MULTIFRACTAL SCENARIOS FOR PRODUCTS OF GEOMETRIC LÉVY-BASED STATIONARY MODELS

D.E. DENISOV AND N.N. LEONENKO

ABSTRACT. We investigate the properties of multifractal products of geometric Gaussian processes with possible long-range dependence and geometric Ornstein-Uhlenbeck processes driven by Lévy motion and their finite and infinite superpositions. We construct the multifractal, such as log-gamma, log-tempered stable or log-normal tempered stable scenarios. For that we use the general conditions for the $\mathcal{L}_q$ convergence of cumulative processes to multifractal limiting processes established in Denisov, Leonenko (2015).

1. INTRODUCTION

Multifractal models have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic, etc. (see, for example, Kolmogorov (1941, 1962), Kahane (1985, 1987), Novikov (1994), Frisch (1995), Mandelbrot (1997), Falconer (1997), Schertzer et al. (1997), Harte (2001), Riedi (2003)). There are many ways to construct random multifractal models ranging from simple binomial cascades to measures generated by branching processes and the compound Poisson process (Kahane (1985, 1987), Falconer (1997), Schmitt (2003), Harte (2001), Barral, Mandelbrot (2002), Barral, Mandelbrot (2010), Bacry, Muzy (2003), Riedi (2003), Moerters, Shieh (2004), Shieh, Taylor (2002), Schmitt (2003), Schertzer et al. (1997), Barral et al. (2009), Ludena (2008),, Jaffard et al. (2010), Schmitt, Marsan (2001)). Jaffard (1999) showed that Lévy processes (except Brownian motion and Poisson processes) are multifractal; but since the increments of a Lévy process are independent, this class excludes the effects of dependence structures. Moreover, Lévy processes have a linear singularity spectrum while real data often exhibit a strictly concave spectrum.

Anh et al. (2008a,b, 2009a,b, 2010a) considered multifractal products of stochastic processes as defined in Kahane (1985, 1987) and Mannersalo et al. (2002). Especially Anh et al. (2008a) constructed multifractal processes based on products of geometric Ornstein-Uhlenbeck (OU) processes driven by Lévy motion with inverse Gaussian or normal inverse Gaussian distribution. They also described the behaviour of the q-th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. In these papers a number of scenarios were obtained for $q \in Q \cap [1, 2]$, where Q is a set of parameters of marginal distribution of an OU processes driven by Lévy motion. The simulations show that for q outside this range, the scenarios still hold (see Anh et al. (2010b)). In Denisov, Leonenko (2015) we gave a rigorous proof that the above scenarios indeed hold outside of this range. We also

2000 *Mathematics Subject Classification.* Primary 60G57; 60G10; 60G17.

Key words and phrases. Multifractal products, stationary processes, short-range dependence, long-range dependence, multifractal scenarios, Rényi function, geometric Gaussian process, geometric Ornstein-Uhlenbeck processes, Lévy processes, superpositions, scaling of moments, log-normal scenario, log-gamma scenario, log normal tempered stable scenario, log-variance gamma scenario.

constructed new scenarios which generalize those corresponding to the inverse Gaussian and normal inverse Gaussian distributions obtained in Anh, Leonenko (2008), Anh et al. (2008a). In the present paper we continue the research of Denisov, Leonenko (2015) and construct new multifractal scenarios.

Section 2 recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987), Mannersalo et al. (2002) and Denisov, Leonenko (2015). In particular Subsection 2.1 contains the general $\mathcal{L}_q$ bounds for cumulative process of multifractal products of stationary processes proved in Denisov, Leonenko (2015). The novelty of the paper is an extension of the results of Denisov, Leonenko (2015) to the class of supOU processes, which possesses long-range dependence by providing new scenarios. The scenarios of this paper are largely based on the results of Subsection 2.1. Section 3 establishes the general results on the scaling moments of multifractal products of geometric OU processes in terms of the marginal distributions of OU processes and their Lévy measures. Similar results for the finite and infinite superpositions of OU processes are proved in the Section 4. The number of multifractal scenarios with exact forms of the scaling function are given in the Sections 5-10.

2. Multifractal products of stochastic processes

This Section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987),Mannersalo et al. (2010) and Denisov, Leonenko (2015). We provide an interpretation of their conditions based on the moment generating functions, which is useful for our exposition. Throughout the text the notation C, c is used for the generic constants which do not necessarily coincide.

Following Denisov, Leonenko (2015) we introduce the following conditions:

A′. Let $\Lambda(t),\ t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $\mathrm{E}\Lambda(t) = 1$.

We call this process the mother process and consider the following setting:

A″. Let $\Lambda(t) = \Lambda^{(i)},\ i = 0, 1, ...$ be independent copies of the mother process Λ, and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$:

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \ldots,$$

where the scaling parameter $b > 1$, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions.

Moreover, in the examples, the stationary mother process satisfies the following conditions:

A‴. Let $\Lambda(t) = \exp\{X(t)\}$, $t \in \mathbb{R}_+$, where $X\ (t)$ is a strictly stationary process, such that there exist a marginal probability density function $\pi(x)$ and a bivariate probability density function $p(x_1, x_2; t_1 - t_2)$. Moreover, we assume that the moment generating function

$$M(\zeta) = \mathrm{E}\exp\{\zeta X(t)\} \tag{2.1}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \tag{2.2}$$

exist.

The conditions $\mathbf{A}'$-$\mathbf{A}'''$ yield

$$\mathrm{E}\Lambda_b^{(i)}(t) = M(1) = 1; \mathrm{Var}\Lambda_b^{(i)}(t) = M(2) - 1 = \sigma_\Lambda^2 < \infty;$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1,1;(t_1 - t_2)b^i) - 1, \ b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = \exp\left\{\sum_{i=0}^{n} X^{(i)}(tb^i)\right\}, t \in [0,1], \tag{2.3}$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s)ds, \quad n = 0,1,2,\dots, t \in [0,1], \tag{2.4}$$

where $X^{(i)}(t), i = 0, ..., n, ...,$ are independent copies of a stationary process $X(t), t \geq 0$.

We also consider the corresponding positive random measures defined on Borel sets B of $\mathbb{R}_+$:

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0,1,2,\dots \tag{2.5}$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $\mathbb{R}_+$, it holds that $\lim_{n\to\infty}\mu_n(B_j) = \mu(B_j)$ for all j with probability one. The almost sure convergence of $A_n(t)$ in countably many points of $\mathbb{R}_+$ can be extended to all points in $\mathbb{R}_+$ if the limit process $A(t)$ is almost surely continuous. In this case, $\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987), there are two extreme cases: (i) $A_n(t) \to A(t)$ in $\mathcal{L}_1$ for each given t, in which case $A(t)$ is not almost surely zero and and is said to be fully active (non-degenerate) on $\mathbb{R}_+$; (ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on $\mathbb{R}_+$. Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987) (Eqs. (18) and (19) respectively.) The condition for complete degeneracy is detailed in Theorem 3 of Kahane (1987). In our work we present general conditions for non-degeneracy in Theorem 2.

The Rényi function of a random measure μ, also known as the deterministic partition function, is defined for $t \in [0,1]$ as

$$T(q) = \liminf_{n\to\infty} \frac{\log \mathrm{E}\sum_{k=0}^{2^n-1} \mu^q\left(I_k^{(n)}\right)}{\log\left|I_k^{(n)}\right|} = \liminf_{n\to\infty}\left(-\frac{1}{n}\right)\log_2 \mathrm{E}\sum_{k=0}^{2^n-1}\mu^q\left(I_k^{(n)}\right),$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}], \quad k = 0,1,\dots,2^n-1, \left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

In Denisov, Leonenko (2015) we established convergence

$$A_n(t) \overset{\mathcal{L}_q}{\to} A(t), \quad n \to \infty. \tag{2.6}$$

For the limiting process we show that for some constants $\overline{C}$ and $\underline{C}$,

$$\underline{C}t^{q-\log_b \mathrm{E}\Lambda^q(t)} \leqslant \mathrm{E}A^q(t) \leqslant \overline{C}t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \tag{2.7}$$

which will be written as

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}.$$

This allows us to find the scaling function

$$\varsigma(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q - \log_b M(q). \tag{2.8}$$

As is shown in Leonenko, Shieh (2013) for the exponentially decreasing correlations and $q \in [1,2]$ there is a connection between Rényi function and the scaling function given by

$$T(q) = \varsigma(q) - 1. \tag{2.9}$$

The exact conditions are stated in Theorem 1 and Theorem 2.

An important contribution of Denisov, Leonenko (2015) is that we proved (2.6) for general $q > 0$. In comparison, in Mannersalo et al. (2002) convergence (2.6) was shown for $q \in [1,2]$ under an additional assumption $A(t) \in \mathcal{L}_q$. Additionally we simplified significantly the conditions under which equations (2.6) and (2.7) hold. Finally we provide a number of scenarios where scaling function can be written explicitly.

2.1. **Scaling function for multifractal products.** In this Subsection we give the main results proved in Denisov, Leonenko (2015). Consider an integer $q > 2$. Now we assume additionally that $A_n(t)$ is a cadlag process. Let

$$\rho(u_1,\dots,u_{q-1}) = \mathrm{E}\Lambda(0)\Lambda(u_1)\dots\Lambda(u_1+\dots+u_{q-1}) \tag{2.10}$$

We require that the function $\rho(u_1,\dots,u_{q-1})$ satisfies certain mixing conditions. Namely, let $m < q-1$ and $\mathcal{C} = \{i_1,\dots i_m\}$ be a subset of indices ordered in the increasing order $1 \le i_1 < \dots < i_m \le q-1$. Consider the vector $(u_1,\dots,u_{q-1})$ such that $u_j = A$ if $j \in \mathcal{C}$ and $u_j = 0$ otherwise. Then we assume that for any set $\mathcal{C}$ the following mixing condition holds

$$\lim_{A\to\infty} \rho(u_1,\dots,u_{q-1}) = \mathrm{E}\Lambda(0)^{i_1}\mathrm{E}\Lambda(0)^{i_2-i_1}\cdot\ldots\cdot\mathrm{E}\Lambda(0)^{q-i_m}. \tag{2.11}$$

The following result was proved in (Denisov, Leonenko, 2015, Theorem 2)

Theorem 1. *Suppose that conditions $\boldsymbol{A}'$-$\boldsymbol{A}''$ hold. Assume that $\rho(u_1,\dots,u_{q-1})$ defined in (2.10) is monotone decreasing in all variables. Let*

$$b^{q-1} > \mathrm{E}\Lambda(0)^q \tag{2.12}$$

for some integer $q \ge 2$, and

$$\sum_{n=1}^{\infty} \left(\rho(b^n,\dots,b^n) - 1\right) < \infty. \tag{2.13}$$

Finally assume that the mixing condition (2.11) holds. Then,

$$\mathrm{E}A(t)^q < \infty, \tag{2.14}$$

and $A_n(t)$ converges to $A(t)$ in $\mathcal{L}_q$ (and hence in $\mathcal{L}_{\widetilde{q}}$ for $\widetilde{q} \in [0,q]$).

Now , for $q > 1$ let

$$\rho_q(s) = \inf_{u\in[0,1]} \left(\frac{\mathrm{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathrm{E}\Lambda(0)^q} - 1\right). \tag{2.15}$$

Note that $\rho_q(s) \le 0$. For $q \in (0,1)$ let

$$\rho_q(s) = \sup_{u\in[0,1]} \left(\frac{\mathrm{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathrm{E}\Lambda(0)^q} - 1\right). \tag{2.16}$$

For $q \le 1$ it is easy to see that $\rho_q(s) \ge 0$. Next result, see (Denisov, Leonenko, 2015, Theorem 3) established the form of the scaling function.

Theorem 2. *Assume that $A(t) \in \mathcal{L}_q$ and $\rho_q(s)$ defined in (2.15) and (2.16) is such that*

$$\sum_{n=1}^{\infty} |\rho_q(b^{-n})| < \infty. \tag{2.17}$$

Then,

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \quad t \in [0,1]. \tag{2.18}$$

and process $A(t)$ is non-degenerate, that is $\mathbb{P}(A(t) > 0) > 0$.

Using the above Theorems Denisov, Leonenko (2015) discusses further the log-normal scenario, see Theorem 4 of Denisov, Leonenko (2015). Furthermore, the case of geometric OU processes is considered in Theorem 5 of Denisov, Leonenko (2015). We discuss the latter scenario in more details below

3. Geometric Ornstein-Uhlenbeck processes

This section reviews a number of known results on Lévy processes (see Bertoin (1996), Kyprianou (2006)) and OU type processes (see Barndorff-Nielsen (1998, 2001), Barndorff-Nielsen, Shephard (2001)) The geometric OU type processes have been studied also by Matsui, Shieh (2009).

As standard notation we will write

$$\kappa(z) = C\{z; X\} = \log \mathrm{E} \exp\{izX\}, \quad z \in \mathbb{R}$$

for the cumulant function of a random variable X, and

$$K\{\zeta; X\} = \log \mathrm{E} \exp\{\zeta X\}, \quad \zeta \in D \subseteq \mathbb{C}$$

for the Lévy exponent or Laplace transform or cumulant generating function of the random variable X. Its domain D includes the imaginary axis and frequently larger areas.

A random variable X is infinitely divisible if its cumulant function has the Lévy-Khintchine form

$$C\{z; X\} = iaz - \frac{d}{2}z^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\mathbf{1}_{[-1,1]}(u)\right) \nu(du), \tag{3.1}$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, that is, a non-negative measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} \min\left(1, u^2\right) \nu(du) < \infty. \tag{3.2}$$

The triplet (a, d, ν) uniquely determines the random variable X. For a Gaussian random variable $X \sim N(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

A random variable X is self-decomposable if, for all $c \in (0, 1)$, the characteristic function $f(z)$ of X can be factorized as $f(z) = f(cz) f_c(z)$ for some characteristic function $f_c(z)$, $z \in \mathbb{R}$. A homogeneous Lévy process $Z = \{Z(t), t \geq 0\}$ is a continuous (in probability), càdlàg process with independent and stationary increments and $Z(0) = 0$ (recalling that a càdlàg process has right-continuous sample paths with existing left limits.) For such processes we have $C\{z; Z(t)\} = tC\{z; Z(1)\}$ and $Z(1)$ has the Lévy-Khintchine representation (3.1).

