Grounded Semantics and Infinitary Argumentation Frameworks

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Abstract

Computing the grounded extension of an argumentation framework can be done using the well-known inductive procedure of Dung's landmark paper. However, this procedure has only been proven to be correct for finitary argumentation frameworks, that is, frameworks in which every argument has only a finite number of defeaters. The problem is that formalisms like ASPIC$^+$ and ASPIC$^-$ can easily generate frameworks in which arguments have an infinite number of defeaters. In the current paper, we will therefore broaden the applicability of the proof procedures for grounded semantics, and weaken the condition that the argumentation framework has to be finitary.

1 Introduction

Rule-based instantiated argumentation formalisms, such as ASPIC [1], ASPIC$^+$ [9, 10] and the argument-interpretation of ABA [6] have enjoyed an increasing popularity within the formal argumentation community. Their main advantage over abstract argumentation (c.f. [5]) is that they enable nonmonotonic entailment to be defined as rule-based inference. This can have advantages when it comes to the ability to explain formal nonmonotonic inference in terms that human actors can relate to, as observed in [2].

One particular difficulty that several rule-based argumentation formalisms are subject to is that even a finite set of rules can lead to an infinite set of arguments. For instance, in an ASPIC type framework [1, 3, 9, 10], and adopting ASPIC notation, given a set of strict rules $\{\rightarrow a; \ \rightarrow b; \ \neg c \rightarrow \neg d; \ \neg d \rightarrow \neg c\}$ and a set of defeasible rules $\{a \Rightarrow c; \ \ b \Rightarrow \neg c\}$ the argument $A : (\rightarrow a) \Rightarrow c$ has an infinite number of defeaters: $B_1 : (\rightarrow b) \Rightarrow \neg c$, $B_2 : (((\rightarrow b) \Rightarrow \neg c) \rightarrow \neg d) \rightarrow \neg c$, $B_3 : (((((\rightarrow b) \Rightarrow \neg c) \rightarrow \neg d) \rightarrow \neg c) \rightarrow \neg d) \rightarrow \neg c$, etc. Similar observations can be made for the argumentation interpretation of ABA [6] and the argumentation interpretation of logic programming [12].

The above example illustrates that even a finite knowledge base can lead to an argumentation framework that is not only infinite, but that is even infinitary in the sense of [5].[1] This can be problematic, as some of the fundamental results of abstract argumentation have only been proven for finitary argumentation frameworks, for example the existence of semi-stable and stage extensions [11] and the correctness of the inductive procedure for computing the grounded extension [5].

One particular way in which this problem has been dealt with is by adding an extra constraint to the argument construction process, so that each rule can be used at most once within each "branch" of an argument.[2] Hence, in the above example B_3 would no longer form a well-formed argument. The disadvantage of this approach, however, is that argument construction loses some of its modular aspects. For instance, if one has an argument A with conclusion a, an argument B with conclusion b, and a rule $a, b \rightarrow c$ then one can no longer be sure that $A, B \rightarrow c$ is a well-formed argument. This can cause difficulties for work like [3] where part of the technical results relies on modular argument construction.

[1] Recall that an argumentation framework is finite when it has a finite number of arguments. It is finitary when each argument has a finite number of defeaters.

[2] Recall that the recursive definition of an argument in ASPIC, ASPIC$^+$ and ASPIC$^-$ essentially defines a tree of rules, similar to what is done in the argumentation interpretation of ABA.

Ideally, one would like to have a solution that does not in any way restrict the construction of arguments. However, this requires the broadening of some of the fundamental results of abstract argumentation theory to particular classes of infinitary argumentation frameworks. In the current paper, we introduce such a broadening. In particular, we show that for argumentation frameworks generated by the ASPIC$^-$ formalism (which, as we have seen, can be infinitary) the iterative procedure for computing the grounded extension is correct as far as the conclusions are concerned. Hence, when it comes to determining the outcome of ASPIC$^-$ under grounded semantics (in terms of the conclusions yielded) one is free to apply the inductive definition of grounded semantics, even though there may be differences on the argument level.

The remainder of this paper is structured as follows. First, in Section 2, we introduce the formal preliminaries of abstract argumentation. Then, in Section 3, we study the effects of omitting what we call "superseded" arguments. In Section 4 we then use our results to show that under ASPIC$^-$ the inductive definition of grounded semantics yields the same conclusions as the grounded extension itself, even though the underlying argumentation framework may not be finitary. We then round off with a discussion of the obtained results in Section 5.

2 Formal Preliminaries

In the current section, we briefly restate some of the key concepts of abstract argumentation theory.

