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Citation for final published version:

Leonenko, Nikolai, Ruiz-Medina, M. Dolores and Taqqu, Murad S. 2017. Non-central limit theorems for random fields subordinated to gamma-correlated random fields. Bernoulli 23 (4B) , pp. 3469-3507. 10.3150/16-BEJ853 file

Publishers page: http://dx.doi.org/10.3150/16-BEJ853 <http://dx.doi.org/10.3150/16-BEJ853>

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Submitted to Bernoulli
arXiv: 1504.00813 [math.SP]

Non-Central Limit Theorems for Random Fields Subordinated to Gamma-Correlated Random Fields

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A reduction theorem is proved for functionals of Gamma-correlated random fields with long-range dependence in d-dimensional space. As a particular case, integrals of non-linear functions of chi-squared random fields, with Laguerre rank being equal to one and two, are studied. When the Laguerre rank is equal to one, the characteristic function of the limit random variable, given by a Rosenblatt-type distribution, is obtained. When the Laguerre rank is equal to two, a multiple Wiener-Itô stochastic integral representation of the limit distribution is derived and an infinite series representation, in terms of independent random variables, is obtained for the limit.

Keywords: Hermite expansion, Laguerre expansion, multiple Wiener-Itô stochastic integrals, non-central limit results, reduction theorems, series expansions.

1. Introduction

This paper considers the family of Gamma-correlated random fields within the general class of *Lancaster-Sarmanov random fields*. Such a class includes non-Gaussian random fields with given marginal distributions and given covariance structure. The bivariate densities of these fields have diagonal expansions. These expansions were idenpendently discovered in [21] and [36], in the context of Markov processes, namely, for dimension $d = 1$, and correlation function $\gamma(|x-y|) = \exp(-c|x-y|)$, $c > 0$. This line of research was also continued by [44], where Laguerre polynomials were used as well as Hermite and Jacobi polynomials, in Markovian settings.

The extension of these limit theorems, based on bilinear expansions, to the context of long-range dependent (LRD) processes was considered in [6] and [7]. Additionally, the extension to the context of LRD random fields was studied in [23], [2] and [3], among others. In this context, the initial motivation for studying Gamma-correlated random fields is to have a class of LRD random fields with Gamma-distributed marginals.

This model setting is important for several applications, where the marginal distributions are positive. Even in the case d=1, Gamma-marginally-distributed stochastic processes arise in financial mathematics in the so-called fractal activity time models (see [15]; [19], and the references therein). The Minkowski functional, moreover (see, for example, [25], p. 1457, 1463; [9], pp. 4–5) constitutes an interesting example of Gamma-correlated functionals. It is defined as

$$\int_{\Delta(t)} \chi(S(x) > a(r))dx, \tag{1}$$

where $\Delta(r)$, $r > 0$, is the homothetic image of a set Δ, χ is the indicator function, $S : \mathbb{R}^d \to \mathbb{R}^1_+$ is a Gamma-correlated random field and $a(r)$ is a continuous non-decreasing function. The indicator function is used to represent the area in excess of some fixed level. This has been widely applied, for example, in Astrophysics, where Minkowski type functionals are studied in the context of spherical random fields. Specifically, these functionals are used in the analysis of the Cosmic Microwave Background (CMB) radiation data (see, for example, [32]). In [33], Minkowski functionals are applied to the characterization of hot regions (i.e., the excursion sets), where the normalized temperature fluctuation field exceeds a given threshold. The normalized temperature fluctuation field, associated with CMB temperature on the sky, is represented in terms of a chi-squared random field (see also [27]). Furthermore, Minkowski functionals are attractive due to their geometrical interpretation in two dimensions, in relation to the total area of all hot regions, the total length of the boundary between hot and cold regions, and the Euler characteristic, which counts the number of isolated hot regions minus the number of isolated cold regions. Minkowski functionals have also been applied to brain mapping. For example, in [41], Minkowski functionals are used to detect local differences in cortical thickness, due to gender, thresholding a smooth T statistics random field at a suitable high value. Minkowski functionals of such a test statistics are used to evaluate, for instance, the expected Euler characteristic of its excursion set (see also [45] for applications to fMRI).