If X is self-decomposable, then there exists a stationary stochastic process $\{X(t), t \geq 0\}$, such that $X(t) \stackrel{d}{=} X$ and

$$X(t) = e^{-\lambda t} X(0) + \int_{(0,t]} e^{-\lambda(t-s)} dZ(\lambda s), \tag{3.3}$$

for all $\lambda > 0$ (see Barndorff-Nielsen (1998)). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $Z(t)$ satisfy the Itô stochastic differential equation

$$dX(t) = -\lambda X(t)\, dt + dZ(\lambda t), \tag{3.4}$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an OU type process. The process $Z(t)$ is termed the background driving Lévy process (BDLP) corresponding to the process $X(t)$. In fact (3.3) is the unique (up to indistinguishability) strong solution to Eq. (3.4) ((Saito, 1992, Section 17)).

Let $X(t)$ be a square integrable OU process. Then $X(t)$ has the correlation function

$$\mathrm{Corr}(X(0), X(t)) = r_X(t) = \exp\{-\lambda |t|\}. \tag{3.5}$$

The cumulant transforms of $X = X(t)$ and $Z(1)$ are related by

$$C\{z; X\} = \int_0^\infty C\left\{e^{-s}z; Z(1)\right\} ds = \int_0^z C\{\xi; Z(1)\} \frac{d\xi}{\xi}, C\{z; Z(1)\} = z\frac{\partial C\{z; X\}}{\partial z}.$$

Suppose that the Lévy measure ν of X has a density function $p(u), u \in \mathbb{R}$, which is differentiable. Then the Lévy measure $\tilde{\nu}$ of $Z(1)$ has a density function $q(u), u \in \mathbb{R}$, and p and q are related by

$$q(u) = -p(u) - up'(u) \tag{3.6}$$

(see Barndorff-Nielsen (1998)).

The logarithm of the characteristic function of a random vector $(X(t_1), ..., X(t_m))$ is of the form

$$\log \mathrm{E} \exp\{i(z_1 X(t_1) + ... + z_m X(t_m)\} = \int_{\mathbb{R}} \kappa\Big(\sum_{j=1}^m z_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s)\Big) ds, \tag{3.7}$$

where

$$\kappa(z) = \log \mathrm{E} \exp\{izZ(1)\} = C\{z; Z(1)\},$$

and the function (3.7) has the form (3.1) with Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$.

The logarithms of the moment generation functions (if they exist) take the forms

$$\log \mathrm{E} \exp\{\zeta X(t)\} = \zeta a + \frac{d}{2}\zeta^2 + \int_{\mathbb{R}} (e^{\zeta u} - 1 - \zeta u \mathbf{1}_{[-1,1]}(u)) \nu(du),$$

where (a, d, ν) is the Lévy triplet of $X(0)$, or in terms of the Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$

$$\begin{aligned} \log \mathrm{E} \exp\{\zeta X(t)\} &= \tilde{a} \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)) ds + \frac{\tilde{d}}{2}\zeta^2 \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp\{u\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\} - 1 - u\left(\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\right) \mathbf{1}_{[-1,1]}(u)] \tilde{\nu}(du)\, ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\log \mathrm{E} \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \\
&= \tilde{a} \int_{\mathbb{R}} \left(\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s)) ds + \frac{\tilde{d}}{2} \zeta^2 \int_{\mathbb{R}} (\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))^2 \right) ds \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp \left\{ u \sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s) \right\} - 1 \\
&\quad - u \left(\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s) \right) \mathbf{1}_{[-1,1]}(u)] \tilde{\nu}(du)\, ds. \qquad (3.9)
\end{aligned}$$

Let us consider a geometric OU-type process as the mother process:

$$\Lambda(t) = e^{X(t) - c_X}, c_X = \log \mathrm{E} e^{X(0)}, M(\zeta) = \mathrm{E} e^{\zeta(X(t) - c_X)}, M_0(\zeta) = \mathrm{E} e^{\zeta X(t)}$$

where $X(t), t \in \mathbb{R}_+$, is the OU-type stationary process (3.3). Note that

$$\frac{M_0(q)}{M_0(1)^q} = \frac{M(q)}{M(1)^q}.$$

Then the correlation function of the mother process is of the form.

$$\mathrm{Corr}(\Lambda(t), \Lambda(t+\tau)) = \frac{M(1,1;\tau) - 1}{M(2) - 1}, \qquad (3.10)$$

where now

$$\begin{aligned}
M(\zeta_1, \zeta_2; \tau) &= \mathrm{E} \exp\{\zeta_1 (X(t_1) - c_X) + \zeta_2 (X(t_2) - c_X)\} \\
&= \exp\{-(\zeta_1 + \zeta_2) c_X\} \mathrm{E} \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}, \qquad (3.11)
\end{aligned}$$

and $\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$ is defined by (3.9).

To prove that a geometric OU process satisfies the covariance decay condition (2.17) in Theorem 2, the expression given by (3.9) is not ready to yield the decay as $t_2 - t_1 \to \infty$.

The following result plays a key role in multifractal analysis of geometric OU processes.

Theorem 3. *Let $X(t), t \in \mathbb{R}_+$ be an OU-type stationary process (3.3) such that the Lévy measure ν in (3.1) of the random variable $X(0)$ satisfies the condition: for an integer $q^* \geq 2$,*

$$\int_{|x| \geq 1} x e^{q^* x} \nu(dx) < \infty. \qquad (3.12)$$

Then, for any fixed b such that

$$b > \left\{ \frac{M_0(q^*)}{M_0(1)^{q^*}} \right\}^{\frac{1}{q^*-1}}, \qquad (3.13)$$

the sequence of stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(s b^j\right) ds, t \in [0,1] \qquad (3.14)$$

converges in $\mathcal{L}_q$ *to the stochastic process* $A(t) \in \mathcal{L}_q$*, as* $n \to \infty$*, for every fixed* $t \in [0,1]$*. The limiting process* $A(t), t \in [0,1]$ *satisfies*

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \quad q \in [0, q^*].$$

The scaling function is given by

$$\varsigma(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q\left(1 + \frac{c_X}{\log b}\right) - \log_b M_0(q), \quad q \in [0, q^*]. \tag{3.15}$$

In addition,

$$\mathrm{Var}A(t) \geqslant 2t \int_0^t \left(1 - \frac{s}{t}\right)(M(1,1;s) - 1)ds, \tag{3.16}$$

where the bivariate moment generating function $M(\zeta_1, \zeta_2; t_1 - t_2)$ *is given by (3.11)*

For the proof see (Denisov, Leonenko, 2015, Theorem 5). The proof of Theorem 3 relies on the following auxiliary result, which we will need as well.

Lemma 1. *For* $s \in [0,1]$*, the following estimate holds*

$$\frac{M_0(1+s)}{M_0(1)M(s)} \leq \left(\frac{M_0(2)}{M_0(1)e^{\mathrm{E}X(1)}}\right)^s. \tag{3.17}$$

The proof of Lemma 1 can be found in (Denisov, Leonenko, 2015, Lemma 1).

4. Superpositions of geometric Ornestein-Uhlenbeck processes

The correlation structures found in applications may be more complex than the exponential decreasing autocorrelation of the form (3.5). Barndorff-Nielsen (1998) (see also Barndorff-Nielsen, Shephard (2001)) proposed to consider the following class of autocovariance functions:

$$R_m(t) = \sum_{j=1}^{m} \sigma_j^2 \exp\{-\lambda_j |t|\}, \tag{4.1}$$

which is flexible and can be fitted to many autocovariance functions arising in applications. The role of an integer $m \geq 1$ is discussed in Barndorff-Nielsen, Shephard (2001).

In order to obtain models with dependence structure (4.1) and given marginal density with finite variance, we consider stochastic processes defined by

$$dX_j(t) = -\lambda_j X_j(t)\,dt + dZ_j(\lambda_j t), \; j = 1, 2, ..., m, ...$$

and their finite superposition

$$X_{m\sup}(t) = X_1(t) + ... + X_m(t), \; t \geq 0, \tag{4.2}$$

where $Z_j, \; j = 1, 2, ..., m, ...$ are mutually independent Lévy processes. Then the solution $X_j = \{X_j(t), t \geq 0\}, \; j = 1, 2, ..., m,$ is a stationary process. Its correlation function is of the exponential form (assuming finite variance of the components).

The superposition (4.2) has its marginal density given by that of the random variable

$$X_{m\sup}(0) = X_1(0) + ... + X_m(0), \tag{4.3}$$

and autocovariance function (4.1). One can generalize Theorem 3 to the case of finite superposition process (4.2).

We are interested in the case when the distribution of (4.3) is tractable, for instance when $X_m(0)$ belongs to the same class as $X_j(0), j = 1, ..., m$ (see the examples in Sections 5–10 below). We denote the class of stochastic processes (4.1) of finite superpositions with marginal law D by

$$\mathbb{FS}_m\{D; \mathrm{E}X_j(t); \mathrm{Var}X_j(t)\}. \tag{4.4}$$

Define the mother process as the geometric process

$$\Lambda(t) = e^{X_{m\sup}(t)-c_X}, c_X = \log \mathrm{E}e^{X_{m\sup}(0)}, M(\zeta) = \mathrm{E}e^{\zeta(X_{m\sup}(t)-c_X)}, M_0(\zeta) = \mathrm{E}e^{\zeta X_{m\sup}(t)},$$

where $X_{m\sup}(t), t \in \mathbb{R}_+$, is the finite superposition process (4.2). Note that

$$\log \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^{m} \log \mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

where $\log \mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (3.9).

Denote

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\}. \tag{4.5}$$

We can formulate the following theorem which can be proved similar to Theorem 3.

Theorem 4. *Let $X_{m\sup}(t), t \in \mathbb{R}_+$ be a finite superposition of OU-type stationary processes (4.2) such that the Lévy measure ν in (3.1) of the random variable $X_m(t)$ satisfies the condition that for a positive integer $q^* \in \mathbf{N}$,*

$$\int_{|x|\geq 1} x e^{q^* x} \nu(dx) < \infty. \tag{4.6}$$

Then, for any fixed b such that

$$b > \left\{ \frac{M_0(q^*)}{M_0(1)^{q^*}} \right\}^{\frac{1}{q^*-1}}, \tag{4.7}$$

stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^j\right) ds,$$

converge in $\mathcal{L}_q$ to the stochastic process $A(t) \in \mathcal{L}_q$, as $n \to \infty$. The limiting process $A(t)$ satisfies

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \quad q \in [0, q^*], t \in [0, 1].$$

The scaling function is given by

$$\varsigma(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q(1 + \frac{c_X}{\log b}) - \log_b M_0(q), \quad q \in [0, q^*]. \tag{4.8}$$

In addition,

$$\mathrm{Var}A(t) \geqslant 2t \int_0^t \left(1 - \frac{u}{t}\right) M(\zeta_1, \zeta_2; u) du, \tag{4.9}$$

where the bivariate moment generating function $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (4.5)

We are interested in generalization of the above result to the case of infinite superposition of OU-type processes which has a long-range dependence property.

Note that an infinite superposition ($m \to \infty$) gives a complete monotone class of covariance functions

$$R_{\sup}(t) = \int_0^\infty e^{-tu} dU(u), t \geq 0,$$

for some finite measure U, which display long-range dependence (see Barndorff-Nielsen (1998, 2001), Barndorff-Nielsen, Leonenko (2005) for possible covariance structures and spectral densities and Barndorff-Nielsen et al. (2011) for multivariate generalizations).

We are going to consider an infinite superposition of the OU processes, which corresponds to $m \to \infty$, that is now

$$X_{\sup}(t) = \sum_{j=1}^{\infty} X_j(t), \tag{4.10}$$

assuming that

$$\sum_{j=1}^{\infty} \mathrm{E}X_j(t) < \infty, \sum_{j=1}^{\infty} \mathrm{Var}X_j(t) < \infty. \tag{4.11}$$

In this case

$$R_{\sup}(t) = \sum_{j=1}^{\infty} \sigma_j^2 \exp\left\{-\lambda_j |t|\right\}, \tag{4.12}$$

and if we assume that for some $\delta_j > 0$

$$\mathrm{E}X_j(t) = \delta_j C_1, \mathrm{Var}X_j(t) = \sigma_j^2 = \delta_j C_2, \delta_j = j^{-(1+2(1-H))}, \frac{1}{2} < H < 1, \tag{4.13}$$

where the constants $C_1 \in \mathbb{R}$ and $C_2 > 0$ represent some other possible parameters (see examples in the Sections 5–10 below), then

$$\mathrm{E}X_{\sup}(t) = C_1 \sum_{j=1}^{\infty} \delta_j = C_1 \zeta(1 + 2(1-H)) < \infty, \tag{4.14}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\mathrm{Re}s > 1$, is the Riemann zeta-function, and with $\lambda_j = \lambda/j$, we have

$$R_{\sup}(t) = \sum_{j=1}^{\infty} \sigma_j^2 \exp\left\{-\lambda_j |t|\right\} = C_2 \sum_{j=1}^{\infty} \delta_j \exp\left\{-\lambda |t| / j\right\} = \frac{L_2(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1, \tag{4.15}$$

where L_2 is a slowly varying at infinity function, bounded on every bounded interval. Thus we obtain a long range dependence property:

$$\int_{\mathbb{R}} R_{\sup}(t) dt = \infty.$$

We denote the class of stochastic processes (4.10) of infinite superpositions with marginal law D as

$$\mathbb{IS}\{D; \mathrm{E}X_j(t); \mathrm{Var}X_j(t)\}. \tag{4.16}$$

We are going to make an additional assumption that there exists parameters $\bar{\delta}_j$ such that

$$\mathrm{E}e^{\zeta X_j(0)} = \mathrm{E}e^{\zeta\bar{\delta}_j Y} \tag{4.17}$$

for some random variable Y. The sum

$$\sum_{j=1}^{\infty} \bar{\delta}_j < \infty \tag{4.18}$$

must be finite. When we specialize (4.12) to this situation we obtain

$$R_{\sup}(t) = C_2 \sum_{j=1}^{\infty} \bar{\delta}_j \exp\left\{-\lambda_j |t|\right\}, \tag{4.19}$$

for some $C_2 > 0$. This approach allows also to treat the case of several parameters.