Definition 1 ([5])**.** *An* argumentation framework *is a pair* (Ar, def) *where* Ar *is set of entities, called arguments, whose internal structure can be left unspecified, and* def *a binary relation on* Ar*. We say that* A defeats B *iff* $(A, B) \in def$*. We say that the argumentation framework is* finite *iff* Ar *is finite. We say that the argumentation framework is* finitary *iff for every* $A \in Ar$*,* $\{B \in Ar \mid (B, A) \in def\}$ *is finite.*

Definition 2. *Let* $AF = (Ar, def)$ *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*. We define* A^+ *as* $\{B \in Ar \mid A \textit{ defeats } B\}$*,* A^- *as* $\{B \in Ar \mid B \textit{ defeats } A\}$*,* $\mathcal{A}rgs^+$ *as* $\bigcup\{A^+ \mid A \in \mathcal{A}rgs\}$*, and* $\mathcal{A}rgs^-$ *as* $\bigcup\{A^- \mid A \in \mathcal{A}rgs\}$*.* $\mathcal{A}rgs$ *is said to be* conflict-free *iff* $\mathcal{A}rgs \cap \mathcal{A}rgs^+ = \emptyset$*.* $\mathcal{A}rgs$ *is said to* defend A *iff* $A^- \subseteq \mathcal{A}rgs^+$*. The characteristic function* $F_{AF} : 2^{Ar} \rightarrow 2^{Ar}$ *is defined as* $F_{AF}(\mathcal{A}rgs) = \{A \in Ar \mid \mathcal{A}rgs \textit{ defends } A\}$*.*

Definition 3. *Let* $AF = (Ar, def)$ *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is said to be:*

- *an admissible set iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs \subseteq F_{AF}(\mathcal{A}rgs)$
- *a complete extension iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs = F_{AF}(\mathcal{A}rgs)$
- *a grounded extension iff* $\mathcal{A}rgs$ *is the smallest (w.r.t.* $\subseteq$*) complete extension*

3 Omitting Superseded Arguments

The idea of superseded arguments is to identify those arguments that can be omitted from the argumentation framework without significantly affecting its outcome, as long as for each argument one omits, one keeps an argument that supersedes it.

Definition 4 (argument superseding)**.** *An argument* A *is* superseded *by an argument* B *iff* $A^+ \subseteq B^+$ *and* $A^- \supseteq B^-$*.*

Please notice that the supersedes relationship among arguments is not a partial order because it does not satisfy anti-symmetry. Hence, it does *not* satisfy Postulate 3.1 of [7], so we cannot apply their theory. We now proceed to define the supersedes relationship between argumentation frameworks.

Definition 5 (AF superseding)**.** *Let* $AF = (Ar, def)$ *be an argumentation framework, and let* $Ar' \subseteq Ar$ *be such that for each* $A \in Ar$ *there exists an* $A' \in Ar'$ *that supersedes it. Let* AF' *be* (Ar', def') *with* $def' = def \cap (Ar' \times Ar')$*. We say that* AF' *supersedes* AF*.*

Notice that the supersedes relationship among argumentation frameworks *does* constitute a partial order.

Proposition 1. *Let* $AF = (Ar, def)$ *and* $AF' = (Ar', def')$ *be argumentation frameworks such that* AF' *supersedes* AF*, and let* $\mathcal{A}rgs' \subseteq Ar'$*. It holds that* $F_{AF'}(\mathcal{A}rgs') \subseteq F_{AF}(\mathcal{A}rgs')$*.*

Proof. Let $A \in F_{AF'}(\mathcal{A}rgs')$. So each $B' \in Ar'$ that defeats A is defeated by some $C \in \mathcal{A}rgs'$. Let $B \in Ar$ be an argument that defeats A. Let $B' \in Ar'$ be an argument that supersedes B. Then, from the fact that $B^+ \subseteq B'^+$ it follows that B' also defeats A. Hence, B' is defeated by some $C \in \mathcal{A}rgs'$. Since $B^- \supseteq B'^-$ it follows that this C also defeats B. Hence, A is defended by $\mathcal{A}rgs'$ under AF. That is, $A \in F_{AF}(\mathcal{A}rgs')$. □

Proposition 2. *Let $AF = (Ar, def)$ and $AF' = (Ar', def')$ be argumentation frameworks such that AF' supersedes AF, and let $\mathcal{A}rgs' \subseteq Ar'$. It holds that $F_{AF}(\mathcal{A}rgs') \cap Ar' = F_{AF'}(\mathcal{A}rgs')$.*

Proof.

$F_{AF'}(\mathcal{A}rgs') \subseteq F_{AF}(\mathcal{A}rgs') \cap Ar'$

Proposition 1 states that $F_{AF'}(\mathcal{A}rgs') \subseteq F_{AF}(\mathcal{A}rgs')$, so from $F_{AF'}(\mathcal{A}rgs') \subseteq Ar'$ it then follows that $F_{AF'}(\mathcal{A}rgs') \subseteq F_{AF}(\mathcal{A}rgs') \cap Ar'$.