The class of Gamma-correlated random fields, as a subclass of the general family of *Lancaster-Sarmanov random fields*, is not empty, as follows from the results given in [20], who constructed the system of finite-dimensional distributions for a given bivariate distribution, consistent with their marginal distributions, using the calculus of variations and the maximum entropy principle. Examples of this class of random fields can be found, for instance, in [36], and, more recently, in [24]. Note that Gaussian homogeneous and isotropic random fields, and χ^2-random fields belong to the *Lancaster-Sarmanov random field class* (see Section 2). Thus, this class is not empty. Properties of stationary sequences with bivariate densities having diagonal expansions, as well as limit theorems were obtained by [18] and [31]. Specifically, in [18], long-range dependence sequences $\{Z_i\}_{i=1}^{\infty}$ with exponential marginal distributions and its subordinated sequences are studied. In particular, processes of the form $Z_i = (X_i^2 + Y_i^2)/2$, $i = 1, 2 \ldots$, where $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are independent copies of a zero-mean stationary Gaussian process with long-range dependence, are investigated. The asymptotic behaviour of a partial-sum process of the long-range-dependent sequence $\{G(Z_i)\}_{i=1}^{\infty}$, constructed by subordination from $\{Z_i\}_{i=1}^{\infty}$, is the same as that of the first nonvanishing term of its Laguerre expansion (see also [39] and [40], in relation to central and noncentral limit theorems for long-range dependence processes in discrete time). In [31], different properties of bivariate densities (not necessarily associated with stochastic processes) are studied, when they admit a diagonal expansion, referred to as Lancaster-Sarmanov expansion, including Mehler's formula for bivariate Gaussian distributions, Myller–Lebedev or Hille–Hardy formula for bivariate Gamma distributions, among others (see, for example, [5], Chapter 10). In particular, Mehler's equality and Gebelein's inequality are generalized. In addition, conditions are established for defining long-range dependence sequences satisfying the reduction principle, by subordination to discrete time stationary processes.

The present paper extends the above-referred results to the general setting of random fields with continuous d-dimensional parameter space, defined by a regular bounded open domain of $\mathbb{R}^d$. In particular, a reduction theorem is derived for Gamma-correlated random fields with long-range dependence. Some non-central limit results can then be established for LRD random fields constructed by subordination from chi-squared random fields. Here, we consider the cases of Laguerre rank equal one and two. In the case of Laguerre rank being equal to one, non-central limit results for non-linear functionals of LRD Gaussian random fields with Hermite rank equal two are obtained in [26], providing the convergence to a Rosenblatt-type limit distribution. Applying the methodology presented in this paper, the convergence to the same family of distributions can be proved for integrals of non-linear functions of chi-squared random fields with Laguerre rank equal one. In addition, in this paper, we obtain the multiple Wiener-Itô stochastic integral representation of the limit random variable of a sequence of functionals, constructed by integration of non-linear functions of chi-squared random fields, with Laguerre rank equal two. Its series expansion, in terms of independent random variables, is also established. These results constitute an extension of the results derived in [26], on the double Wiener-Itô stochastic integral representation of the limit random variable of a sequence of non-linear functionals of LRD Gaussian random fields with Hermite rank equal two. Its series expansion in terms of independent chi-squared random variables is obtained as well. We study in some detail functionals of chi-squared random fields, since an explicit representation is available for them.

The paper is organized as follows. In Section 2, we define the Lancaster-Sarmanov random field class. In Section 3, we consider the case of Gamma and chi-squared random fields. In Section 4, we prove the reduction principle for Gamma-correlated random fields. A non-central limit theorem is obtained for the case of integrals of non-linear functions of chi-squared random fields, with Laguerre rank equal one, proving the convergence of their characteristic functions. In Section 5, a multiple Wiener-Itô stochastic integral representation of the limit random variable is obtained for functionals of chi-squared random fields with Laguerre rank equal two. Finally, the infinite series representation, in terms of independent random variables, of the limit random variable for the case of Laguerre rank equal two is derived in Section 6. The Appendix establishes the infinitely divisibility property of the limit of a sequence of non-linear functionals of a chi-squared random field, when the Laguerre rank is equal to one.