We are starting our considerations with $\mathcal{L}_q$ convergence. Let

$$\rho_j(u_1, \ldots, u_{q-1}) = \mathrm{E}e^{X_j(0)+X_j(u_1)+X_j(u_1+u_2)+\cdots+X_j(u_1+\cdots+u_{q-1})}$$

correspond to the process $X_j(t)$. Then, since $X_j(\cdot)$ are independent of each other,

$$\rho(u_1, \ldots, u_{q-1}) = \prod_{j=1}^{\infty} \rho_j(u_1, \ldots, u_{q-1}). \tag{4.20}$$

We have shown above that $\rho_j(u_1, \ldots, u_{q-1})$ is monotone decreasing in $u_1, \ldots, u_{q-1}$. Therefore $\rho(u_1, \ldots, u_{q-1})$, being a product of monotone decreasing functions is monotone decreasing itself. Next we prove finiteness of the series. It follows from Lemma 1 (see also(Denisov, Leonenko, 2015, Equation (6.19)))

$$1 \le \rho_j(b^n, \ldots, b^n) \le \left(\frac{M_0(1+e^{-\lambda_j b^n})}{M_0(1)M_0(e^{-\lambda_j b^n})}\right)^q \le \left(\frac{\mathrm{E}e^{2X_j(0)}}{\mathrm{E}e^{X_j(0)}e^{\mathrm{E}X_j(0)}}\right)^{qe^{-\lambda_j b^n}} = C^{q\delta_j e^{-\lambda_j b^n}}$$

where $C = \frac{\mathrm{E}e^{2Y}}{\mathrm{E}e^{Y}e^{\mathrm{E}Y}}$, and we denote

$$M_{0j}(\zeta) = \mathrm{E}e^{\zeta X_j(t)}.$$

Then, using (4.19), we obtain

$$1 \le \rho(b^n, \ldots, b^n) \le C^{q\sum_{j=1}^{\infty}\delta_j e^{-\lambda_j b^n}} = C^{\frac{q}{\sigma^2}R_{sup}(b^n)} \le 1 + o(1)\frac{q}{\sigma^2}\ln C R_{sup}(b^n).$$

Then $\sum_{n=1}^{\infty} \rho(b^n, \ldots, b^n)$ is finite if the sum $\sum_{n=1}^{\infty} R_{sup}(b^n)$ is finite.

We are left to check the mixing condition (2.11). But this condition follows from the fact that it holds for ρ_j, representation (4.20) and monotonicity of ρ_j. Indeed let $1 \le i_1 < i_2 < \ldots \le i_m$ and put $\delta_{i_1,\ldots,i_m}(A) = (u_1, \ldots, u_{q-1})$, where $u_i = A$ if $i \in \{i_1, \ldots, i_m\}$ and 0 otherwise. Let N be a number which we let tend to ∞ later. Then, for fixed N,

$$\prod_{j=1}^{N} \rho_j(\delta_{i_1,\ldots,i_m}(A)) \to \prod_{j=1}^{N} \mathrm{E}e^{i_1 X_j(0)}\mathrm{E}e^{(i_2-i_1)X_j(0)} \ldots \mathrm{E}e^{(q-i_m)X_j(0)}$$

by the corresponding property of the geometric Ornestein-Uhlenbeck process. The product,

$$1 \le \prod_{j=N}^{\infty} \rho_j(\delta_{i_1,\ldots,i_m}(A)) \le \prod_{j=N}^{\infty} \rho_j(\delta_{i_1,\ldots,i_m}(1)) \to 1,$$

as $N\to\infty$, uniformly in $A>1$. Therefore, the mixing property holds.

Now we turn to proving the scaling property. For that we use Theorem 2. Let $M_0(\zeta)=\mathrm{E}e^{\zeta Y}$ be the moment generating function of Y and $f(\zeta)=\ln M_0(\zeta)$. Then, similarly to (Denisov, Leonenko, 2015, Equation (6.22)),

$$|\rho_q(s)|=1-\prod_{j=1}^{\infty}\left(\frac{M_0(q-1+e^{-\lambda_j s})M_0(1)}{M_0(q)M_0(e^{-\lambda_j s})}\right)^{\delta_j}\leq 1-\prod_{j=1}^{\infty}e^{(-1+e^{-\lambda_j s})(f'(q)-f(1))\delta_j}$$
$$\leq 1-\prod_{j=1}^{\infty}e^{-\lambda_j s(f'(q)-f(1))\delta_j}\leq s(f'(q)-f(1))\sum_{j=1}^{\infty}\lambda_j\delta_j.$$

The convergence of the series immediately follows from this estimate and we can apply Theorem 2.

Note that

$$\log \mathrm{E}\exp\{\zeta_1 X_{\sup}(t_1)+\zeta_2 X_{\sup}(t_2)\}=\sum_{j=1}^{\infty}\log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

where $\log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\}, j=1,2,..$ are given by (3.9).

Define the mother process as the geometric process

$$\Lambda(t)=e^{X_{\sup}(t)-c_X},\, c_X=\log \mathrm{E}e^{X_{\sup}(0)},\, M(\zeta)=\mathrm{E}e^{\zeta(X_{\sup}(0)-c_X)},$$

where $X_{\sup}(t), t\in\mathbb{R}$, is the infinite superposition process (4.10).

Denote

$$M(\zeta_1,\zeta_2;t_1-t_2)=\exp\{-c_X(\zeta_1+\zeta_2)\}\,\mathrm{E}\exp\{\zeta_1 X_{\sup}(t_1)+\zeta_2 X_{\sup}(t_2)\}. \tag{4.21}$$

We arrive to the following result.

Theorem 5. *Let $X_{\sup}(t), t\in\mathbb{R}_+$ be an infinite superposition of OU-type stationary processes (4.10) such that (4.11),(4.13),(4.17) are satisfied as well as (4.18). Assume that the Lévy measure ν in (3.1) of the random variable $X_{\sup}(t)$ satisfies the condition that for a positive integer $q^*\in\mathbf{N}$,*

$$\int_{|x|\geq 1} x e^{q^* x}\nu(dx)<\infty. \tag{4.22}$$

Then, for any fixed b such that

$$b>\left\{\frac{M(q^*)}{M(1)^{q^*}}\right\}^{\frac{1}{q^*-1}}, \tag{4.23}$$

stochastic processes

$$A_n(t)=\int_0^t\prod_{j=0}^{n}\Lambda^{(j)}\left(sb^j\right)ds,\, t\in[0,1]$$

converge in $\mathcal{L}_q$ to the stochastic process $A(t)\in\mathcal{L}_q, t\in[0,1]$, as $n\to\infty$. The limiting process $A(t)$ satisfies

$$\mathrm{E}A^q(t)\sim t^{\zeta(q)},\quad q\in[0,q^*].$$

The scaling function is given by

$$\varsigma(q)=q-\log_b \mathrm{E}\Lambda^q(t),\quad q\in[0,q^*], t\in[0,1]. \tag{4.24}$$

In addition,

$$\mathrm{Var}A(t) \geqslant \int_0^t \int_0^t M(\zeta_1, \zeta_2; t_1 - t_2)dtds, \tag{4.25}$$

where the bivariate moment generating functions $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (4.21).

5. Log-tempered stable scenario

This section introduces a scenario which generalize the log-inverse Gaussian scenario obtained in Anh et al. (2008a, 2010a). Note that the tempered stable distribution (up to constants) arises in the theory of Vershik-Yor subordinator (see Donati-Martin, Yor (2006), and the references therein). This section constructs a multifractal process based on the geometric tempered stable OU process. In this case, the mother process takes the form $\Lambda(t) = \exp\{X(t) - c_X\}$, where $X(t), t \geq 0$ is a stationary OU type process (3.3) with tempered stable marginal distribution and c_X is a constant depending on the parameters of its marginal distribution. This form is needed for the condition $\mathrm{E}\Lambda(t) = 1$ to hold. The log-tempered stable scenario appeared in Novikov (1994) in a physical setting and in Anh et al. (2001) in a genomic setting under different terminology. So, we present here a rigorous proofs regarded these scenarios. Some applications of the log-tempered stable scenario and other related multifractal scenarios considered below in a subordinated models for currency exchange rates can be found in Leonenko et al. (2013).

We consider the stationary OU process whose marginal distribution is the tempered stable distribution $TS(\kappa, \delta, \gamma)$ (see, for example, Barndorff-Nielsen, Shephard (2002), G.Terdik, Woyczynski (2004)). This distribution is the exponentially tilted version of the positive κ-stable law $S(\kappa, \delta)$ whose cumulant transform is of the form: $-\delta(2\zeta)^\kappa, \zeta > 0, \kappa \in (0,1), \delta > 0$.We denote its probability density function (pdf) as $s_{\kappa,\delta}(x), x > 0$. The pdf of the tempered stable distribution $TS(\kappa, \delta, \gamma)$ is

$$\pi(x) = \pi(x; \kappa, \delta, \gamma) = e^{\delta\gamma} s_{\kappa,\delta}(x) e^{-\frac{x}{2}\gamma^{1/\kappa}}, x > 0, \kappa \in (0,1), \delta > 0, \gamma > 0. \tag{5.1}$$

It is clear that $TS(\frac{1}{2}, \delta, \gamma) = IG(\delta, \gamma)$, the inverse Gaussian distribution with pdf

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \frac{\delta e^{\delta\gamma}}{x^{3/2}} \exp\left\{-\left(\frac{\delta^2}{x} + \gamma^2 x\right)\frac{1}{2}\right\} \mathbf{1}_{[0,\infty)}(x), \delta > 0, \gamma \geq 0.$$

In general the pdf of tempered stable distribution is given in form of a series representation (see, i.e., Anh et al. 2010).

The cumulant transform of a random variable $X \sim TS(\kappa, \delta, \gamma)$ is of the form.

$$\log \mathrm{E}e^{\zeta X} = \delta\gamma - \delta\left(\gamma^{\frac{1}{\kappa}} - 2\zeta\right)^\kappa, 0 < \zeta < \frac{\gamma^{1/\kappa}}{2}. \tag{5.2}$$

Note that

$$\mathrm{E}X(t) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}}, \mathrm{Var}X(t) = 4\kappa(1-\kappa)\delta\gamma^{\frac{\kappa-2}{\kappa}}.$$

We will consider a stationary OU type process (3.4) with marginal distribution $TS(\kappa, \delta, \gamma)$. This distribution is self-decomposable (and hence infinitely divisible) with the Lévy triplet $(a, 0, \nu)$, where

$$\nu(du) = b(u)du, b(u) = 2^\kappa \delta \frac{\kappa}{\Gamma(1-\kappa)} u^{-1-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}}, u > 0.$$

The BDLP $Z(t)$ in (3.4) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, with

$$\tilde{\nu}(du) = \lambda\omega(u)du, \tag{5.3}$$

$$\omega(u) = 2^{\kappa}\delta\frac{\kappa}{\Gamma(1-\kappa)}\left(\frac{\kappa}{u} + \frac{\gamma^{1/\kappa}}{2}\right)u^{-\kappa}e^{-\frac{u\gamma^{1/\kappa}}{2}}, u > 0. \tag{5.4}$$

Consider a mother process of the form

$$\Lambda(t) = \exp\left\{X(t) - c_X\right\}$$

with

$$c_X = \left[\delta\gamma - \delta\left(\gamma^{\frac{1}{\kappa}} - 2\right)^{\kappa}\right], \gamma > 2^{\kappa}, \tag{5.5}$$

where $X(t)$ is OU processes with $TS(\kappa,\delta,\gamma)$ marginal distribution and correlation function $R_X(t) = \exp\{-\lambda|t|\}$.

The correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1,1;\tau) - 1}{M(2) - 1}, \gamma > 4^{\kappa}, \tag{5.6}$$

where

$$M(\zeta) = e^{-\zeta c_X}\mathrm{E}e^{\zeta X(t)},$$

and the bivariate moment generating function $M(1,1,\tau)$ is given by (3.11), in which the Lévy measure $\tilde{\nu}$ is defined by (5.3), (5.4), and c_X is given by (5.5).

Condition (3.12) becomes

$$\int_1^{\infty} ue^{q^*u}u^{-1-\kappa}e^{-\frac{u\gamma^{1/\kappa}}{2}}du = \int_1^{\infty} u^{-\kappa}e^{-u(\frac{\gamma^{1/\kappa}}{2} - q^*)}du < \infty,$$

if $0 < q^* < \frac{\gamma^{1/\kappa}}{2}, \kappa \in (0,1)$. Note that $M(q)$ exists if $(2q)^{\kappa} < \gamma$.