$F_{AF}(\mathcal{A}rgs') \cap Ar' \subseteq F_{AF'}(\mathcal{A}rgs')$

Let $A \in F_{AF}(\mathcal{A}rgs') \cap Ar'$. The fact that $A \in F_{AF}(\mathcal{A}rgs')$ means that each $B \in Ar$ that defeats A is defeated by some $C \in \mathcal{A}rgs'$. The fact that $Ar' \subseteq Ar$ implies that also each $B' \in Ar'$ that defeats A is defeated by some $C \in \mathcal{A}rgs'$. Hence, $\mathcal{A}rgs'$ defends $A \in Ar'$ under AF'. That is, $A \in F_{AF'}(\mathcal{A}rgs')$. □

The complete extensions of a superseded argumentation framework can be converted to the extensions of the superseding argumentation framework, and vice versa.

Theorem 1. *Let $AF = (Ar, def)$ and $AF' = (Ar', def')$ be argumentation frameworks such that AF' supersedes AF.*

1. *if CE is a complete extension of AF, then $CE \cap Ar'$ is a complete extension of AF'*
2. *if CE' is a complete extension of AF', then $F_{AF}(CE')$ is a complete extension of AF*
3. *if CE is a complete extension of AF, then $F_{AF}(CE \cap Ar') = CE$*
4. *if CE' is a complete extension of AF', then $F_{AF}(CE') \cap Ar' = CE'$*

Proof.

1. Let CE be a complete extension of AF and let CE' be $CE \cap Ar'$. We need to prove that CE' is a conflict-free fixed-point of $F_{AF'}$. Conflict-freeness follows from the fact that CE is conflict-free and $CE' \subseteq CE$. To prove that CE' is a fixed-point of $F_{AF'}$ we need to show two things:

   $CE' \subseteq F_{AF'}(CE')$

   Let $A \in CE'$. Then the facts that $A \in CE$ and CE is a complete extension imply that each $B \in Ar$ that defeats A is defeated by some $C \in CE$. From $Ar' \subseteq Ar$ it then follows that each $B' \in Ar'$ that defeats A is defeated by some $C \in CE$. The fact that AF' supersedes AF implies that there is a $C' \in Ar'$ with $C^+ \subseteq C'^+$, so C' defeats B'. Since this C' is defended by CE (since the facts that CE is a complete extension and $C \in CE$ imply that C is defended by CE, so the fact that $C^- \supseteq C'^-$ implies that C' is also defended by CE) it follows that $C' \in CE$, so $C' \in CE \cap Ar'$. That is, $C' \in CE'$, so $A \in F_{AF'}(CE')$.

   $F_{AF'}(CE') \subseteq CE'$

   Let $A \in F_{AF'}(CE')$. From $CE' \subseteq CE$ it follows that $F_{AF}(CE') \subseteq F_{AF}(CE)$ (since F_{AF} is a monotonic function). As $F_{AF'}(CE') \subseteq F_{AF}(CE')$ (Proposition 1) it follows (by transitivity of $\subseteq$) that $F_{AF'}(CE') \subseteq F_{AF}(CE)$. As CE is a complete extension of AF, it holds that $F_{AF}(CE) = CE$, so we obtain $F_{AF'}(CE') \subseteq CE$. Since, by definition, $F_{AF'}(CE') \subseteq Ar'$ it then follows that $F_{AF'}(CE') \subseteq CE \cap Ar'$. That is, $F_{AF'}(CE') \subseteq CE'$.

2. Let CE' be a complete extension of AF'. We need to prove that $F_{AF}(CE')$ is a conflict-free fixed-point of F_{AF}. We first show that $F_{AF}(CE')$ is conflict-free. Suppose, towards a contradiction, that $F_{AF}(CE')$ is not conflict-free. That is, there exist $A, B \in F_{AF}(CE')$ such that A defeats B. Then CE' contains an argument C that defeats A (this is because CE' defends B).

However, the fact that CE' also defends A implies that CE' also contains an argument D that defeats C. But then CE' is not conflict-free, so CE' is not a complete extension of AF'. Contradiction.

We proceed to show that $F_{AF}(CE')$ is a fixed-point of F_{AF}. That is, $F_{AF}(CE') = F_{AF}(F_{AF}(CE'))$.

$F_{AF}(CE') \subseteq F_{AF}(F_{AF}(CE'))$

From the fact that CE' is a complete extension of AF' it follows that $CE' \subseteq F_{AF'}(CE)$. Since $F_{AF'}(CE') \subseteq F_{AF}(CE')$ (Proposition 1) it then follows (transitivity $\subseteq$) that $CE' \subseteq F_{AF}(CE')$. From the fact that F_{AF} is a monotonic function it then follows that $F_{AF}(CE') \subseteq F_{AF}(F_{AF}(CE'))$.