2. The Lancaster-Sarmanov random fields

The class of Lancaster-Sarmanov random fields with given one-dimensional marginal distributions and general covariance structure is now introduced. Denote by $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ the Hilbert space of zero-mean second-order random variables defined on the complete probability space $(\Omega, \mathcal{F}, P)$. For a probability density function p on the interval (l, r), with $-\infty \leq l < r \leq \infty$, we consider the Hilbert space $L^2((l,r),\ p(u)du)$ of equivalence classes of Lebesgue measurable functions $h : (l,r) \to \mathbb{R}$ satisfying

$$\int_l^r h^2(u)\ p(u)\ du < \infty, \quad p(u) \geq 0.$$

Let us also consider a complete orthonormal system $\{e_k(u)\}_{k=0}^{\infty}$ of functions in $L^2((l,r),\ p(u)du)$, that is,

$$\int_l^r e_k(u)\ e_m(u)\ p(u)du = \delta_{k,m}, \tag{2}$$

where $\delta_{k,m}$ denotes the Kronecker delta function. We introduce the following condition:

Condition A0. Let $\{\xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$ be a mean-square continuous zero-mean homogeneous isotropic random field with correlation function

$$\gamma(\|\mathbf{x}\|) = \frac{B(\|\mathbf{x}\|)}{B(0)}, \quad B(\|\mathbf{x}\|) = \mathrm{Cov}(\xi(0), \xi(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^d.$$

We assume that the densities

$$\begin{aligned} p(u) &= \frac{d}{du} P\{\xi(\mathbf{x}) \leq u\},\ u \in (l,r), \\ p(u, w, \|\mathbf{x}-\mathbf{y}\|) &= \frac{\partial^2}{\partial u \partial w} P\{\xi(\mathbf{x}) \leq u,\ \xi(\mathbf{y}) \leq w\}, \quad (u,w) \in (l,r) \times (l,r), \end{aligned}$$

exist, and that the bilinear expansion

$$p(u, w, \|\mathbf{x}-\mathbf{y}\|) = p(u)\ p(w)\ \left(1 + \sum_{k=1}^{\infty} \gamma^k(\|\mathbf{x}-\mathbf{y}\|)\ e_k(u)\ e_k(w)\right) \tag{3}$$

holds, where

$$\sum_{k=1}^{\infty} \gamma^{2k}(\|\mathbf{x}\|) < \infty, \quad \|\mathbf{x}\| = \sqrt{\sum_{m=1}^{d} x_m^2}, \quad \mathbf{x} \in \mathbb{R}^d,$$

and $\{e_k(u)\}_{k=0}^{\infty}$ is, as before, a complete orthonornal system in the Hilbert space $L^2((l,r), p(u)du)$. Assume also that $e_0(u) \equiv 1$.

The symmetric kernel

$$Q(u,w,\|\mathbf{x}-\mathbf{y}\|)=\frac{p(u,w,\|\mathbf{x}-\mathbf{y}\|)}{p(u)p(w)}=1+\sum_{k=1}^{\infty}\gamma^{k}(\|\mathbf{x}-\mathbf{y}\|)e_{k}(u)e_{k}(w) \tag{4}$$

plays an important role.

The series (4) converges in the space $L^2((l,r)\times(l,r),p\otimes p(u,w)dudw)$ if the integral

$$\begin{aligned} I^2 &= \int_l^r\int_l^r Q^2(u,w,\|\mathbf{x}-\mathbf{y}\|)p(u)p(w)dudw \\ &= \int_l^r\int_l^r Q^2(u,w,\|\mathbf{x}-\mathbf{y}\|)dP\left\{\xi(\mathbf{x})\leq u\right\}dP\left\{\xi(\mathbf{y})\leq w\right\}<\infty, \end{aligned}$$

where I^2-1 is known as the Pearson functional for the bivariate density $p(u,w,\|\mathbf{x}\|)$ (see, for example, [22]). Then, the symmetric kernel $Q(u,w)$ belongs to the product space $L^2((l,r)\times(l,r),p\otimes p(u,w)dudw)$ of square integrable functions on $(l,r)\times(l,r)$, with respect to the measure $p\otimes p(u,w)dudw$. Thus, the kernel Q defines an integral Hilbert-Schmidt operator on the space $L^2((l,r),p(u)du)$. In particular, from the spectral theorem for compact and self-adjoint operators (see, for example, [12], p.112), the integral operator $\mathcal{Q}$ defined by kernel Q admits the diagonal spectral expansion

$$\begin{aligned} \mathcal{Q}(f)(g) &= \int_l^r\int_l^r Q(u,w,\|\mathbf{x}-\mathbf{y}\|)f(u)g(w)p(u)p(w)dudw \\ &= \int_l^r\int_l^r p(u,w,\|\mathbf{x}-\mathbf{y}\|)f(u)g(w)dudw \\ &= \sum_{k=0}^{\infty}\int_l^r\int_l^r r_k(\|\mathbf{x}-\mathbf{y}\|)e_k(u)e_k(w)f(u)g(w)p(u)p(w)dudw, \end{aligned} \tag{5}$$

for all

$$f,g\in L^2((l,r)\times(l,r),p\otimes p(u,w)dudw).$$

Here,

$$\left\{r_k\left(\|\mathbf{x}-\mathbf{y}\|\right)\right\}_{k=0}^{\infty}$$

is the sequence of eigenvalues, associated with the orthonormal system of eigenfunctions $\{e_k(u)\}_{k=0}^{\infty}$, which could also depend on $\mathbf{x}$ and $\mathbf{y}$ in a general setting.

