{W_p}) = k$ and so the fibers W_p are connected. By Prop. 4.8 we have $\omega_{W/X} \simeq \pi^*L \otimes \omega_{W/Z}$ for some $L \in \mathrm{Pic} Z$. Restricting this to every fiber gives the assertion (2).

Finally, suppose that ξ is Arinkin-Caldararu. Then $\xi^*\xi_*\mathcal{O}_W \simeq \bigoplus_i \wedge^i \mathcal{N}^\vee[-i]$, so

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) &\simeq \mathbf{R}\mathrm{Hom}_W(\xi^*\xi_*\mathcal{O}_W, \iota_{W_p*}\mathcal{O}_{W_p}) \simeq \\ &\simeq \bigoplus_i \mathbf{R}\mathrm{Hom}_W(\wedge^i \mathcal{N}^\vee, \iota_{W_p*}\mathcal{O}_{W_p})[i] \simeq \bigoplus_i \mathbf{R}\mathrm{Hom}_{W_p}(\mathcal{O}_{W_p}, \wedge^i \mathcal{N})[i] \end{aligned}$$

and we see that the assertion (1) is equivalent to

$$\mathbf{R}\mathrm{Hom}_X(\xi_*\mathcal{O}_W, \xi_{p*}\mathcal{O}_{W_p}) \simeq k \oplus k[-(\dim X - \dim Z)].$$

□

APPENDIX A. AN EXAMPLE

It is well-known that the derived category $D(T^*Fl_n)$ where Fl_n is the full flag variety for some Lie algebra \mathfrak{g} carries an action of the affine braid group [KT07], [Bez06]. It is shown in [KT07] that the action of the usual braid group Br_n is by spherical twists T_i , $i = 1, \dots, n - 1$ in spherical functors $S_i: D(T^*\mathcal{P}_i) \rightarrow D(T^*Fl_n)$, where \mathcal{P}_i are the partial flag varieties with the space of dimension i missing from the flag. The functor S_i is obtained as the composition $\iota_*\pi^*$, where $\iota: D_i \hookrightarrow T^*Fl_n$ is the embedding of the divisor $D_i = Fl_n \times_{\mathcal{P}_i} T^*\mathcal{P}_i$, and $\pi: D_i \rightarrow T^*\mathcal{P}_i$ is a \mathbb{P}^1 -bundle. The Fourier–Mukai kernel of S_i is an example of a spherical fibration, being the structure sheaf of $D_i \subset T^*Fl_n \times T^*\mathcal{P}_i$ where D_i embeds into T^*Fl_n and is fibered over $T^*\mathcal{P}_i$.

Recall that the usual braid group is generated by $n - 1$ “crossings” t_1, \dots, t_{n-1} , with the relations $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$. The affine braid group is generated by the same t_1, \dots, t_{n-1} , plus a “rotation” generator r (if the affine braid group is viewed as the group of braids in an annulus, this generator shifts strands, say, counterclockwise). The relations then are $rt_i r^{-1} = t_{i+1}$ and $r^2 t_n r^{-2} = t_1$. One can add one more “crossing” $r^{-1} t_1 r = t_0 = t_n = r t_{n-1} r^{-1}$, keeping the relations $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$. In the above affine braid group action on $D(T^*Fl_n)$ the action of the functor corresponding to t_n is not known to have an interpretation as a spherical twist. This can be mended in a specific case, and the relative spherical object that induces the twist will not be a structure sheaf of a subscheme. For the details and proofs please see [Ann08].

Let \mathfrak{g} be $\mathfrak{sl}_n(\mathbb{C})$. Consider the Grothendieck–Springer resolution $\tilde{\pi}: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. It provides a resolution of singularities $\pi: T^*Fl_n \rightarrow \mathcal{N}$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. Let z_{2n} be a nilpotent element of $\mathfrak{sl}_{2n}(\mathbb{C})$, with two Jordan blocks of rank n , let $\mathcal{S}_{2n} \subset \mathfrak{g}$ be a transversal slice to the orbit of z_{2n} under the adjoint action of $\mathrm{SL}_{2n}(\mathbb{C})$, and let $\mathcal{U}_{2n} \subset \tilde{\mathfrak{g}}$ be the preimage of \mathcal{S}_{2n} under the resolution π . By [Bez06, Remark 2.2] the action of the affine braid group on $D(\tilde{\mathfrak{g}})$ restricts to an action of the same group on $D(\mathcal{U}_{2n})$; one can construct this action explicitly in a manner similar to [KT07]. The variety \mathcal{U}_{2n} is smooth symplectic of complex dimension $2n$. The preimage \mathcal{X}_{2n} of z_{2n} is a projective variety of dimension n . It is a union of smooth components intersecting normally. For simplicity, denote the derived category $D_{\mathcal{X}_{2n}}(\mathcal{U}_{2n})$ by \mathcal{D}_{2n} .

The non-affine braid group action on \mathcal{D}_{2n} is generated by twists in functors S_i , $1 \leq i \leq 2n - 1$ defined by certain spherical fibrations, cf. [AN] for explicit formulas; it is the special property of the nilpotent element z_{2n} that the sources of these functors S_i are all equivalent to \mathcal{D}_{2n-2} . Apart from these functors, there is an

autoequivalence $R : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$, cf. [Ann08, §4.1] that corresponds to the affine generator r described above. The remaining twist T_{2n} can be obtained by conjugating T_1 or T_{2n-1} by R .

It is proven in [Ann08] that the generator T_n is indeed a twist in some functor $S_{2n} : \mathcal{D}_{2n-2} \rightarrow \mathcal{D}_{2n}$. In fact, S_{2n} is isomorphic to RS_1 or $R^{-1}S_{2n-1}$. The remarkable thing about S_{2n} is that being a composition of S_1 or S_{2n-1} and an autoequivalence of \mathcal{D}_{2n} , it retains many properties of S_i 's. In particular, its kernel $\mathcal{K} \in D(\mathcal{U}_{2n-2} \times \mathcal{U}_{2n})$ is orthogonally spherical over \mathcal{U}_{2n-2} . At the same time \mathcal{K} is a genuine object of the derived category $D(\mathcal{U}_{2n-2} \times \mathcal{U}_{2n})$, that is, not isomorphic to the direct sum of its cohomology sheaves. It may be seen in the computation carried out in [Ann08], section 7.2, for $n = 2$; in this case $\mathcal{U}_{2n-2} = \mathcal{U}_2 \simeq T^*\mathbb{P}^1$, and while the image of $\mathcal{O}_{\mathbb{P}^1}$ is a sheaf on \mathcal{U}_4 , the image of $\mathcal{O}_{\mathbb{P}^1}(-1)$ is not. If \mathcal{K} was actually a spherical fibration, that is, a structure sheaf of some $D \subset \mathcal{U}_4$ fibered over \mathcal{U}_2 , this would not be possible.

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E-mail address: anno@pitt.edu

DEPARTMENT OF MATHEMATICS, SCHOOL OF ARTS AND SCIENCES, UNIVERSITY OF PITTSBURGH, 301 THACKERAY HALL, PITTSBURGH, PA 15260, USA

E-mail address: LogvinenkoT@cardiff.ac.uk

CARDIFF SCHOOL OF MATHEMATICS, CARDIFF UNIVERSITY, SENGHENNYDD ROAD, CARDIFF, CF24 4AG, UK