$F_{AF}(F_{AF}(CE')) \subseteq F_{AF}(CE')$

Let $A \in F_{AF}(F_{AF}(CE'))$. Then each $B \in Ar$ that defeats A is defeated by some $C \in F_{AF}(CE')$. Let $C' \in Ar'$ be an argument that supersedes C. From the facts that C is defended by CE' and $C^- \supseteq C'^-$ it follows that C' is also defended by CE'. That is, $C' \in F_{AF}(CE')$. Since $C' \in Ar'$ it then follows that $C' \in F_{AF}(CE') \cap Ar'$. So A is defended by $F_{AF}(CE') \cap Ar'$. That is, $A \in F_{AF}(F_{AF}(CE') \cap Ar')$. Proposition 2 states that $F_{AF}(CE') \cap Ar' = F_{AF'}(CE')$ so we obtain that $A \in F_{AF}(F_{AF'}(CE'))$. But since CE' is a complete extension of AF' it holds that $F_{AF'}(CE') = CE'$, so $A \in F_{AF}(CE')$.

3. Let CE be a complete extension of AF. We need to prove that $CE = F_{AF}(CE \cap Ar')$

$CE \subseteq F_{AF}(CE \cap Ar')$

Let $A \in CE$. Then, from the fact that CE is a complete extension of AF, it follows that for each $B \in Ar$ that defeats A, there is a $C \in CE$ that defeats B. Let $C' \in Ar'$ be an argument that supersedes C. From the fact that $C^+ \subseteq C'^+$ it follows that C' defeats B. The fact that $C \in CE$ means that C is defended by CE (as CE is a complete extension) so from the fact that $C^- \supseteq C'^-$ it follows that C' is also defended by CE. Hence, $C' \in CE$, so $C' \in CE \cap Ar'$. So A is defended by $CE \cap Ar'$. That is, $A \in F_{AF}(CE \cap Ar')$.

$F_{AF}(CE \cap Ar') \subseteq CE$

It trivially holds that $CE \cap Ar' \subseteq CE$. Since F_{AF} is a monotonic function, it then follows that $F_{AF}(CE \cap Ar') \subseteq F_{AF}(CE)$. Since CE is a complete extension of AF, it holds that $F_{AF}(CE) = CE$. Hence, $F_{AF}(CE \cap Ar') \subseteq CE$.

4. Let CE' be a complete extension of AF'. We need to prove that $F_{AF}(CE') \cap Ar' = CE'$.

$CE' \subseteq F_{AF}(CE') \cap Ar'$

Let $A \in CE'$. Then, by definition, $A \in Ar'$. The fact that CE' is a complete extension of AF' means that A is defended by CE' (under AF'). So each $B' \in Ar'$ that defeats A is defeated by some $C \in CE'$. We now show that each $B \in Ar$ that defeats A is defeated by some $C \in CE'$. Let $B \in Ar$ be an argument that defeats A. Let $B' \in Ar'$ be an argument that supersedes B. From the fact that $B^+ \subseteq B'^+$ it follows that B' defeats A. So there exists a $C \in CE'$ that defeats B'. Since $B^- \supseteq B'^-$ it follows that C defeats B. So A is defended (under AF) by CE'. That is, $A \in F_{AF}(CE')$. This, together with the earlier observed fact that $A \in Ar'$ implies that $A \in F_{AF}(CE') \cap Ar'$.

$F_{AF}(CE') \cap Ar' \subseteq CE'$

Let $A \in F_{AF}(CE') \cap Ar'$. Then, the fact that $A \in F_{AF}(CE')$ implies that each $B \in Ar$ that defeats A is defeated by some $C \in CE'$. From the fact that $Ar' \subseteq Ar$ it follows that also each $B' \in Ar'$ that defeats A is defeated by some $C \in CE'$, so CE' defends A under AF'. That is, $A \in F_{AF'}(CE')$. But since CE' is a complete extension of AF', it holds that $F_{AF'}(CE') = CE'$. Hence, $A \in CE'$. □

The grounded extension of a superseded argumentation framework can be converted to the grounded extension of the superseding argumentation framework, and vice versa.

Theorem 2. *Let $AF = (Ar, def)$ and $AF' = (Ar', def')$ be argumentation frameworks such that AF' supersedes AF.*

1. If GE is the grounded extension of AF, then $GE \cap Ar'$ is the grounded extension of AF'.

2. *If GE' is the grounded extension of AF', then $F_{AF}(GE')$ is the grounded extension of AF.*

Proof.