Thus, **Condition A0** postulates the expansion (5) for the case where

$$r_k(\|\mathbf{x}-\mathbf{y}\|)=\gamma^k(\|\mathbf{x}-\mathbf{y}\|),$$

and where $e_k(u)$ does not depend on $\mathbf{x}$ and $\mathbf{y}$. **Condition A0** then implies

$$\begin{aligned}
&\mathrm{E}[e_k(\xi(\mathbf{x}))] = \int_l^r e_k(u)p(u)du = 0, \quad k \geq 1 \\
&\mathrm{E}[e_n(\xi(\mathbf{x}))e_m(\xi(\mathbf{y}))] = \int_l^r \int_l^r e_n(u)e_m(w)p(u,w,\|\mathbf{x}-\mathbf{y}\|)dudw \\
&= \int_l^r \int_l^r e_n(u)e_m(w)p(u)p(w)\left(1+\sum_{k=1}^{\infty}\gamma^k(\|\mathbf{x}-\mathbf{y}\|)e_k(u)e_k(w)\right)dudw \\
&= \delta_{n,m}\ \gamma^n(\|\mathbf{x}-\mathbf{y}\|), \quad n,m \geq 1. \qquad (6)
\end{aligned}$$

We will call the random fields satisfying **Condition A0** *Lancaster-Sarmanov random fields*, due to Lancaster [21] and Sarmanov [36], in the context of Markov processes. In the next section, we will refer to the special case of Gamma-correlated random fields, and, in particular, to the case of chi-squared random fields. We will also let (l, r) in (2) to be $(0, \infty)$.

Remark 1 *As indicated in the Introduction, the class of* Lancaster-Sarmanov random fields *is not empty. In [20] (pp. 120-124), using the calculus of variations and the maximum entropy principle, the system of finite-dimensional distributions, consistent with their marginal distributions, is constructed from a given bivariate distribution. Homogeneous and isotropic zero-mean Gaussian random fields, and χ^2-random fields constitute two examples of* Lancaster-Sarmanov random fields*, which can be constructively defined from the bivariate distribution (3) (see, for example, [3], p. 465).*

3. Gamma-correlated random fields

In this paper, all random fields considered are assumed to be measurable and mean-square continuous. We now refer to the class of random fields with Gamma marginal distribution, and given correlation function. For details see [6], [7], [23] and [3], among others.

3.1. General formulation

Following the ideas of [21] and [36], we introduce a homogeneous and isotropic random field $\{\xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$, with given one-dimensional Gamma distributions, and given correlation structure

$$\gamma(\|\mathbf{x}-\mathbf{y}\|) = \mathrm{Corr}\left(\xi(\mathbf{x}), \xi(\mathbf{y})\right),\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Let

$$p_\beta(u) = \frac{1}{\Gamma(\beta)} u^{\beta-1}\exp(-u), \quad u > 0, \quad \beta > 0, \qquad (7)$$

be a Gamma probability density with shape parameter β and scale parameter one, and let $L^2((0,\infty), p_\beta(u)du)$ be the Hilbert space of square integrable functions with respect to the measure $p_\beta(u)du$, i.e., the space of functions F such that

$$\int_0^\infty F^2(u)p_\beta(u)du < \infty. \tag{8}$$

An orthogonal basis of the Hilbert space $L^2((0,\infty), p_\beta(u)du)$ can be constructed from generalized Laguerre polynomials $L_k^{(\beta)}$, $k \geq 0$, of index β (see [5]). Specifically, the elements of such a basis are defined as follows: For $k, m \geq 0$,

$$e_k(u) = e_k^{(\beta)}(u) = L_k^{(\beta-1)}(u)\left[\frac{k!\Gamma(\beta)}{\Gamma(\beta+k)}\right]^{1/2}, \quad \int_0^\infty e_k^{(\beta)}(u)\ e_m^{(\beta)}(u)\ p_\beta(u)du = \delta_{k,m}, \tag{9}$$

where by Rodríguez formula for Laguerre polynomials

$$L_k^{(\beta)} = L_k^{(\beta)}(u) = (k!)^{-1}u^{-\beta}\exp(u)\frac{d^k}{du^k}\left\{\exp(-u)u^{\beta+k}\right\}. \tag{10}$$