We can formulate the following

Theorem 6. *Let $X(t)$ be an OU processes with $TS(\kappa,\delta,\gamma)$ marginal distributions, $\lambda > 0$ and*

$$Q = \{q : 0 < q < \frac{\gamma^{1/\kappa}}{2}, \gamma \geq \max\{(2q^*)^{\kappa}, 4^{\kappa}\}, \kappa \in (0,1), \delta > 0\} \cap [0, q^*],$$

where q^* is a fixed integer. Then, for any

$$b > \exp\left\{-\gamma\delta + \frac{\delta}{1-q^*}\left(\gamma^{\frac{1}{\kappa}} - 2q^*\right)^{\kappa} - \frac{q^*}{1-q^*}\delta\left(\gamma^{\frac{1}{\kappa}} - 2\right)^{\kappa}\right\}, \gamma \geq \max\{(2q^*)^{\kappa}, 4^{\kappa}\},$$

the stochastic processes (3.14) converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ for each fixed $t \in [0,1]$ as $n \to \infty$ such that, $A_q(1) \in \mathcal{L}_q$, for $q \in Q$, and

$$\mathrm{E}A^q(t) \sim t^{\varsigma(q)},$$

where the scaling function $\varsigma(q)$ is given by

$$\varsigma(q) = q\left(1 + \frac{\delta\gamma}{\log b} - \frac{\delta}{\log b}\left(\gamma^{\frac{1}{\kappa}} - 2\right)\right)^{\kappa} + \frac{\delta}{\log b}\left(\gamma^{\frac{1}{\kappa}} - 2q\right)^{\kappa} - \frac{\delta\gamma}{\log b}, q \in Q. \tag{5.7}$$

Moreover, (5.7) holds, where $M(1,1,\tau)$ is given by (3.9), in which the Lévy measure $\tilde{\nu}$ is defined by (5.3), (5.4).

Theorem 6 follows from Theorem 3. Note that for $\kappa = 1/2$ Theorem 6 is an extension of Theorem 4 ofAnh et al. (2008a).

In this particular case we arrive to log-inverse Gaussian scenario where the scaling function is of the form:

$$\varsigma(q) = q\left(1 + \frac{\delta\left[\gamma - \sqrt{\gamma^2 - 2}\right]}{\log b}\right) + \frac{\delta}{\log b}\sqrt{\gamma^2 - 2q} - \frac{\gamma\delta}{\log b}, q \in Q,$$

and

$$Q = \{q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2, \delta > 0\} \cap (0, q^*) \tag{5.8}$$

if

$$b > \exp\left\{-\gamma\delta - \frac{\delta}{1 - q^*}\sqrt{\gamma^2 - 2q} - \frac{q^*}{1 - q^*}\delta\sqrt{\gamma^2 - 2}\right\}$$

and q^* is a fixed integer.

Note that the set (5.8) is an extension of the log-inverse Gaussian scenario in Theorem 4 of Anh et al. (2008a) which is obtained for the set

$$Q = \{q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2, \delta > 0\} \cap [1, 2].$$

In this case the Rényi function $T(q) = \varsigma(q) + 1$.

We can construct log-tempered stable scenarios for a more general class of finite superpositions of stationary tempered stable OU-type processes (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $X_j(t) \sim TS(\kappa, \delta_j, \gamma), j = 1, ..., m$, and parameters $\delta_j, j = 1, ..., m$. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{TS(\kappa, \sum_{j=1}^{m} \delta_j, \gamma); 2\kappa\delta_j\gamma^{\frac{\kappa-1}{\kappa}}; 4\kappa\,(1-\kappa)\,\gamma^{\frac{\kappa-2}{\kappa}}\delta_j)$$

It follows from Theorem 4 that the statement of Theorem 6 can be reformulated for $X_{m\sup}(t)$ with $\delta = \sum_{j=1}^{m} \delta_j$, and

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\}\,\mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\},$$

where

$$\log \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^{m} \log \mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (3.9).

Moreover, one can construct log-tempered stable scenarios for a more general class of infinite superpositions of stationary tempered stable OU-type processes (4.10), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $X_j(t) \sim TS(\kappa, \delta_j, \gamma), j = 1, 2..$ and parameters $\delta_j, j = 1, 2....$ .Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{TS(\kappa, \sum_{j=1}^{\infty} \delta_j, \gamma); 2\kappa\delta_j\gamma^{\frac{\kappa-1}{\kappa}}; 4\kappa\,(1-\kappa)\,\gamma^{\frac{\kappa-2}{\kappa}}\delta_j\}.$$

It follows from Theorem 5 that the statement of Theorem 6 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j$, and

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \operatorname{E} \exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2 X_{\sup}(t_2)\},$$

where

$$\log \operatorname{E} \exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2 X_{\sup}(t_2)\} = \sum_{j=1}^{\infty} \log \operatorname{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \operatorname{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (3.9) with Lévy triplet $(\widetilde{a}, 0, \tilde{\nu})$ given by (5.3) and (5.4).

6. Log-normal tempered stable scenario

This subsection constructs a multifractal process based on the geometric normal tempered stable (NTS) OU process. The log-normal tempered stable scenario is important for risky asset modelling, see Leonenko et al. (2013).

We consider a random variable $X = \mu + \beta Y + \sqrt{Y}\epsilon$, where the random variable Y follows the $TS(\kappa, \delta, \gamma)$ distribution, ϵ has a standard normal distribution, and Y and ϵ are independent. We then say that X follows the normal tempered stable law $NTS(\kappa, \gamma, \beta, \mu, \delta)$ (see, for example, Barndorff-Nielsen, Shephard (2002)). In particular, for $\kappa = 1/2$ we have that $NTS(\frac{1}{2}, \gamma, \beta, \mu, \delta)$ is the same as the normal inverse Gaussian law $NIG(\alpha, \beta, \mu, \delta)$ with $\alpha = \sqrt{\beta^2 + \gamma^2}$ (see Barndorff-Nielsen (1998)). We assume that

$$\mu \in \mathbb{R},\ \delta > 0,\ \gamma > 0, \beta > 0, \kappa \in (0, 1).$$

It was pointed out by Barndorff-Nielsen, Shephard (2002) that $NTS(\kappa, \gamma, \beta, \mu, \delta)$ is self-decomposable. Thus, there exists a stationary OU-type process $X(t), t \geq 0$, with stationary $NTS(\kappa, \gamma, \beta, \mu, \delta)$ marginal distribution and the correlation function $r_X(t) = \exp\{-\lambda |t|\}$ .Note that

$$\operatorname{E}X(t) = \mu + 2\kappa\beta\delta\gamma^{\frac{\kappa-1}{\kappa}}, \operatorname{Var}X(t) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa\beta^2\delta(\kappa - 1)\gamma^{\frac{\kappa-2}{\kappa}}.$$

We see that the variance can be factorized in a similar manner as in Section 4, and thus superposition can be used to create multifractal scenarios with more elaborate dependence structures.

The cumulant transform of the random variable $X(t)$ with $NTS(\kappa, \gamma, \beta, \mu, \delta)$ distribution is equal to

$$\log \operatorname{E} e^{\zeta X(t)} = \mu\zeta + \delta\gamma - \delta\left(\alpha^2 - (\beta + \zeta)^2\right)^{\kappa}, |\beta + \zeta| < \alpha = \sqrt{\beta^2 + \gamma^{1/\kappa}}. \tag{6.1}$$

The Lévy triplet of $NTS(\kappa, \gamma, \beta, \mu, \delta)$ is $(a, 0, \nu)$, where

$$\nu(du) = b(u)du, \tag{6.2}$$

$$b(u) = \frac{\delta}{\sqrt{2\pi}}\alpha^{\kappa+\frac{1}{2}} \frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)} |u|^{-(\kappa+\frac{1}{2})} K_{\kappa+\frac{1}{2}}(\alpha|u|)e^{\beta u}, u \in \mathbb{R}, \tag{6.3}$$

where here and below ${}_{\nu}(z) = \int_0^{\infty} e^{-z\cosh(u)} \cosh(\nu z)du,\ z > 0$, is the modified Bessel function of the third kind of index $\nu, \operatorname{Re}\nu > 0$.

From (3.6), (6.1) and the formulae

$$K_\nu(x) = K_\nu(-x),\ K_{-\nu}(x) = K_\nu(x), \frac{d}{dx}K_\nu(x) = -\frac{\lambda}{x}K_\nu(x) - K_{\nu-1}(x),$$

we obtain that the BDLP $Z(t)$ in (3.4) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, with

$$\tilde{\nu}(du) = \lambda\omega(u)du,$$

where

$$\omega(u) = -b(u) - ub'(u) = \frac{\delta}{\sqrt{2\pi}}\alpha^{\kappa+\frac{1}{2}}\frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)}$$
$$\times \{(\kappa - \frac{1}{2})|u|^{-(\kappa+\frac{1}{2})} K_{\kappa+\frac{1}{2}}(\alpha|u|)e^{\beta u} + |u|^{-(\kappa-\frac{1}{2})}[-\frac{\kappa+\frac{1}{2}}{|u|}K_{\kappa+\frac{1}{2}}(\alpha|u|)e^{\beta u}$$
$$- K_{\kappa-\frac{1}{2}}(\alpha|u|)e^{\beta u}\alpha + K_{\kappa+\frac{1}{2}}(\alpha|u|)e^{\beta u}\beta]\}. \tag{6.4}$$

Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\},$$

with

$$c_X = \mu + \delta\gamma - \delta\left(\beta^2 + \gamma^{1/\kappa} - (\beta+1)^2\right)^{\kappa}, \beta < \frac{\gamma^{1/\kappa}-1}{2},$$

where $X(t)$ is a stationary $NTS(\kappa,\gamma,\beta,\mu,\delta)$ OU-type process.

Under condition B$''$, we obtain the following moment generating function

$$M(\zeta) = \mathrm{E}\exp\{\zeta(X(t) - c_X)\} = e^{-c_X\zeta}e^{\mu\zeta+\delta\gamma-\delta\left(\beta^2+\gamma^{1/\kappa}-(\beta+\zeta)^2\right)^{\kappa}}, \quad |\beta+\zeta| < \alpha, \tag{6.5}$$

and bivariate moment generating function

$$M(\zeta_1,\zeta_2;(t_1-t_2)) = \mathrm{E}\exp\{\zeta_1(X(t_1)-c_X) + \zeta_2(X(t_2)-c_X)\}$$
$$= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2(X(t_2)\}, \tag{6.6}$$

and $\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2(X(t_2)\}$ is given by (3.9) with Lévy measure $\tilde{\nu}$ having density (6.4). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1,1;\tau)-1}{M(2)-1}, \tag{6.7}$$

where we assumed that $\beta < (\gamma^{1/\kappa} - 4)/4$.

Note that as $z \to \infty$ the modified Bessel function of the third kind of index ν.

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}z^{-1/2}e^{-z}(1 + \frac{4\nu^2-1}{8z} + ...), z > 0,$$

Condition (3.12) now becomes

$$\int_{|u|>1} ue^{q^* u}|u|^{-(\kappa+\frac{1}{2})}K_{\kappa+\frac{1}{2}}(\alpha|u|)e^{\beta u}du < \infty, if\, |\beta+q^*| < \alpha = \sqrt{\beta^2+\gamma^{1/\kappa}}.$$

Theorem 7. *Let $X(t)$ be a stationary $NTS(\kappa,\gamma,\beta,\mu,\delta)$ OU-type process, $\lambda > 0$ and*

$$q \in Q = \left\{q : 0 < q < q^* \le \sqrt{\beta^2+\gamma^{1/\kappa}} - \beta, \beta < (\gamma^{1/\kappa}-1)/2, \mu \in \mathbb{R}, \delta > 0, \kappa \in (0,1)\right\},$$

where q^* is a fixed integer.

Then, for any

$$b > \exp\left\{-\delta\gamma + \frac{\delta\left(\beta^2+\gamma^{1/\kappa}-(\beta+q^*)^2\right)^{\kappa} - q^*\delta\left(\beta^2+\gamma^{1/\kappa}-(\beta+1)^2\right)^{\kappa}}{1-q^*}\right\},$$

the sequence of stochastic processes (3.14) *converge in* $\mathcal{L}_q$ *to the stochastic process* $A(t)$ *for each fixed* $t \in [0,1]$ *as* $n \to \infty$ *such that* $A(1) \in \mathcal{L}_q$ *for* $q \in Q$, *and*

$$\mathrm{E}A^q(t) \sim t^{\varsigma(q)}, \quad q \in Q,$$

where the scaling function $\varsigma(q)$ *is given by*

$$\varsigma(q) = \left(1 - \frac{\delta\left[\left(\beta^2 + \gamma^{1/\kappa} - (\beta+1)^2\right)\right]^\kappa - \gamma}{\log b}\right) q + \\ + \frac{\delta}{\log b}\left(\beta^2 + \gamma^{1/\kappa} - (\beta+q)^2\right)^\kappa - \frac{\delta\gamma}{\log b}, q \in Q. \tag{6.8}$$

Moreover, (3.14) *holds, where* M *is given by (6.6).*

Theorem 7 follows from Theorem 3.

Note that, for $\kappa = 1/2$, Theorem 7 is an extension to Theorem 5 of Anh et al. (2008a), which is now extended to present a log-normal inverse Gaussian scenarios with scaling function:

$$\varsigma(q) = \left(1 - \frac{\delta\left[\sqrt{\beta^2 + \gamma^2 - (\beta+1)^2} - \gamma\right]}{\log b}\right) q + \\ + \frac{\delta}{\log b}\sqrt{\beta^2 + \gamma^2 - (\beta+q)^2} - \frac{\delta\gamma}{\log b}, q \in Q,$$

where we use the notation $NTS(\frac{1}{2}, \gamma, \beta, \mu, \delta) = NIG(\alpha, \beta, \delta, \mu)$, $\gamma = \sqrt{\alpha^2 - \beta^2}$,

$$Q = \left\{q : 0 < q < q^* \le \sqrt{\beta^2 + \gamma^2} - \beta, \beta < (\gamma^2 - 1)/2, \mu \in \mathbb{R}, \delta > 0\right\},$$

and

$$b > \exp\left\{-\delta\gamma + \frac{\delta\sqrt{\beta^2 + \gamma^2 - (\beta+q^*)^2} - q^*\delta\sqrt{\beta^2 + \gamma^2 - (\beta+1)^2}}{1 - q^*}\right\}.$$

We can construct log-normal tempered stable scenarios for a more general class of finite superpositions of stationary tempered stable OU-type processes (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $X_j(t) \sim NTS(\kappa, \gamma, \beta, \mu_j, \delta_j)$, $j = 1, ..., m$, and parameters $\mu_j, \delta_j, j = 1, ..., m$.. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{NTS(\kappa, \gamma, \beta, \sum_{j=1}^m \mu_j, \sum_{j=1}^m \delta_j); \mu_j + 2\kappa\beta\delta_j\gamma^{\frac{\kappa-1}{\kappa}}; \left[2\kappa\gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa\beta^2(\kappa-1)\gamma^{\frac{\kappa-2}{\kappa}}\right]\delta_j\}$$

Then the statement of Theorem 7 can be reformulated for $X_{m\sup}$ with $\mu = \sum_{j=1}^m \mu_j, \delta = \sum_{j=1}^m \delta_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E}\exp\left\{\zeta_1\left(X_{m\sup}(t_1) - c_X\right) + \zeta_2\left(X_{m\sup}(t_2) - c_X\right)\right\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1 X_{m\sup}(t_1) + \zeta_2\left(X_{m\sup}(t_2)\right)\right\}, \end{aligned}$$

where

$$\log \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^m \log \mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (3.9) with Lévy measure $\tilde{\nu}$ having density (6.4).