1. Let GE be the grounded extension of AF and let GE' be $GE \cap Ar'$. From the fact that GE is also a complete extension of AF, it follows (Theorem 1, point 1) that GE' is a complete extension of AF'. In order to prove that GE' is also the grounded extension of AF', we show that for each complete extension CE' of AF', it holds that $GE' \subseteq CE'$. Let CE' be a complete extension of AF'. Then from Theorem 1 (point 2) it follows that $F_{AF}(CE')$ is a complete extension of AF, so $GE \subseteq F_{AF}(CE')$, which implies that $GE \cap Ar' \subseteq F_{AF}(CE') \cap Ar'$. Theorem 1 (point 4) states that $F_{AF}(CE') \cap Ar' = CE'$, so we obtain that $GE \cap Ar' \subseteq CE'$, so (as $GE' = GE \cap Ar'$) $GE' \subseteq CE'$.

2. Let GE' be the grounded extension of AF', and let GE be $F_{AF}(GE')$. From the fact that GE' is also a complete extension of AF', it follows that GE is a complete extension of AF (Theorem 1, point 2). In order to prove that GE is also the grounded extension of AF, we show that for each complete extension CE of AF, it holds that $GE \subseteq CE$. Let CE be a complete extension of AF. Then (Theorem 1, point 1) $CE \cap Ar'$ is a complete extension of AF'. From the fact that GE' is the grounded extension of AF', it then follows that $GE' \subseteq CE \cap Ar'$. Since F_{AF} is a monotonic function, we obtain $F_{AF}(GE') \subseteq F_{AF}(CE \cap Ar')$. Since $F_{AF}(GE') = GE$ (by definition) and $F_{AF}(CE \cap Ar') = CE$ (Theorem 1, point 3) we obtain that $GE \subseteq CE$. □

4 Omitting C-Superseded Arguments

So far, we have proved equivalence purely on the semantic level (for complete and grounded semantics). The next step is to examine things at the level of proof procedures. Our aim is to examine to what extent one can still apply the iterative procedure for determining grounded semantics in the presence of a possibly infinite argumentation framework that is superseded by a finite argumentation framework. We start with a lemma.

Lemma 1. *Let $AF = (Ar, def)$ and $AF' = (Ar', def')$ be argumentation frameworks such that AF' supersedes AF. For every $i \in \{0, 1, 2, \ldots\}$ it holds that $F^i_{AF'}(\emptyset) \subseteq F^i_{AF}(\emptyset)$.*

Proof. By induction over i:

basis $i = 0$. In that case $F^i_{AF'}(\emptyset) \subseteq F^i_{AF}(\emptyset)$, as $F^0_{AF'}(\emptyset) = \emptyset = F^0_{AF}(\emptyset)$.

step Suppose that $F^i_{AF'}(\emptyset) \subseteq F^i_{AF}(\emptyset)$ for some $i \in \{0, 1, 2, \ldots\}$. As F_{AF} is a monotonic function, it follows that $F_{AF}(F^i_{AF'}(\emptyset)) \subseteq F_{AF}(F^i_{AF}(\emptyset))$. As $F_{AF'}(F^i_{AF'}(\emptyset)) \subseteq F_{AF}(F^i_{AF'}(\emptyset))$ (Proposition 1) we obtain that $F_{AF'}(F^i_{AF'}(\emptyset)) \subseteq F_{AF}(F^i_{AF}(\emptyset))$. That is, $F^{i+1}_{AF'}(\emptyset) \subseteq F^{i+1}_{AF}(\emptyset)$. □

In the context of this work, we are interested in equivalence at the level of conclusions rather than equivalence purely at the level of arguments. For this, we need the following two definitions. Note that if A is an argument we write $\texttt{Conc}(A)$ for its conclusion, and if $\mathcal{A}rgs$ is a set of arguments we write $\texttt{Concs}(\mathcal{A}rgs)$ for $\{\texttt{Conc}(A) \mid A \in \mathcal{A}rgs\}$ as is done in ASPIC$^-$ [3].

Definition 6. *An argument A is* c-superseded *by an argument B iff A is superseded by B and* $\texttt{Conc}(A) = \texttt{Conc}(B)$.

Definition 7. *Let $AF = (Ar, def)$ be an argumentation framework, and let $Ar' \subseteq Ar$ be such that for each $A \in Ar$ there exists an $A' \in Ar'$ that c-supersedes it. Let AF' be (Ar', def') with $def' = def \cap (Ar' \times Ar')$. We say that AF' c-supersedes AF.*

Trivially, it holds that if A is c-superseded by B then A is superseded by B (but not vice versa) and that if AF is c-superseded by AF' then AF is superseded by AF' (but not vice versa). We now come to one of the main results of this paper.

Theorem 3. *Let $AF = (Ar, def)$ be an argumentation framework for which there exists a finitary argumentation framework $AF' = (Ar', def')$ that c-supersedes it. Let GE be the grounded extension of AF. It holds that* $\texttt{Concs}(GE) = \texttt{Concs}(\cup_{i=0}^{\infty} F^i_{AF}(\emptyset))$.