The first three polynomials are then given by

$$\begin{aligned} e_0^{(\beta)}(u) &\equiv 1, \quad e_1^{(\beta)}(u) = \sqrt{\frac{1}{\beta}}\,(\beta-u) \\ e_2^{(\beta)}(u) &= \left(u^2 - 2\,(\beta+1)\ u + (\beta+1)\,\beta\right)\ \left[2\,(\beta+1)\,\beta\right]^{-1/2}. \end{aligned} \tag{11}$$

Applying Myller-Lebedev or Hille-Hardy formula (see [5], Chapter 10) we obtain

$$\begin{aligned} p_\beta(u,w,\|\mathbf{x}-\mathbf{y}\|) &= p_\beta(u)\ p_\beta(w)\ \left[1+\sum_{k=1}^\infty \gamma^k(\|\mathbf{x}-\mathbf{y}\|)\ e_k^{(\beta)}(u)\ e_k^{(\beta)}(w)\right] \\ &= \left(\frac{uw}{\gamma(\|\mathbf{x}-\mathbf{y}\|)}\right)^{(\beta-1)/2}\exp\left\{-\frac{u+w}{1-\gamma(\|\mathbf{x}-\mathbf{y}\|)}\right\} \\ &\times\ I_{\beta-1}\left(2\frac{\sqrt{uw\gamma(\|\mathbf{x}-\mathbf{y}\|)}}{1-\gamma(\|\mathbf{x}-\mathbf{y}\|)}\right)\frac{1}{\Gamma(\beta)\ (1-\gamma(\|\mathbf{x}-\mathbf{y}\|))}, \end{aligned} \tag{12}$$

where $\gamma(\|\mathbf{x}-\mathbf{y}\|)$ is a continuous non-negative definite kernel on $\mathbb{R}^d \times \mathbb{R}^d$, depending on $\|\mathbf{x}-\mathbf{y}\|$, and $I_\varrho(z)$ is the modified Bessel function of the first kind of order ϱ, with

$$I_\varrho(z) = \frac{(z/2)^\varrho}{\sqrt{\pi}\Gamma\left(\varrho+\frac{1}{2}\right)}\int_{-1}^1 (1-t^2)^{\varrho-1/2}\exp(zt)dt, \quad z>0.$$

Summarizing, one can define a homogeneous and isotropic gamma-correlated random field as a random field $\left\{\xi(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\right\}$, such that its one dimensional densities

$$\frac{d}{du}P\left[\xi(\mathbf{x})\leq u\right]$$

and two-dimensional densities

$$p\left(u,w,\|\mathbf{x}-\mathbf{y}\|\right)=\frac{\partial^2}{\partial u\partial w}P\left[\xi(\mathbf{x})\leq u,\xi(\mathbf{y})\leq w\right]$$

are defined by (7) and (12), respectively. In addition, the correlation function γ satisfies

$$\sum_{k=1}^{\infty}\gamma^{2k}(\|\mathbf{z}\|)<\infty,\quad \mathbf{z}\in\mathbb{R}^d.$$

A function F satisfying (8) can be expanded into the series

$$F(u)=\sum_{q=0}^{\infty}C_q^L e_q^{(\beta)}(u),\quad C_q^L=\int_0^{\infty}F(u)\; e_q^{(\beta)}(u)p_\beta(u)\;du,\quad q=0,1,2,\dots, \tag{13}$$

which converges in the Hilbert space $L_2((0,\infty),p_\beta(u)du)$. In particular,

$$C_0^L=\int_0^{\infty}F(u)e_0^{(\beta)}(u)p_\beta(u)\;du=\mathrm{E}[F(\xi(\mathbf{x}))]. \tag{14}$$

The *Laguerre rank* of the function F is defined as the smallest $k\geq 1$ such that

$$C_1^L=0,\dots,C_{k-1}^L=0,\; C_k^L\neq 0.$$

From equation (12), for a homogeneous and isotropic Gamma-correlated random field $\{\xi(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\}$, with correlation function γ, the following identities hold:

$$E[e_k^{(\beta)}(\xi(\mathbf{x}))]=0,\quad E[e_m^{(\beta)}(\xi(\mathbf{x}))e_k^{(\beta)}(\xi(\mathbf{y}))]=\delta_{m,k}\gamma^k(\|\mathbf{x}-\mathbf{y}\|). \tag{15}$$