Moreover, one can construct log-normal tempered stable scenarios for a more general class of infinite superpositions of stationary normal tempered stable OU-type processes (4.10), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $X_j(t) \sim NTS(\kappa, \gamma, \beta, \mu_j, \delta_j), j = 1, 2..$ and parameters $\mu_j, \delta_j, j = 1, 2....$. Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{NTS(\kappa, \gamma, \beta, \sum_{j=1}^{\infty} \mu_j, \sum_{j=1}^{\infty} \delta_j); \mu_j + 2\kappa\beta\delta_j\gamma^{\frac{\kappa-1}{\kappa}}; \left[2\kappa\gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa\beta^2 (\kappa - 1) \gamma^{\frac{\kappa-2}{\kappa}}\right] \delta_j\}.$$

Then the statement of Theorem 7 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j, \mu = \sum_{j=1}^{\infty} \mu_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E} \exp\{\zeta_1 (X_{\sup}(t_1) - c_X) + \zeta_2 (X_{\sup}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1+\zeta_2)} \mathrm{E} \exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2 (X_{\sup}(t_2)\}, \end{aligned}$$

where

$$\log \mathrm{E} \exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2 X_{\sup}(t_2)\} = \sum_{j=1}^{\infty} \log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m, ...$ are given by (3.8) with Lévy measure $\tilde{\nu}$ having density (6.4).

7. Log-gamma scenario

The log-gamma multifractal scenario is well-known in the theory of turbulence and multiplicative cascades (Saito (1992)). In this section, we propose a stationary version of the log-gamma scenario. We will use a stationary OU type process $X(t), t \in \mathbb{R}_+$,(see, (3.4)) with marginal gamma distribution $\Gamma(\beta, \alpha)$. It is known that the gamma distribution with the moment generating function $\mathrm{E} \exp\{\zeta X(t)\} = \left(1 - \frac{\zeta}{\alpha}\right)^{-\beta}$, $\zeta < \alpha, \alpha > 0, \beta > 0$, is self-decomposable. The Lévy triplet is of the form $(0, 0, \nu)$, where

$$\nu(du) = \frac{\beta e^{-\alpha u}}{u} \mathbf{1}_{[0,\infty)}(u) du,$$

while the BDLP $Z(t)$ in (3.4) is a compound Poisson subordinator, that is

$$\kappa(z) = \log \mathrm{E} e^{izZ(1)} = \frac{i\beta z}{\alpha - iz}, z \in \mathbb{R},$$

and the (finite) Lévy measure $\tilde{\nu}$ of $Z(1)$ is

$$\tilde{\nu}(du) = \alpha\beta e^{-\alpha u} \mathbf{1}_{(0,\infty)}(u) du. \tag{7.1}$$

The covariance function is then $r_X(t) = (\beta/\alpha^2) \exp(-\lambda |t|)$.

Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X), \quad c_X = \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^{\beta}}, \alpha > 1,$$

where $X(t)$ is a stationary gamma OU type stochastic process.

We obtain the following moment generating function:

$$M(\zeta) = \mathrm{E} \exp(\zeta (X(t) - c_X)) = \frac{e^{-c_X \zeta}}{\left(1 - \frac{\zeta}{\alpha}\right)^{\beta}}, \ \zeta < \alpha, \alpha > 1, \tag{7.2}$$

and the bivariate moment generating function is given by the formula (3.9), in which the measure $\tilde{\nu}$ is given by (7.1), since

$$\begin{aligned} M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \mathrm{E}\exp\left(\zeta_1\left(X(t_1)-c_X\right)+\zeta_2\left(X(t_2)-c_X\right)\right) \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left(\zeta_1 X(t_1)+\zeta_2\left(X(t_2)\right)\right) \\ &= e^{-c_X(\zeta_1+\zeta_2)}\exp\left(\int_{\mathbb{R}}\frac{\beta\sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j-s)}{\alpha-\sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j-s)}ds\right), \end{aligned} \tag{7.3}$$

or

$$\begin{aligned} M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \exp\left(-c_X\left(\zeta_1+\zeta_2\right)\right) \\ &\times\exp\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg(\exp\Bigg(u\sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j-s)\Bigg)-1 \\ &-u\Bigg(\sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j-s)\Bigg)\mathbf{1}_{[-1,1]}\left(u\right)\Bigg)\alpha\beta e^{-\alpha u}\mathbf{1}_{(0,\infty)}(u)duds\Bigg). \end{aligned} \tag{7.4}$$

Thus, the correlation function of the mother process takes the form (6.7), where $M(2)$ is given by (7.2) and $M(1,1;\tau)$ is given by (7.4). It turns out that, in this case,

$$\log_b \mathrm{E}\Lambda\left(t\right)^q = \frac{1}{\log b}\left(-q\log\frac{1}{\left(1-\frac{1}{\alpha}\right)^{\beta}}-\beta\log\left(1-\frac{q}{\alpha}\right)\right),$$

and the condition (3.12) of Theorem 3 holds, since

$$\int_{|u|\geq 1} ue^{q^* u}\nu(du) = \frac{\alpha^{\beta}\beta}{\Gamma\left(\beta\right)}\int_1^{\infty} e^{q^* u}e^{-\alpha u}du < \infty, q^* < \alpha.$$

We formulate the following

Theorem 8. *Let $X\left(t\right)$ be a stationary gamma OU type stochastic process and let $Q = \{q: 0<q<q^* <\alpha, \alpha>2, \beta>0\}$, where q^* is a fixed integer. Then, for any*

$$b > \left[\left(1-\frac{1}{\alpha}\right)^{\beta q^*}/\left(1-\frac{q^*}{\alpha}\right)^{\beta}\right]^{\frac{1}{q^*-1}},$$

the stochastic processes $A_n\left(t\right)$ defined by (3.14) for the mother process as in condition $\boldsymbol{B'''}$ converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ as $n\to\infty$, such that $A\left(t\right)\in\mathcal{L}_q$ and

$$\mathrm{E}A(t)^q \sim t^{\varsigma(q)},$$

where the scaling function $\varsigma\left(q\right)$ is given by

$$\varsigma\left(q\right) = q\left(1+\frac{1}{\log b}\log\frac{1}{\left(1-\frac{1}{\alpha}\right)^{\beta}}\right)+\frac{\beta}{\log b}\log\left(1-\frac{q}{\alpha}\right), q\in Q. \tag{7.5}$$

Moreover, (3.16) holds, where M is given by (7.3) or (7.4).

Proof. Theorem 8 follows from Theorem 3 □

Note that Theorem 8 is an extension of the log-gamma scenario in Theorem 3 of Anh et al. (2008a), in which the set $Q = \{q : 0 < q \le 2, \alpha = 2, \beta > 0\}$.

We can construct log-tempered stable scenarios for a more general class of finite superpositions of stationary gamma OU-type processes (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $\Gamma(\beta_j, \alpha), j = 1, ..., m$. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{\Gamma(\sum_{j=1}^{m}\beta_j),\alpha);\frac{\beta_j}{\alpha};\frac{\beta_j}{\alpha^2}).$$

Theorem 8 can be reformulated for the process of superposition $X_{m\sup}$ with $\beta = \sum_{j=1}^{m}\beta_j$ and

$$\begin{aligned}M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \mathrm{E}\exp\left\{\zeta_1\left(X_{m\sup}(t_1)-c_X\right)+\zeta_2\left(X_{m\sup}(t_2)-c_X\right)\right\}\\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1 X_{m\sup}(t_1)+\zeta_2\left(X_{m\sup}(t_2)\right\}\right.,\end{aligned}$$

where

$$\log\mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1)+\zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^{m}\log\mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

and $\log\mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (7.3) or (7.4).

Moreover, one can construct log-gamma scenarios for a more general class of infinite superpositions of stationary gamma OU-type processes (4.10), where $X_j(t), j = 1, ...,$ are independent stationary processes with marginals $\Gamma(\beta_j, \alpha), j = 1, 2...$ Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{\Gamma((\sum_{j=1}^{\infty}\beta_j),\alpha);\frac{\beta_j}{\alpha};\frac{\beta_j}{\alpha^2})\}.$$

Then the statement of Theorem 8 remains true with $\beta = \sum_{j=1}^{\infty}\beta_j$ and

$$\begin{aligned}M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \mathrm{E}\exp\left\{\zeta_1\left(X_{\sup}(t_1)-c_X\right)+\zeta_2\left(X_{\sup}(t_2)-c_X\right)\right\}\\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1 X_{\sup}(t_1)+\zeta_2\left(X_{\sup}(t_2)\right\}\right.,\end{aligned}$$

where

$$\log\mathrm{E}\exp\{\zeta_1 X_{\sup}(t_1)+\zeta_2 X_{\sup}(t_2)\} = \sum_{j=1}^{\infty}\log\mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

and $\log\mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\}, j = 1, 2, ...$ are given by (7.3) or (7.4).

8. Log-variance gamma scenario

The next example of a hyperbolic OU process is based on the variance-gamma distribution (see, for example, Madan et al. (1998), Finlay, Seneta (2006), Carr et al. (2007)). We will use a stationary OU type process (3.4)) with marginal variance gamma distribution $VG(\kappa, \alpha, \beta, \mu)$, which has the moment generating function

$$\log\mathrm{E}e^{\zeta X(t)} = \mu\zeta + 2\kappa\log\left(\gamma/\sqrt{\alpha^2-(\beta+\zeta)^2}\right),\quad |\beta+\zeta|<\alpha,$$

where the set of parameters is of the form

$$\gamma^2 = \alpha^2 - \beta^2, \kappa > 0, \alpha > |\beta| > 0, \mu \in \mathbb{R}.$$

It is known that this distribution is self-decomposable. Note that

$$\mathrm{E}X(t) = \mu + \frac{2\beta\kappa}{\gamma^2}, \quad \mathrm{Var}X(t) = \frac{2\kappa}{\gamma^2}\left(1 + 2\left(\frac{\beta}{\gamma}\right)^2\right).$$

Thus, if $X_{j,}(t)\, j = 1, ..., m$, are independent so that $X_j \sim VG(\kappa_j, \alpha, \beta, \mu_j)$, $j = 1, ..., m$, then we have that

$$X_1(t) + ... + X_m(t) \sim VG(\kappa_1 + ... + \kappa_n, \alpha, \beta, \mu_1 + ... + \mu_n).$$

The Lévy measure ν of $X(t)$ has density

$$p(u) = \frac{\kappa}{|u|} e^{\beta u - \alpha|u|}, u \in \mathbb{R}. \tag{8.1}$$

By (3.6) the Lévy measure $\tilde{\nu}$ of the BDLP $Z(t)$ in (3.4) has density

$$q(u) = -p(u) - up'(u), \tag{8.2}$$

$$p'(u) = \begin{cases} -\frac{\kappa}{u} e^{u(\beta+\alpha)}(\beta+\alpha) + \frac{\kappa}{u^2} e^{u(\beta+\alpha)}, & u < 0, \\ \frac{\kappa}{u} e^{u(\beta-\alpha)}(\beta-\alpha) - \frac{\kappa}{u^2} e^{u(\beta-\alpha)}, & u > 0. \end{cases}$$

Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X), c_X = \mu + 2\kappa \log\left(\gamma / \sqrt{\alpha^2 - (\beta+1)^2}\right), \quad |\beta + 1| < \alpha,$$

where $X(t)$ is a stationary $VG(\kappa, \alpha, \beta, \mu)$ OU type process with covariance function

$$R_X(t) = \frac{2\kappa}{\gamma^2}\left(1 + 2\left(\frac{\beta}{\gamma}\right)^2\right) \exp(-\lambda|t|).$$

We obtain the moment generating function

$$M(\zeta) = \mathrm{E}\exp(\zeta(X(t) - c_X)) = e^{-c_X\zeta} e^{\mu\zeta + 2\kappa\log\left(\gamma/\sqrt{\alpha^2 - (\beta+\zeta)^2}\right)}, \quad |\beta + \zeta| < \alpha, \tag{8.3}$$

and the bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E}\exp(\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)) \\ &= e^{-c_X(\zeta_1+\zeta_2)} \mathrm{E}\exp(\zeta_1 X(t_1) + \zeta_2(X(t_2))), \end{aligned} \tag{8.4}$$

where $\mathrm{E}\exp(\zeta_1 X(t_1) + \zeta_2(X(t_2))$ is given by (3.9) with Lévy measure $\tilde{\nu}$ having density (8.2). Thus, the correlation function of the mother process takes the form (6.7), where $M(2)$ is given by (8.3) and $M(1, 1; \tau)$ is given by (8.4).

The condition (3.12) of Theorem *3* holds for $q < \alpha - |\beta|$.