Proof. Let GE' be the grounded extension of AF'. We need to show two things:

$\texttt{Concs}(GE) \subseteq \texttt{Concs}(\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset))$
Let $a \in \texttt{Concs}(GE)$. Then there is an $A \in GE$ with $\texttt{Conc}(A) = a$. Let $A' \in Ar'$ be an argument that c-supersedes A. From the fact that A is defended by GE (as GE is a complete extension) and $A^{-} \supseteq A'^{-}$ it follows that A' is defended by GE, so $A' \in GE$. That is, $A' \in GE \cap Ar'$, so (Theorem 2, point 1) $A' \in GE'$. Since AF' is finitary, it holds that $GE' = \cup_{i=0}^{\infty} F_{AF'}(\emptyset)$, so $A' \in \cup_{i=0}^{\infty} F_{AF'}^{i}(\emptyset)$. From Lemma 1 it follows that $\cup_{i=0}^{\infty} F_{AF'}^{i}(\emptyset) \subseteq \cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset)$ so $A' \in \cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset)$ so $\texttt{Conc}(A') \in \texttt{Concs}(\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset))$. Since $\texttt{Conc}(A') = \texttt{Conc}(A)$ (as A' c-supersedes A) it then follows that $a \in \texttt{Concs}(\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset))$.

$\texttt{Concs}(\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset)) \subseteq \texttt{Concs}(GE)$
As proven by Dung [5], it holds for any argumentation framework AF (finitary or infinitary) that $\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset) \subseteq GE$. From the fact that Concs is a monotonic function, it then directly follows that $\texttt{Concs}(\cup_{i=o}^{\infty} F_{AF}^{i}(\emptyset)) \subseteq \texttt{Concs}(GE)$. □

To illustrate the applicability of our theory, we show that any argumentation framework obtained by the ASPIC^{-} formalism (assuming a finite defeasible theory [3]) is c-superseded by a finitary argumentation framework. We refer to the finitary argumentation framework as a *finited* version of the original framework. Unfortunately, space restrictions prevent us from including all relevant definitions of ASPIC^{-}. For these, we refer the reader to [3] instead.

Proposition 3. *Let $\mathcal{A}rgs$ be an infinite set of arguments of a particular* ASPIC^{-} *theory. There exists a rule r that has no upper bound in the number of times it can occur in the same branch of an argument in $\mathcal{A}rgs$.*

Proof. Suppose, towards a contradiction, that there exists an upper bound, say n. This means that each argument in the infinite set $\mathcal{A}rgs$ has each rule in the defeasible theory occurring at most n times in the same branch. This implies that the depth of each argument in $\mathcal{A}rgs$ is at most $n \cdot |\mathcal{R}|$. Let m be the size of the largest antecedent of the rules in $\mathcal{R}$ (that is, m is the biggest "fan-out" factor one can get when constructing an argument). Then the maximal number of rule-occurrences in each argument is $m^{n \cdot |\mathcal{R}|}$. Even if one takes into account all possible permutations of the rules in an argument, the result is still finite. But this means it is impossible to obtain an infinite number of arguments in $\mathcal{A}rgs$. □

Theorem 4. *Let $AF = (Ar, def)$ be generated by a finite* ASPIC^{-} *theory. There exists a finitary argumentation framework $AF' = (Ar', def')$ that c-supersedes it.*

Proof. We distinguish two cases: weakest link and last link.

weakest link Assume that AF has been generated using Ewl or Dwl. We first observe that for each argument A with a same-branch repeating rule, there exists an argument A^{*} without any same-branch rule, such that A^{*} c-supersedes A. The idea is to construct this A^{*} by iteratively applying subargument substitution. Let A be an argument that has a same-branch repeating rule. That is, $\exists A_1, A_2 : A_1 \in \texttt{Sub}(A) \wedge A_2 \in \texttt{Sub}(A_1) \wedge \texttt{TopRule}(A_1) = \texttt{TopRule}(A_2)$. Substitute A_2 for A_1 in A. Keep on doing substitutions like this until there are no same-branch repeating rules anymore. Call the resulting argument A^{*}. At each substitution step, the argument after the step (say A'') c-supersedes the argument before the step (say A') for the following reasons.