In order to introduce long-range dependence for Gamma-correlated random fields, we assume the following condition:

Condition A1. The non-negative definite function

$$\gamma(\|\mathbf{z}\|)=\frac{\mathcal{L}(\|\mathbf{z}\|)}{\|\mathbf{z}\|^{\delta}},\quad \mathbf{z}\in\mathbb{R}^d,\quad 0<\delta<d, \tag{16}$$

where $\mathcal{L}$ is a slowly varying function at infinity.

3.2. The chi-squared random fields

One can construct examples of random fields with marginal density (7) and bivariate probability density (12) considering the class of chi-squared random fields. The chi-squared random fields are given by

$$\chi_r^2(\mathbf{x})=\frac{1}{2}\left(Y_1^2(\mathbf{x})+\cdots+Y_r^2(\mathbf{x})\right),\quad \mathbf{x}\in\mathbb{R}^d, \tag{17}$$

where $Y_1(\mathbf{x}), \ldots, Y_r(\mathbf{x})$ are independent copies of Gaussian random field $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$ with covariance function $B(\|\mathbf{x}\|)$ with $B(\|\mathbf{0}\|) = 1$. In this case

$$\gamma(\|\mathbf{x}-\mathbf{y}\|) = \frac{\mathrm{Cov}(\chi_r^2(\mathbf{x}), \chi_r^2(\mathbf{y}))}{\mathrm{Var}(\chi_r^2(\mathbf{0}))} = B^2(\|\mathbf{x}-\mathbf{y}\|), \quad \beta = r/2. \tag{18}$$

Note that by construction, the correlation function of chi-squared random fields is always non-negative. Moreover,

$$\mathrm{E}\chi_r^2(\mathbf{x}) = \frac{r}{2}, \quad \mathrm{Var}\chi_r^2(\mathbf{x}) = \frac{r}{4}\mathrm{Var}\ Y_1^2(\mathbf{x}) = \frac{r}{2}, \quad \mathrm{Cov}(\chi_r^2(0), \chi_r^2(\mathbf{x})) = \frac{r}{2}B^2(\|\mathbf{x}\|).$$

and

$$\mathrm{E}[e_k^{(r/2)}(\chi_r^2(\mathbf{x}))\ e_m^{(r/2)}(\chi_r^2(\mathbf{y}))] = \delta_{m,k}\ B^{2m}(\|\mathbf{x}-\mathbf{y}\|), \tag{19}$$

since as noted in (2),

$$\int_0^\infty e_k^{(r/2)}(u)\ e_m^{(r/2)}(u)\ p_{r/2}(u)\ du = \delta_{k,m}.$$

In the case of chi-squared random fields (17), the analogous of **Condition A1** setting in (16) is the following **Condition A2**.

Condition A2. The random field $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$, whose independent copies define the chi-squared random field (17), is a measurable homogeneous and isotropic mean-square continuous zero-mean Gaussian random field on a probability space $(\Omega, \mathcal{A}, P)$, with $\mathrm{E}Y^2(\mathbf{x}) = 1$, for all $\mathbf{x} \in \mathbb{R}^d$, and correlation function $\mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})] = B(\|\mathbf{x}-\mathbf{y}\|)$ of the form:

$$B(\|\mathbf{z}\|) = \frac{\mathcal{L}(\|\mathbf{z}\|)}{\|\mathbf{z}\|^\alpha}, \quad \mathbf{z} \in \mathbb{R}^d, \quad 0 < \alpha < d/2. \tag{20}$$

Since, from **Condition A2**, the correlation function B of Y is continuous, it follows that $\mathcal{L}(r) = \mathcal{O}(r^\alpha)$, as $r \longrightarrow 0$.

4. Reduction principle for Gamma-correlated random fields

In this section, the reduction principle formulated for Gamma-correlated random fields is an analogous, in spirit, to the reduction principle of [39] and [40] (see also [6], [7]; [23], among others).

In the following, let us denote by $\mathcal{D}(T)$ a homothetic transformation of a set $\mathcal{D} \subset \mathbb{R}^d$ with center at the point $\mathbf{0} \in \mathcal{D}$ and coefficient or scale factor $T > 0$. In addition, $\mathcal{D}$ is assumed to be a regular bounded open domain, whose interior has positive Lebesgue measure, and with boundary having null Lebesgue measure. Dirichlet-regularity here is understood in the general setting established, for example, in [16] (p. 253), as given in the following definition.