Theorem 9. *Let $X(t)$ be a stationary $VG(\kappa, \alpha, \beta, \mu)$ OU type process and let*

$$Q = \{q : 0 < q < q^* < |\alpha| - |\beta|, \kappa > 0\},$$

where q^* is a fixed integer.

Then, for any

$$b > \exp\left\{2\kappa\left[\frac{1}{1-q^*} \log\frac{\gamma}{\sqrt{\alpha^2 - (\beta+q^*)^2}} + \frac{q^*}{1-q^*} \log\frac{\gamma}{\sqrt{\alpha^2 - (\beta+1)^2}}\right]\right\},$$

the stochastic processes $A_n(t)$ defined by (3.14) for the mother process as in condition $\boldsymbol{B}''''$ *converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ as $n \to \infty$ such that, if $A(1) \in \mathcal{L}_q$ for $q \in Q$,*

$$\mathrm{E}A(t)^q \sim t^{\varsigma(q)},$$

where the scaling function is given by

$$\varsigma(q) = q\left(1 + \frac{2\kappa}{\log b}\log\frac{\gamma}{\sqrt{\alpha^2 - (\beta+1)^2}}\right) + \frac{2\kappa}{\log b}\log\sqrt{\alpha^2 - (\beta+q)^2} - \frac{2\kappa}{\log b}\log\gamma. \tag{8.5}$$

Moreover, (3.16) *holds, where M is given by (8.4).*

Proof. Theorem 9 follows from Theorem 3. □

We can construct log-variance-gamma scenarios for a more general class of finite superpositions of stationary variance gamma OU-type processes (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $VG(\kappa_j, \alpha, \beta, \mu_j), j = 1, ..., m$. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{VG(\kappa_1 + ... + \kappa_m, \alpha, \beta, \delta, \mu_1 + ... + \mu_m); \mu_j + \frac{2\beta}{\gamma^2}\kappa_j; \frac{2}{\gamma^2}\left(1 + 2\left(\frac{\beta}{\gamma}\right)^2\right)\kappa_j\}.$$

The generalization of Theorem 9 remains true for this situation with $\kappa = \sum_{j=1}^m \kappa_j$, $\mu = \sum_{j=1}^m \mu_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E}\exp\{\zeta_1(X_{m\sup}(t_1) - c_X) + \zeta_2(X_{m\sup}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2(X_{m\sup}(t_2)\}, \end{aligned}$$

where

$$\log\mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^m \log\mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log\mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (8.4).

We can construct log-variance-gamma scenarios for a more general class of infinite superpositions of stationary variance gamma OU-type processes (4.10), where $X_j(t), j = 1, ...,$ are independent stationary processes with marginals $VG(\kappa_j, \alpha, \beta, \mu_j)$. Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{VG\left(\sum_{j=1}^\infty \kappa_j, \alpha, \beta, \sum_{j=1}^\infty \mu_j\right); \mu_j + \frac{2\beta}{\gamma^2}\kappa_j; \frac{2}{\gamma^2}\left(1 + 2\left(\frac{\beta}{\gamma}\right)^2\right)\kappa_j\}.$$

Then the statement of Theorem 15 remains true with $\mu = \sum_{j=1}^\infty \mu_j, \kappa = \sum_{j=1}^\infty \kappa_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E}\exp\{\zeta_1(X_{\sup}(t_1) - c_X) + \zeta_2(X_{\sup}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2(X_{\sup}(t_2)\}, \end{aligned}$$

where

$$\log\mathrm{E}\exp\{\zeta_1 X_{\sup}(t_1) + \zeta_2 X_{\sup}(t_2)\} = \sum_{j=1}^\infty \log\mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log\mathrm{E}\exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., ...$ are given by (8.4).

9. Log-Euler's gamma multifractal scenario

This section presents a new scenario which is based on Euler's gamma distribution (see, for example, Grigelionis (2001)). We consider the random variable Y with the gamma distribution $\Gamma(\beta,\alpha)$ and the random variable

$$X_\gamma = \gamma \log Y, \gamma \neq 0,$$

which has the pdf

$$\pi\left(x\right) = \frac{\alpha^\beta}{\left|\gamma\right| \Gamma\left(\beta\right)} \exp\left\{\frac{\beta x}{\gamma} - \alpha e^{\frac{x}{\gamma}}\right\}, \quad x\in\mathbb{R}, \tag{9.1}$$

where the parameters satisfy

$$\alpha>0, \beta>0, \gamma\neq 0.$$

The characteristic function of random variable X with pdf (9.1) is

$$\mathrm{E}e^{izX_\gamma} = \frac{\Gamma(\beta+i\gamma z)}{\Gamma\left(\beta\right)\alpha^{i\gamma z}}, z\in\mathbb{R}.$$

Grigelionis (2003) proved that for

$$\delta>0, \alpha>0, \beta>0, \gamma\neq 0, \tag{9.2}$$

the function

$$\mathrm{E}e^{izX} = \left(\frac{\Gamma(\beta+i\gamma z)}{\Gamma\left(\beta\right)\alpha^{i\gamma z}}\right)^\delta, z\in\mathbb{R} \tag{9.3}$$

is a self-decomposable characteristic function. We denote the distribution of the random variable X by $\Gamma(\gamma,\alpha,\beta,\delta)$.

We note that $\Gamma(\gamma, e^{-\frac{\theta}{\gamma}}, 1, 1), \theta\in\mathbb{R}$, is the Gumbel distribution with location parameter θ and scale parameter $\left|\gamma\right|$, since with

$$\Lambda(x) = \exp\left\{-e^{-x}\right\}, \bar{\Lambda}(x) = 1-\Lambda(-x), x\in\mathbb{R},$$

$$\mathrm{P}\left\{X\leq x\right\} = \begin{cases} \Lambda\left(\frac{x-\theta}{\left|\gamma\right|}\right), & \gamma<0, \\ \bar{\Lambda}\left(\frac{x-\theta}{\gamma}\right), & \gamma>0 \end{cases}, x\in\mathbb{R}.$$

We will use a stationary OU-type process (3.4) with marginal distribution $\Gamma(\gamma,\alpha,\beta,\delta)$, which is self-decomposable, and hence infinitely divisible. It means that the characteristic function of $X(t), t\in[0,1]$ is of the form (9.3) under the set of parameters (9.2). Note that, for $\beta>0$, we have

$$\Gamma(\beta+iz) = \Gamma(\beta)\exp\{iz\int\limits_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-\beta x}}{1-e^{-x}} 1_{\{0\leq x\leq 1\}}\right)dx$$

$$+ \int\limits_{-\infty}^0 (e^{izx} - 1 - izx 1_{\{-1\leq x<0\}}) \frac{e^{\beta x}}{\left|x\right|\left(1-e^x\right)} dx\},$$

and thus the distribution corresponding to the characteristic function (9.3) has the Lévy triplet $(\delta a, 0, \nu)$, where

$$a = \gamma \int_0^{\frac{1}{\left|\gamma\right|}} \left(\frac{e^{-x}}{x} - \frac{e^{-\beta x}}{1-e^{-x}}\right)dx + \gamma\int_{\frac{1}{\left|\gamma\right|}}^\infty \frac{e^{-x}}{x}dx - \gamma\log\alpha,$$

and

$$\nu(du) = \delta b(u)du, \tag{9.4}$$

$$b(u) = \begin{cases} \frac{e^{\frac{\beta}{\gamma}u}}{|u|\left(1-e^{\frac{1}{\gamma}u}\right)}, & u<0, \gamma>0, \\ \frac{e^{\frac{\beta}{\gamma}u}}{u\left(1-e^{\frac{1}{\gamma}u}\right)}, & u>0, \gamma<0. \end{cases}$$

Thus, if $X_j(t),\ \ j = 1, ..., m$, are independent so that $X_j(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_j)$, $j = 1, ..., m$, then we have that

$$X_1(t) + ... + X_m(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_1 + ... + \delta_m)$$

and if $X_j(t),\ \ j = 1, ..., m$, are independent so that $X_j(t) \sim \Gamma(\gamma, \alpha_j, \beta, \delta)$, $j = 1, ..., m$, then

$$X_1(t) + ... + X_m(t) \sim \Gamma(\gamma, \prod_{j=1}^{m} \alpha_j, \beta, \delta).$$

The BDLP $Z(t)$ in (3.4) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, where

$$\tilde{a} = \gamma\lambda\delta \frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)} + \gamma\lambda\delta\log\alpha - \lambda\delta\int_{|x|>1} x\omega(x)dx,$$

with the density of $\tilde{\nu}$ given by

$$\tilde{\nu}(du) = \lambda\delta\omega(u)du, \tag{9.5}$$

$$\omega(u) = \begin{cases} \frac{\beta}{\gamma}e^{\frac{\beta}{\gamma}u}\left(1 - e^{\frac{1}{\gamma}u} + \frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1-e^{\frac{1}{\gamma}u}\right)^2}, & \gamma>0, u<0, \\ \frac{\beta}{|\gamma|}e^{\frac{\beta}{\gamma}u}\left(1 - e^{\frac{1}{\gamma}u} + \frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1-e^{\frac{1}{\gamma}u}\right)^2}, & \gamma<0, u>0. \end{cases}$$

The correlation function of the stationary process $X(t)$ then takes the form

$$r_X(t) = \exp\{-\lambda|t|\}.$$

Note that

$$\mathrm{E}X(t) = \gamma\delta\frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)} - \gamma\delta\log\alpha, \mathrm{Var}X(t) = \delta\gamma^2\int_0^\infty \frac{xe^{-\beta x}}{1-e^{-x}}dx.$$

Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\},$$

with

$$c_X = \delta\log\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)e^{\gamma}}, \beta>0, \gamma<0, \beta>-\gamma.$$

where $X(t)$ is a stationary $\Gamma(\gamma, \alpha, \beta, \delta)$ OU-type stochastic process with covariance function

$$R_X(t) = \mathrm{Var}X(t)\exp\{-\lambda|t|\},$$

The logarithm of the moment generating function of $\Gamma(\gamma, \alpha, \beta, \delta)$ is

$$\log \mathrm{E}e^{\zeta X(t)} = \delta\log\frac{\Gamma(\beta+\gamma\zeta)}{\Gamma(\beta)e^{\gamma\zeta}}, 0<\zeta<-\frac{\beta}{\gamma}, \beta>0, \gamma<0.$$

We obtain the following moment generating function

$$M\left(\zeta\right)=\mathrm{E}\exp\left\{\zeta\left(X\left(t\right)-c_X\right)\right\}=e^{-c_X\zeta}e^{M(\zeta)},\quad 0<\zeta<-\frac{\beta}{\gamma},\tag{9.6}$$

and bivariate moment generating function

$$\begin{aligned}M\left(\zeta_1,\zeta_2;(t_1-t_2)\right)&=\mathrm{E}\exp\left\{\zeta_1\left(X(t_1)-c_X\right)+\zeta_2\left(X(t_2)-c_X\right)\right\}\\&=e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1X(t_1)+\zeta_2\left(X(t_2)\right\}\right.,\end{aligned}\tag{9.7}$$

where $\mathrm{E}\exp\left\{\zeta_1X(t_1)+\zeta_2\left(X(t_2)\right\}\right.$ is given by (3.8) with Lévy measure $\tilde{\nu}$ having density (9.5). Thus, the correlation function of the mother process takes the form (6.7), where $M_\theta(2)$ is given by (9.6) and $M_\theta(1,1;\tau)$ is given by (9.7).

Theorem 10. *Let $X\left(t\right)$ be a stationary $\Gamma(\gamma,\alpha,\beta,\delta)$ OU type process and let*

$$q\in Q=\left\{q:0<q<q^*\leq-\frac{\beta}{\gamma},\beta>0,\gamma<0,\beta>-\gamma,\delta>0,\alpha>0\right\}.$$

where q^* is a fixed integer. Then, for any

$$b>\left[\frac{\Gamma^{\delta}\left(\beta+\gamma q^*\right)\Gamma^{q^*}(\beta)}{\Gamma(\beta)\Gamma^{q^*}\left(\beta+\gamma\right)}\right]^{\frac{1}{q^*-1}}$$

the stochastic processes (3.14) converge in $\mathcal{L}_q$ to the stochastic process $A(t)$, $t\in[0,1]$ as $n\to\infty$ such that, if $A(1)\in\mathcal{L}_q$, and $q\in Q$,

$$\mathrm{E}A^q(t)\sim t^{\varsigma(q)},$$

where the scaling function is given by

$$\varsigma\left(q\right)=q\left(1+\frac{\delta}{\log b}\log\Gamma\left(\beta+\gamma\right)-\frac{\delta}{\log b}\log\Gamma\left(\beta\right)\right)-\frac{\delta}{\log b}\log\Gamma\left(\beta+q\gamma\right),q\in Q.\tag{9.8}$$

Moreover, (3.16) holds, where M_θ is given by (9.7).

Proof. Theorem 10 follows from Theorem 3. $\square$

We can construct log-Euler's gamma scenarios for a more general class of finite superpositions of Euler's gamma OU-type processes (4.2), where $X_j(t), j=1,...,m$, are independent stationary processes with marginals $\Gamma(\gamma,\alpha,\beta,\delta), j=1,...,m$. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{\Gamma(\gamma,\alpha,\beta,\delta_j);\delta_j(\gamma\frac{\frac{d}{d\beta}\Gamma\left(\beta\right)}{\Gamma\left(\beta\right)}-\gamma\log\alpha);\delta_j\gamma^2\int_0^\infty\frac{xe^{-\beta x}}{1-e^{-x}}dx\}.$$

The generalization of Theorem 9 remains true for this situation with $\delta=\sum_{j=1}^m\delta_j$, and

$$\begin{aligned}M\left(\zeta_1,\zeta_2;(t_1-t_2)\right)&=\mathrm{E}\exp\left\{\zeta_1\left(X_{m\sup}(t_1)-c_X\right)+\zeta_2\left(X_{m\sup}(t_2)-c_X\right)\right\}\\&=e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1X_{m\sup}(t_1)+\zeta_2\left(X_{m\sup}(t_2)\right\}\right.,\end{aligned}$$

where

$$\log\mathrm{E}\exp\{\zeta_1X_{m\sup}(t_1)+\zeta_2X_{m\sup}(t_2)\}=\sum_{j=1}^m\log\mathrm{E}\exp\{\zeta_1X_j(t_1)+\zeta_2X_j(t_2)\},$$

and $\log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (8.4).