1. $\texttt{Conc}(A'') = \texttt{Conc}(A')$
2. $A'^{+} \subseteq A''^{+}$. Suppose A' defeats B. We distinguish two cases.
   - A' undercuts B. Then A'' also undercuts B (since $\texttt{Conc}(A'') = \texttt{Conc}(A')$)
   - A' rebuts B and $\texttt{DefRules}(A') \not\prec_{\{\texttt{Ewl},\texttt{Dwl}\}} \texttt{DefRules}(B')$ (where B' is the subargument of B whose top-conclusion is defeated). Since $\texttt{Conc}(A'') = \texttt{Conc}(A')$ it follows that A'' rebuts B. Since $\texttt{DefRules}(A'') \subseteq \texttt{DefRules}(A')$ it follows that $\texttt{DefRules}(A'') \not\prec_{\{\texttt{Ewl},\texttt{Dwl}\}} \texttt{DefRules}(B')$.
3. $A'^{-} \supseteq A''^{-}$. Suppose A'' is defeated by B. We distinguish two cases.
   - B undercuts A''. Since $\texttt{DefRules}(A'') \subseteq \texttt{DefRules}(A')$ it follows that B also undercuts A'.

- B rebuts A'' and $\texttt{DefRules}(B) \not\prec_{\{\texttt{Ewl},\texttt{Dwl}\}} \texttt{DefRules}(A''')$ (where A''' is the subargument of A'' whose top-conclusion is defeated). Since $\texttt{DefRules}(A_2) \subseteq \texttt{DefRules}(A_1)$ it follows that $\texttt{DefRules}(A''') \subseteq \texttt{DefRules}(A'''')$ (where A'''' is the subargument of A' whose top-conclusion is defeated). Hence, $\texttt{DefRules}(B) \not\prec_{\{\texttt{Ewl},\texttt{Dwl}\}} \texttt{DefRules}(A'''')$.

Let $AF' = (Ar', def')$ be the argumentation framework where Ar' consist of each A^* resulting from an $A \in Ar$ and def' be $def \cap (Ar' \times Ar')$. From the above, it follows that AF' c-supersedes AF. We now prove that AF' is finite. Suppose, towards a contradiction, that AF' is infinite. Proposition 3 tells us that there is a rule that has no upper bound in the number of times it can occur in the same branch. However, each argument $A^* \in Ar'$ has each rule occurring at most once in each branch. So there actually *is* an upper bound (it's 1). Contradition.

last link Assume that AF has been generated using $\texttt{Ell}$ or $\texttt{Dll}$. We first observe that with last link, we cannot always carry out the same kind of substitutions as with weakest link and still expect the resulting argument to c-supersede the original argument. The reason is that we cannot be sure that $\texttt{LastDefRules}(A_2) \subseteq \texttt{LastDefRules}(A_1)$. It appears that an alternative strategy is needed. Instead of performing a substitution whenever there are two occurrences of the same rule in the same branch, we only perform substitution if, in addition, these two rule-occurrences also have the same $\texttt{LastDefRules}$. That is, let $A \in Ar$ be such that $\exists A_1, A_2 : A_1 \in \texttt{Sub}(A) \wedge A_2 \in \texttt{Sub}(A_1) \wedge \texttt{TopRule}(A_1) = \texttt{TopRule}(A_2) \wedge \texttt{LastDefRules}(A_1) = \texttt{LastDefRules}(A_2)$ then substitute A_2 for A_1 in A. Keep doing substitution steps like these until there are no same-branch repeated rules with the same $\texttt{LastDefRules}$. Call the resulting argument A^* and let $AF' = (Ar', def')$ be the associated argumentation framework. Following similar reasoning as for the weakest link case above, it follows that AF' c-supersedes AF.
We still have to prove that AF' is finite. This requires some additional effort, because now a rule can occur more than once in the same branch (as long as they have different $\texttt{LastDefRules}$). Suppose towards a contradiction that AF' is infinite. Then Proposition 3 tells that there is a rule that has no upper bound in the number of times it can occur in the same branch. But as the number of rules in the defeasible theory is finite, it follows that at some point $\texttt{LastDefRules}$ will start to become the same (this is because there is only a finite number of subsets of $\mathcal{R}_d$ that can serve as $\texttt{LastDefRules}$). But this is impossible, because then this multiple rule-occurrence should have been substituted away during the substitution process. Contradiction. □

Theorem 5. *Let $AF = (Ar, def)$ be generated by a finited* ASPIC$^-$ *theory and let GE be the grounded extension of AF. It holds that* $\texttt{Concs}(GE) = \texttt{Concs}(\cup_{i=0}^{\infty} F_{AF}^{i}(\emptyset))$.

Proof. This follows directly from Theorem 3 and Theorem 4. □

What the above theorem shows is that if we want to compute the conclusions yielded by ASPIC$^-$ under grounded semantics, then we are free to do so using the iterative procedure, even though the argumentation framework generated by the ASPIC$^-$ theory might not be finitary.

5 Discussion and Conclusions

In this paper we formalise the concept of one argument superseding another. Since a finite set of rules can generate an infinite set of arguments, the results presented in this paper are of critical importance — they allow us to reduce such infinite frameworks into finite ones, and enable us to compute the grounded semantics over such frameworks in a standard way. While our results focused on the ASPIC$^-$ framework of [3], they are also directly applicable to other ASPIC style frameworks [1, 10, 9], as well as the argumentation interpretation of ABA [6] and argument-based Logic Programming [12, 4].