Definition 1 *For a connected bounded open domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ we say that $\mathbf{x}_0 \in \partial\mathcal{D}$ is regular if and only if it has a Green kernel $G^{\mathcal{D}}$ such that*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} G^{\mathcal{D}}(\mathbf{x},\mathbf{y}) = 0, \quad \forall \mathbf{y} \in \mathcal{D}. \tag{21}$$

The set $\mathcal{D}$ is regular if every point of $\partial\mathcal{D}$ is regular.

Dirichlet regularity of domain $\mathcal{D}$ ensures that the eigenvectors of the operator $\mathcal{K}_\alpha$, introduced in equation (43) below, vanish continuously at the boundary of domain $\mathcal{D}$ (see, for example, [8], p. 137, in the context of potential theory, and, more recently, [11], for $0 < \alpha < 2$, in the context of subordinate processes in domains).

Let ξ be a homogeneous and isotropic gamma-correlated random field, from equation (15),

$$\mathrm{E}\left[\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)} e_k^{(\beta)}(\xi(\mathbf{x}))e_m^{(\beta)}(\xi(\mathbf{y}))\ d\mathbf{x}d\mathbf{y}\right] = \delta_{k,m}\ \sigma_k^2(T), \tag{22}$$

where, under **Condition A1**, for $0 < \delta < d/k$,

$$\begin{aligned}\sigma_k^2(T) &= \mathrm{Var}\left[\int_{\mathcal{D}(T)} e_k^{(\beta)}(\xi(\mathbf{x}))\ d\mathbf{x}\right]\\ &= \int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)} \gamma^k\left(\|\mathbf{x}-\mathbf{y}\|\right) d\mathbf{x}d\mathbf{y} = [a_{d,k}(\mathcal{D})]^2 T^{2d-k\delta}\mathcal{L}^k(T)(1+o(1)),\end{aligned} \tag{23}$$

as $T \longrightarrow \infty$, with

$$a_{d,k}(\mathcal{D}) = \left[\int_{\mathcal{D}}\int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{k\delta}} d\mathbf{x}d\mathbf{y}\right]^{1/2}, \quad k \geq 1. \tag{24}$$

Note that, for the particular case of chi-squared random fields we have from (19)

$$\mathrm{E}\left[\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)} e_k^{(r/2)}(\chi_r^2(\mathbf{x}))e_m^{(r/2)}(\chi_r^2(\mathbf{y}))\ d\mathbf{x}d\mathbf{y}\right] = \delta_{k,m}\ \sigma_k^2(T),$$

where, under **Condition A2**, for $0 < \alpha < \frac{d}{2k}$, as $T \longrightarrow \infty$,

$$\begin{aligned}\sigma_k^2(T) &= \mathrm{Var}\left[\int_{\mathcal{D}(T)} e_k(\chi_r^2(\mathbf{x}))\ d\mathbf{x}\right]\\ &= \int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)} B^{2k}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y} = [a_{d,k}^{\chi_r^2}(\mathcal{D})]^2 T^{2d-2k\alpha}\mathcal{L}^{2k}(T)(1+o(1)),\end{aligned} \tag{25}$$

with

$$a_{d,k}^{\chi_r^2}(\mathcal{D}) = \left[\int_{\mathcal{D}}\int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2k\alpha}} d\mathbf{x}d\mathbf{y}\right]^{1/2}, \quad k \geq 1, \tag{26}$$

(see equations (40)–(43) in [26], for $k=1$, for more details on the methodological approach adopted in that computations).

The following theorem states the reduction principle.

Theorem 1 *Let* $\{\xi(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\}$ *be a Gamma-correlated random field, and consider* $F\in L^2((0,\infty),p_\beta(u)du)$ *having generalized Laguerre rank* k*, with* $p_\beta(u)$ *given in (7). Assume that* ***Condition A1*** *holds. If, for* $0<\delta<d/k$*, as* $T\to\infty$*,*

$$\frac{C_k^L}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\int_{\mathcal{D}(T)} e_k^{(\beta)}(\xi(\mathbf{x}))\ d\mathbf{x}$$

converges, then

$$S_T^L=\frac{1}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\left[\int_{\mathcal{D}(T)}F(\xi(\mathbf{x}))d\mathbf{x}-C_0^LT^d|\mathcal{D}|\right] \tag{27}$$

converges to the same limit. The constants C_k^L *and* C_0^L *are defined in equations (13) and (14), respectively.*