We can construct log-Euler's-gamma scenarios for a more general class of infinite superpositions of stationary Euler's gamma OU-type processes (4.10), where $X_j(t), j = 1, ...,$ are independent stationary processes with marginals $\Gamma(\gamma, \alpha, \beta, \delta_j)$. Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{\Gamma(\gamma, \alpha, \beta, \delta_j); \delta_j(\gamma \frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)} - \gamma \log \alpha); \delta_j \gamma^2 \int_0^\infty \frac{xe^{-\beta x}}{1 - e^{-x}} dx\}.$$

Then the statement of Theorem 10 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j$.

and

$$M(\zeta_1, \zeta_2; (t_1 - t_2)) = \mathrm{E} \exp\{\zeta_1 (X_{m\sup}(t_1) - c_X) + \zeta_2 (X_{m\sup}(t_2) - c_X)\}$$
$$= e^{-c_X(\zeta_1+\zeta_2)} \mathrm{E} \exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 (X_{m\sup}(t_2)\},$$

where

$$\log \mathrm{E} \exp\{\zeta_1 X_{m\sup}(t_1) + \zeta_2 X_{m\sup}(t_2)\} = \sum_{j=1}^{m} \log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, .., m$ are given by (9.7).

10. Log-z scenario

The next scenario is based on the z-distribution (see, for example, Grigelionis (2001)). We consider a pdf of the form

$$\pi(x) = \frac{2\pi \exp\left(\frac{2\pi\beta_1}{\alpha}(x - \mu)\right)}{\alpha B(\beta_1, \beta_2)\left(1 + \exp\left(\frac{2\pi}{\alpha}(x - \mu)\right)\right)^{\beta_1+\beta_2}}, \quad x \in \mathbb{R}, \tag{10.1}$$

where the set of parameters is

$$\alpha > 0, \beta_1 > 0, \beta_2 > 0, \mu \in \mathbb{R}$$

(see Prentice (1975),Barndorf-Nilsen et al. (1982)). The characteristic function of a random variable X with pdf (10.1) is given by

$$\mathrm{E}e^{izX} = \frac{B(\beta_1 + \frac{i\alpha z}{2\pi}, \beta_2 - \frac{i\alpha z}{2\pi})}{B(\beta_1, \beta_2)} e^{iz\mu}, \ z \in \mathbb{R}.$$

This distribution has semiheavy tails and is known to be self-decomposable (Barndorf-Nilsen et al. (1982)), hence is infinitely divisible. Due to this infinite divisibility of the z-distribution, the following generalization can be suggested.

We will use a stationary OU type process (3.4) with marginal generalized z-distribution $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$. The characteristic function of $X(t), \ t \in \mathbb{R}_+$ is then of the form

$$\mathrm{E}e^{izX} = \left(\frac{B(\beta_1 + \frac{i\alpha z}{2\pi}, \beta_2 - \frac{i\alpha z}{2\pi})}{B(\beta_1, \beta_2)}\right)^{2\delta} e^{iz\mu}, \ z \in \mathbb{R}, \tag{10.2}$$

where the set of parameters is

$$\alpha > 0, \beta_1 > 0, \beta_2 > 0, \delta > 0, \mu \in \mathbb{R}.$$

This distribution is self-decomposable, hence infinitely divisible, with the Lévy triplet $(a, 0, \nu)$, where

$$a = \frac{\alpha\delta}{\pi}\int_0^{\frac{2\pi}{\alpha}} \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1-e^{-x}}dx + \mu,$$

and

$$\nu(du) = b(u)du, \tag{10.3}$$

$$b(u) = \begin{cases} \frac{2\delta e^{-\frac{2\pi\beta_2}{\alpha}u}}{u\left(1-e^{-\frac{2\pi}{\alpha}u}\right)}, & u > 0, \\ \frac{2\delta e^{\frac{2\pi\beta_1}{\alpha}u}}{|u|\left(1-e^{\frac{2\pi}{\alpha}u}\right)}, & u < 0. \end{cases}$$

Thus, if $X_j(t),\ \ j = 1, ..., m$, are independent so that $X_j(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j)$, $j = 1, ..., m$, then we have that

$$X_1(t) + ... + X_m(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_1 + ... + \delta_n, \mu_1 + ... + \mu_n).$$

The BDLP $Z(t)$ has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, where

$$\tilde{a} = \lambda\mu + \frac{\alpha\lambda\delta}{\pi}\int_0^{\infty} \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1-e^{-x}}dx - \lambda\int_{|x|>1} x\omega(x)dx,$$

with the density of $\tilde{\nu}$ being given from

$$\tilde{\nu}(du) = \lambda\omega(u)du,$$

$$\omega(u) = \begin{cases} \frac{4\pi\delta}{\alpha}\left(\beta_2 e^{-\frac{2\pi\beta_2}{\alpha}u}\left(1-e^{-\frac{2\pi}{\alpha}u}\right) + e^{-\frac{2\pi(\beta_2+1)}{\alpha}u}\right)\frac{1}{\left(1-e^{-\frac{2\pi}{\alpha}u}\right)^2}, & u > 0, \\ \frac{4\pi\delta}{\alpha}\left(\beta_1 e^{\frac{2\pi\beta_1}{\alpha}u}\left(1-e^{\frac{2\pi}{\alpha}u}\right) + e^{\frac{2\pi(\beta_1+1)}{\alpha}u}\right)\frac{1}{\left(1-e^{\frac{2\pi}{\alpha}u}\right)^2} & u < 0. \end{cases} \tag{10.4}$$

The correlation function of the stationary process with marginal density (10.1) is then

$$r_X(t) = \exp(-\lambda|t|).$$

The pdf of the generalized z-distribution $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ has semiheavy tails:

$$\pi(x) \sim C_{\pm}|x|^{\rho_{\pm}} e^{-\sigma_{\pm}|x|}, |x| \to \pm\infty,$$

where

$$\rho_{\pm} = 2\delta - 1, \sigma_+ = \frac{2\pi\beta_2}{\alpha}, \sigma_- = \frac{2\pi\beta_1}{\alpha}, C_{\pm} = \left(\frac{2\pi}{\alpha B(\beta_1, \beta_2)}\right)^{2\delta}\frac{e^{\pm\mu\sigma_{\pm}}}{\Gamma(2\delta)}.$$

Note that

$$\mathrm{E}X(t) = \frac{\alpha\delta}{\pi}\int_0^{\infty}\frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1-e^{-x}}dx + \mu, \mathrm{Var}X(t) = \frac{2\alpha^2\delta}{(2\pi)^2}\int_0^{\infty} x\frac{e^{-\beta_2 x} + e^{-\beta_1 x}}{1-e^{-x}}dx.$$

In particular, the generalized z-distribution $Z(\alpha, \frac{1}{2} + \frac{\beta}{2\pi}, \frac{1}{2} - \frac{\beta}{2\pi}, \delta, \mu) = M(\alpha, \beta, \delta, \mu)$ is known as the Meixner distribution (Schoutens, Teugels (1998), Grigelionis (1999), Morales, Schoutens (2003)). The density function of a Meixner distribution is given by

$$\pi(x) = \frac{\left(2\cos\frac{\beta}{2}\right)^{2\delta}}{2\pi\alpha\Gamma(2\delta)}\exp\left(\frac{\beta}{\alpha}(x-\mu)\right)\left|\Gamma\left(\delta + i\frac{x-\mu}{\alpha}\right)\right|^2, \quad x \in \mathbb{R},$$

where

$$\alpha > 0, -\pi < \beta < \pi, \delta > 0, \mu \in \mathbb{R}.$$

Note that

$$|\Gamma(x+iy)|^2 \sim \sqrt{2\pi}\,|y|^{x-1/2}\,e^{-\pi|y|/2} \text{ as } |y| \longrightarrow \infty.$$

This distribution is infinitely divisible and self-decomposable with triplet $(a, 0, \nu)$, where

$$a = \alpha\delta\tan\frac{\beta}{2} - 2\delta\int_1^\infty \frac{\sinh(\beta x/2)}{\sinh(\pi x/2)}dx + \mu$$

and

$$\nu(du) = \frac{\delta\exp\left(\frac{\beta u}{\alpha}\right)}{u\sinh\left(\frac{\pi u}{\alpha}\right)}du.$$

The cumulant function is

$$C\{z; X(t)\} = i\mu z + 2\delta\log\frac{\cos\beta/2}{\cosh((\alpha z - i\beta)/2)},\ z\in\mathbb{R}.$$

In particular, the hyperbolic cosine distribution $Z(\alpha, \frac{1}{2}, \frac{1}{2}, 0, \mu) = M\left(\alpha, 0, \frac{1}{2}, \mu\right)$ has the pdf

$$\pi(x) = \frac{1}{\alpha\cosh\left(\frac{\pi}{\alpha}(x-\mu)\right)},\ x\in\mathbb{R}$$

and characteristic function

$$\mathrm{E}e^{izX(t)} = e^{iz\mu}\frac{1}{\cosh\left(\frac{\alpha z}{2}\right)},\ z\in\mathbb{R},$$

while the logistic distribution $Z(\alpha, 1, 1, 0, \mu)$ has the pdf

$$\pi(x) = \frac{2\pi\exp\left(\frac{\pi}{\alpha}(x-\mu)\right)}{\alpha\left(1+\cosh\left(\frac{\pi}{\alpha}(x-\mu)\right)\right)},\ x\in\mathbb{R}$$

and characteristic function

$$\mathrm{E}e^{izX(t)} = e^{iz\mu}\frac{\alpha z}{2\sinh\left(\frac{\alpha z}{2}\right)}, z\in\mathbb{R}.$$

Another example is the z-distribution $Z(2\pi, \frac{k_1}{2}, \frac{k_2}{2}, 0, \log\frac{k_1}{k_2})$, which is the log F_{k_1,k_2} distribution, where F_{k_1,k_2} is the Fisher distribution Barndorf-Nilsen et al. (1982)). Note that the generalized z-distributions and generalized hyperbolic distributions form non-intersecting sets. However, one can show that some Meixner distributions and corresponding Lévy processes can be obtained by subordination, that is, by random time change in the Brownan motion (see, for instance,Morales, Schoutens (2003)).

Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X),$$

with

$$c_X = 2\delta\left(\log\Gamma\left(\beta_1 + \frac{\alpha}{2\pi}\right) + \log\Gamma\left(\beta_2 - \frac{\alpha}{2\pi}\right) - \log\frac{\Gamma(\beta_1)}{\Gamma(\beta_2)}\right) + \mu,$$

where $X(t),\ t\in\mathbb{R}_+$ is a stationary $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ OU-type process with covariance function

$$R_X(t) = (\mathrm{Var}X(t))\exp(-\lambda|t|).$$

The logarithm of the moment generating function of $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ is

$$\log\mathrm{E}e^{\zeta X(t)} = 2\delta\left(\log\Gamma\left(\beta_1 + \frac{\alpha\zeta}{2\pi}\right) + \log\Gamma\left(\beta_2 - \frac{\alpha\zeta}{2\pi}\right) - \log\frac{\Gamma(\beta_1)}{\Gamma(\beta_2)}\right) + \mu\zeta,\quad \zeta\in\left(-\frac{2\pi\beta_2}{\alpha}, \frac{2\pi\beta_1}{\alpha}\right).$$

Then we obtain the moment generating function

$$M(\zeta) = \mathrm{E}\exp(\zeta(X(t) - c_X)) = e^{-c_X\zeta} e^{K\{\zeta; X(t)\}}, \tag{10.5}$$

and the bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathrm{E}\exp(\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)) \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp(\zeta_1 X(t_1) + \zeta_2(X(t_2))), \end{aligned} \tag{10.6}$$

where $\mathrm{E}\exp(\zeta_1 X(t_1) + \zeta_2(X(t_2)))$ is given by (3.8) with Lévy measure $\tilde{\nu}$ having density (10.4). Thus, the correlation function of the mother process takes the form (6.7), where $M(2)$ is given by (10.5) and $M(1, 1; \tau)$ is given by (10.6).

Theorem 11. *Let $X(t)$ be a stationary OU type process with z-distribution and let*

$$Q = \left\{ q \in (0, q^*) : 0 < \frac{2\pi\beta_1}{\alpha} < q^* < \frac{2\pi\beta_2}{\alpha}, \beta_1 < \beta_2, \beta_1 + \frac{\alpha}{2\pi} > 0, \beta_2 - \frac{\alpha}{2\pi} > 0 \right\},$$

where q^* is a fixed integer.
Then, for any

$$b > \left[\frac{\Gamma(\beta_1)/\Gamma(\beta_2)}{\Gamma\left(\beta_1 + \frac{q^*\alpha}{2\pi}\right)\Gamma\left(\beta_2 - \frac{q^*\alpha}{2\pi}\right)} \right]^{2\delta},$$

the stochastic processes $A_n(t)$ defined by (3.14) for the mother process (5.1) converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ as $n \to \infty$ such that, if $A(1) \in \mathcal{L}_q$ for $q \in Q$,

$$\mathrm{E}A(t)^q \sim t^{\varsigma(q)},$$

where the scaling function is given by

$$\begin{aligned} \varsigma(q) = q &\left(1 + \frac{2\delta\left(\log\Gamma\left(\beta_1 + \frac{\alpha}{2\pi}\right) + \log\Gamma\left(\beta_2 - \frac{\alpha}{2\pi}\right) - \log\frac{\Gamma(\beta_1)}{\Gamma(\beta_2)}\right)}{\log b} \right) \\ &- \frac{2\delta}{\log b}\left(\log\Gamma\left(\beta_1 + \frac{q\alpha}{2\pi}\right) + \log\Gamma\left(\beta_2 - \frac{q\alpha}{2\pi}\right)\right) + \frac{1}{\log b}2\delta\log\frac{\Gamma(\beta_1)}{\Gamma(\beta_2)}. \end{aligned} \tag{10.7}$$

Moreover, (3.16) holds, where M is given by (10.6).