With regards to related work, [7] introduces a redundancy relation between arguments. The aim of [7] was to identify postulates necessary for generic argument systems to be useful. Redundancy was therefore used to *trim* large argument systems, obtained from formalisms such as ABA, into smaller systems which comply with their postulates. [7] showed that such *trimmed* frameworks (i.e. those without redundant arguments) yield the same extensions as untrimmed frameworks. Unlike the present work, [7] did not consider the validity of the inductive definition for the grounded semantics in the presence of infinitary argumentation systems. Furthermore, our results are applicable to instantiated frameworks which make use of unrestricted rebut (such as ASPIC$^-$), and which are therefore arguably more natural to use for the reasoning about argument in real domains (see the discussion in [3]).

Another line of research where our results are relevant is in embedding classical logic into rule based formalisms. Approaches such as ASPIC-lite [13] and [8] seek to embed propositional logic into an ASPIC style system. Classical entailment can lead to an infinite number of attackers, for reasons other than reoccurring rules or propositions. For example, consider the defeasible rules $\Rightarrow a$, $\Rightarrow b$ and $\Rightarrow \neg(a \wedge b)$. When using strict rules as classical inference, the arguments $A_0 :\Rightarrow \neg(a \wedge b)$ has an infinite number of attackers, such as $B_1 : (\Rightarrow a), (\Rightarrow b) \rightarrow a \wedge b$; $B_2 : ((\Rightarrow a), (\Rightarrow b) \rightarrow a \wedge a \wedge b) \rightarrow a \wedge b$; $B_3 : ((\Rightarrow a), (\Rightarrow b) \rightarrow a \wedge a \wedge a \wedge b) \rightarrow a \wedge b$, etc. Our work can potentially be applied to show that the inductive definition of grounded semantics is still applicable in such situations.

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References

[1] M.W.A. Caminada and L. Amgoud. On the evaluation of argumentation formalisms. *Artificial Intelligence*, 171(5-6):286–310, 2007.

[2] M.W.A. Caminada, R. Kutlak, N. Oren, and W.W. Vasconcelos. Scrutable plan enactment via argumentation and natural language generation. In *Proceedings AAMAS'14*, pages 1625–1626, 2014.

[3] M.W.A. Caminada, S. Modgil, and N. Oren. Preferences and unrestricted rebut. In Simon Parsons, Nir Oren, Chris Reed, and Frederico Cerutti, editors, *Computational Models of Argument; Proceedings of COMMA 2014*, pages 209–220. IOS Press, 2014.

[4] M.W.A. Caminada, S. Sá, and J. Alcântara. On the equivalence between logic programming semantics and argumentation semantics. In *Proceedings ECSQARU 2013*, pages 97–108, 2013.

[5] P.M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77:321–357, 1995.

[6] P.M. Dung, R.A. Kowalski, and F. Toni. Assumption-based argumentation. In Guillermo Simari and Iyad Rahwan, editors, *Argumentation in Artificial Intelligence*, pages 199–218. Springer US, 2009.

[7] P.M. Dung, F. Toni, and P. Mancarella. Some design guidelines for practical argumentation systems. In Pietro Baroni, Frederico Cerutti, Massimiliano Giacomin, and Guillermo R. Simari, editors, *Computational Models of Argument; Proceedings of COMMA 2010*, pages 183–194, 2010.

[8] D. Grooters and H. Prakken. Combining paraconsistent logic with argumentation. In Simon Parsons, Nir Oren, Chris Reed, and Frederico Cerutti, editors, *Computational Models of Argument; Proceedings of COMMA 2014*, pages 301–312. IOS Press, 2014.

[9] S. Modgil and H. Prakken. A general account of argumentation with preferences. *Artificial Intellligence*, 195:361–397, 2013.

[10] S. Modgil and H. Prakken. The ASPIC+ framework for structured argumentation: a tutorial. *Argument & Computation*, 5:31–62, 2014.

[11] E. Weydert. Semi-stable extensions for infinite frameworks. In Patrick de Causmaecker, Joris Maervoet, Tommy Messelis, Katja Verbeeck, and Tim Vermeulen, editors, *Proceedings of the 23rd Benelux Conference on Artificial Intelligence (BNAIC 2011)*, pages 336–343, 2011.

[12] Y. Wu, M.W.A. Caminada, and D.M. Gabbay. Complete extensions in argumentation coincide with 3-valued stable models in logic programming. *Studia Logica*, 93(1-2):383–403, 2009. Special issue: new ideas in argumentation theory.

[13] Y. Wu and M. Podlaszewski. Implementing crash-resistance and non-interference in logic-based argumentation. *Journal of Logic and Computation*, 2014. in print.