Proof. Theorem 11 follows from Theorem 3. □

We can construct log-z scenarios for a more general class of finite superpositions of stationary OU-type processes of the form (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals
$X_j(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j)$, $j = 1, ..., m$ and parameters δ_j, μ_j, $j = 1, ..., m$. Then $X_{m\sup}(t), t \in \mathbb{R}_+$ has the marginal distribution $Z(\alpha, \beta_1, \beta_2, \sum_{j=1}^{m}\delta_j, \sum_{j=1}^{m}\mu_j)$.

We can construct log-z scenarios for a more general class of finite superpositions of Euler's gamma OU-type processes (4.2), where $X_j(t), j = 1, ..., m$, are independent stationary processes with marginals $Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j), j = 1, ..., m$. Using notation (4.4), we consider the class of processes

$$\mathbb{FS}_m\{Z(\alpha,\beta_1,\beta_2,\delta_j,\mu_j);(\frac{\alpha}{\pi}\int_0^\infty \frac{e^{-\beta_2 x}-e^{-\beta_1 x}}{1-e^{-x}}dx)\delta_j+\mu_j;(\frac{2\alpha^2}{(2\pi)^2}\int_0^\infty x\frac{e^{-\beta_2 x}+e^{-\beta_1 x}}{1-e^{-x}}dx)\delta_j\}.$$

The generalization of Theorem 11 remains true for this situation with $\delta=\sum_{j=1}^m \delta_j, \mu=\sum_{j=1}^m \mu_j$ and

$$\begin{aligned} M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \mathrm{E}\exp\left\{\zeta_1\left(X_{m\sup}(t_1)-c_X\right)+\zeta_2\left(X_{m\sup}(t_2)-c_X\right)\right\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1 X_{m\sup}(t_1)+\zeta_2\left(X_{m\sup}(t_2)\right\}\right., \end{aligned}$$

where

$$\log \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1)+\zeta_2 X_{m\sup}(t_2)\}=\sum_{j=1}^m \log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\}, j=1,..,m$ are given by (8.4).

We can construct log-z scenarios for a more general class of infinite superpositions of stationary OU-type processes (4.10), where $X_j(t), j=1,...,$ are independent stationary processes with marginals $Z(\alpha,\beta_1,\beta_2,\delta_j,\mu_j)$. Using notation (4.16), we consider the class of processes

$$\mathbb{IS}\{Z(\alpha,\beta_1,\beta_2,\delta_j,\mu_j);(\frac{\alpha}{\pi}\int_0^\infty \frac{e^{-\beta_2 x}-e^{-\beta_1 x}}{1-e^{-x}}dx)\delta_j+\mu_j;(\frac{2\alpha^2}{(2\pi)^2}\int_0^\infty x\frac{e^{-\beta_2 x}+e^{-\beta_1 x}}{1-e^{-x}}dx)\delta_j\}.$$

Then the statement of Theorem 11 remains true with $\delta=\sum_{j=1}^\infty \delta_j, \mu=\sum_{j=1}^\infty \mu_j$, and

$$\begin{aligned} M\left(\zeta_1,\zeta_2;(t_1-t_2)\right) &= \mathrm{E}\exp\left\{\zeta_1\left(X_{m\sup}(t_1)-c_X\right)+\zeta_2\left(X_{m\sup}(t_2)-c_X\right)\right\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\left\{\zeta_1 X_{m\sup}(t_1)+\zeta_2\left(X_{m\sup}(t_2)\right\}\right., \end{aligned}$$

where

$$\log \mathrm{E}\exp\{\zeta_1 X_{m\sup}(t_1)+\zeta_2 X_{m\sup}(t_2)\}=\sum_{j=1}^m \log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

and $\log \mathrm{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\}, j=1,..,m$ are given by (9.7).

In principle, it is possible to obtain log-hyperbolic scenarios for which there exist exact forms of Lévy measures of the OU process and the BDLP Lévy process; however some analytical work is still to be carried out. This will be done elsewhere.

11. Summary of scenarios

For the reader convenience we summarise the scaling functions considered in the paper in the following table

Distribution	Scaling function
$TS(\kappa,\delta,\gamma)$	(5.7)
$NTS(\kappa,\gamma,\beta,\mu,\delta)$	(6.8)
$\Gamma(\alpha,\beta)$	(7.5)
$VG(\kappa,\alpha,\beta,\mu)$	(8.5)
$\Gamma(\gamma,\alpha,\beta,\delta)$	(9.8)
$Z(\alpha,\beta_1,\beta_2,\delta,\mu)$	(10.7)

Acknowledgements

The authors would like to thank the referee for the valuable suggestions.

N. Leonenko was supported in particular by Cardiff Incoming Visiting Fellowship Scheme and International Collaboration Seedcorn Fund, Australian Research Council's Discovery Projects funding scheme (project number DP160101366) and by project MTM2012-32674 (co-funded with FEDER) of the DGI, MINECO, Spain.

References

Anh V., Lau K.-S., Yu Z.-G. Multifractal characterization of complete genomes // J. Phys. A. 2001. 34, 36. 7127–7139.

Anh V. V., Leonenko N. N. Log-normal, log-gamma and log-negative inverted gamma scenarios in multifractal products of stochastic processes // Statist. Probab. Lett. 2008. 78. 1274–1282.

Anh V. V., Leonenko N. N., Shieh N.-R. Multifractality of products of geometric Ornstein-Uhlenbeck-type processes // Adv. in Appl. Probab. 2008a. 40. 1129–1156.

Anh V. V., Leonenko N. N., Shieh N.-R. Log-Euler' s Gamma scenario for products of Ornstein-Uhlenbeck processes // Mathematical Communications. 2008b. 13, 2. 133–148.

Anh V. V., Leonenko N. N., Shieh N.-R. Multifractal scaling of products of birth-death processes // Bernoulli. 2009a. 15. 508–531.

Anh V. V., Leonenko N. N., Shieh N.-R. Multifractal products of stationary diffusion processes // Stoch.Anal. Appl. 2009b. 27. 475–499.

Anh V. V., Leonenko N. N., Shieh N.-R. Multifractal scenarios for products of geometric Ornstein-Uhlenbeck processes // Dependence in Probability and Statistics. Springer Lecture Notes in Statistics. 2010a. 200. 103–122.

Anh V. V., Leonenko N. N., Shieh N.-R., Taufer E. Simulation of multifractal products of Ornstein-Uhlenbeck type processes // Nonlinearity. 2010b. 23. 823–843.

Bacry E., Muzy J.F. Log-infinitely divisible multifractal processes // Comm. Math. Phys. 2003. 236. 449–475.

Barndorf-Nilsen O.E., Kent J., Sorenson M. Normal variance-mean mixtures and z distributions // International Statistical Review. 1982. 50. 145–159.

Barndorff-Nielsen O. E., Eiler O., Stelzer R. Multivariate supOU processes // Ann. Appl. Probab. 2011. 21. 140–182.

Barndorff-Nielsen O. E., Leonenko N. N. Spectral properties of superpositions of Ornstein-Uhlenbeck type processes // Methodol. Comput. Appl. Probab. 2005. 7. 335–352.

Barndorff-Nielsen O. E., Shephard N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics // J. R. Stat. Soc. Ser. B Stat. Methodol. 2001. 63. 167–241.

Barndorff-Nielsen O. E., Shephard N. Normal modified stable processes // Theory Probab. Math. Statist. 2002. 65. 1–20.

Barndorff-Nielsen O.E. Processes of normal inverse Gaussian type // Finance and Stochastics. 1998. 2. 41–68.

Barndorff-Nielsen O.E. Superpositions of Ornstein-Uhlenbeck type processes // Theory Probab. Appl. 2001. 45. 175–194.

Barral J., Mandelbrot B. Multiplicative products of cylindrical pulses // Probab. Theory Related Fields. 2002. 124. 409–430.

Barral J., Mandelbrot B. Convergence of complex multiplicative cascades // Ann. Appl. Probab. 2010. 20. 1219–1252.

Barral J., Peyrière J., Wen Z.-Y. Dynamics of Mandelbrot cascades // Probab. Theory Related Fields. 2009. 144. 615–631.

Bertoin J. Lévy Processes. Cambrdige: Cambridge University Press, 1996.

Carr P., Geman H., Madan D., Yor M. Self-decomposability and option pricing // Math. Finance. 2007. 17. 31–57.

Denisov D., Leonenko N. Limit theorems for Multifractal Products of Geometric Stationary Processes // to appear in Bernoulli. 2016.

Donati-Martin C., Yor M. Some exlicit Krein representations of certain subordinators, including the Gamma process // Pbl. RIMS, Kyoto University. 2006. 42. 879–895.

Falconer K. Techniques in Fractal Geometry. Chichester: Wiley, 1997.

Finlay R., Seneta E. Stationary-Increment Student and Variance-Gamma Processes // J. Appl. Probab. 2006. 2. 441–453.

Frisch U. Turbulence. Cambrdige: Cambridge University Press, 1995.

G.Terdik , Woyczynski W.A. Rosinski measures for tempered stable and related Ornstein-Uhlenbeck processes // Probab. and Math. Stats. 2004. 26. 213–243.

Grigelionis B. Processes of Meixner type // Liet. Mat. Rink. 1999. 39. 40–51.

Grigelionis B. Generalized z-distributions and related stochastic processes // Liet. Mat. Rink. 2001. 41. 303–319.

Harte D. Multifractals: Theory and Applications. London: Chapman & Hall, 2001.

Jaffard S. The multifractal nature of Lévy processes // Probab. Theory Related Fields. 1999. 114. 207–227.

Jaffard S., Abry P., Roux S. G., Vedel B., Wendt H. The contribution of wavelets in multifractal analysis. Wavelet methods in mathematical analysis and engineering. Beijing: Higher Ed. Press, 2010. 51–98. (Ser. Contemp. Appl. Math. CAM, 14).

Kahane J.-P. Sur la chaos multiplicatif // Ann. Sc. Math. Quebec. 1985. 9. 105–150.

Kahane J.-P. Positive martingale and random measures // Chinese Annals of Mathematics. 1987. 8B. 1–12.

Kolmogorov A.N. Local structure of turbulence in fluid for very large Reynolds numbers // Doklady Acad. Sciences of USSR. 1941. 31. 538–540.

Kolmogorov A.N. On refinement of previous hypotheses concerning the local structure in viscous incompressible fluid at high Reynolds number // J. Fluid Mech. 1962. 13. 82–85.

Kyprianou A.E. Introductory Lectures on Fluctuations of Lévy Processes with Applications. Berlin: Springer-Verlag, 2006.

Leonenko N.N., Petherick S., Taufer E. Multifractal scaling for risky asset modelling // Physica A. 2013. 392, 1. 7–16.

Leonenko N.N., Shieh N.-R. Rényi function for multifractal random fields // Fractals. 2013. 21, 2.

Ludena C. $\mathcal{L}_p$-variations for multifractal fractional random walks // Ann. Appl. Probab. 2008. 18. 1138–1163.

Madan D., Carr P., Chang E. The variance gamma process and option pricing // Eur. Finance Rev. 1998. 2. 79–105.

Mandelbrot B. Fractals and Scaling in Finance. New York: Springer, 1997.

Mannersalo P., Norris I., Riedi R. Multifractal products of stochastic processes: construction and some basic properties // Adv. Appl. Prob. 2002. 34. 888–903.

Mannersalo P., Norros I., Riedi R. H. Multifractal analysis of infinite products of stationary jump processes // J. Probab. Stat. 2010. Art. ID 807491.

Matsui M., Shieh N.-R. On the exponentials of fractional Ornstein-Uhlenbeck processes // Electron. J. Probab. 2009. 14. 594–611.

Moerters P., Shieh N.-R. On multifractal spectrum of the branching measure on a Galton-Watson tree // J. Appl. Prob. 2004. 41. 1223–1229.

Morales M., Schoutens W. A risk model driven by Lévy processes // Appl. Stoch. Models Bus. Ind. 2003. 19. 147–167.

Novikov E.A. Infinitely divisible distributions in turbulence // Physical Review E. 1994. 50. 3303–3305.

Prentice R.L. Discrimination among some parametric models // Biometrika. 1975. 62. 607–614.

Riedi R. Multifractal processes, In Theory and Applications of Long-Range Dependence. Boston: Birkhauser, 2003. 625–716.

Saito Y. Log-gamma distribution model of intermittency in turbulence // J. Phys. Soc. Japan. 1992. 61. 403–406.

Schertzer D., Lovejoy S., Schmitt F., Chigirinskaya Y., Marsan D. Multifractal cascade dynamics and turbulent intermittency // Fractals. 1997. 5. 427–471.

Schmitt F.G. A causal multifractal stochastic equation and its statistical properties // European Physical Journal. 2003. 34. 85–98.

Schmitt F.G., Marsan D. Stochastic equation generating continuous multiplicative cascades // Eur. J. Phys. B. 2001. 20. 3–6.

Schoutens W., Teugels J.L. Levy processes, polynomials and martingales. Special Issue in honor of Marcel F. Neuts // Comm. Statist. Stochastic Models. 1998. 14. 335–349.

Shieh N.R., Taylor S. J. Multifractal spectra of branching measure on a Galton-Watson tree // J. Appl. Prob. 2002. 39. 100–111.

(D.E. Denisov) School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK

E-mail address: denis.denisov@manchester.ac.uk

(N.N. Leonenko) School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4YH, UK

E-mail address: LeonenkoN@Cardiff.ac.uk