Proof. The proof is based on the generalized Laguerre polynomial expansion of the function F. Specifically, under **Condition A1**, since $\gamma(\|\mathbf{x}\|)\leq 1$, and $\gamma(0)=1$, we have

$$\gamma^{k+l}(\|\mathbf{x}\|)\leq\gamma^{k+1}(\|\mathbf{x}\|),\quad l\geq 2.$$

Hence, from equation (23), for T sufficiently large,

$$E\left[\frac{1}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\left(\int_{\mathcal{D}(T)}F(\xi(\mathbf{x}))\ d\mathbf{x}-C_0^LT^d\,|\mathcal{D}|-C_k^L\int_{\mathcal{D}(T)}e_k^{(\beta)}(\xi(\mathbf{x}))\ d\mathbf{x}\right)\right]^2$$

$$=\left[\frac{1}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\right]^2\sum_{j=k+1}^{\infty}(C_j^L)^2\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)}\gamma^j(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\leq$$

$$\leq\left[\frac{1}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\right]^2\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\sum_{j=k+1}^{\infty}(C_j^L)^2=K_R.$$

By **Condition A1**, for any $\epsilon>0$, there exists $A_0>0$, such that for$\|\mathbf{x}-\mathbf{y}\|>A_0$, $\gamma(\|\mathbf{x}-\mathbf{y}\|)<\epsilon$. Let $K_1=\{(\mathbf{x},\mathbf{y})\in\mathcal{D}(T):\|\mathbf{x}-\mathbf{y}\|\leq A_0)\}$, and $K_2=\{(\mathbf{x},\mathbf{y})\in\mathcal{D}(T):\|\mathbf{x}-\mathbf{y}\|>A_0)\}$. Then,

$$\begin{aligned}\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y} &= \left\{\int\int_{K_1}+\int\int_{K_2}\right\}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\\ &= S_T^{(1)}+S_T^{(2)}.\end{aligned} \tag{28}$$

Using the bound $\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)\leq 1$ on K_1, and the bound $\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)<\epsilon\gamma^{k}(\|\mathbf{x}-\mathbf{y}\|)$ on K_2, we obtain, again, for T sufficiently large,

$$\left|S_T^{(1)}\right|\leq\left|\int\int_{K_1}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\right|\leq M_1T^d,$$

for a suitable constant $M_1>0$, and

$$\left|S_T^{(2)}\right|\leq\left|\int\int_{K_2}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\right|\leq\epsilon\left|\int\int_{K_2}\gamma^{k}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\right|\leq\epsilon M_2T^{2d-k\delta}\mathcal{L}^k(T),$$

for suitable $M_2>0$ (see equation (23)), and arbitrary $\epsilon>0$. Thus,

$$\begin{aligned}K_R &= \left[\frac{1}{a_{d,k}(\mathcal{D})\mathcal{L}^{k/2}(T)T^{d-(k\delta)/2}}\right]^2\int_{\mathcal{D}(T)}\int_{\mathcal{D}(T)}\gamma^{k+1}(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y}\sum_{j=k+1}^{\infty}(C_j^L)^2\\ &\leq M_1\vee M_2\left[\frac{T^d}{a_{d,k}^2(\mathcal{D})\mathcal{L}^k(T)T^{2d-k\delta}}+\epsilon\frac{T^{2d-k\delta}\mathcal{L}^k(T)}{a_{d,k}^2(\mathcal{D})\mathcal{L}^k(T)T^{2d-k\delta}}\right], \qquad (29)\end{aligned}$$

which can be made arbitrary small together with $\epsilon>0$.
(See also the proof of Theorem 3.2(ii), in [26], in the particular case of functionals of chi-squared random fields with Laguerre rank equal one).

The following additional condition is assumed for the slowly varying function $\mathcal{L}$ in **Condition A2**, in order to ensure the convergence, as $T\to\infty$, of the characteristic functions of the functionals $S_T^{\chi_r^2}$, $T>0$, introduced in equation (33) below, to the characteristic function (34).

Condition A3. Let $\mathcal{L}$ be the slowly varying function introduced in **Condition A2**. Assume that, for every $m\geq 2$ there exists a constant $C>0$, such that

$$\begin{aligned}&\int_{\mathcal{D}}\dots(m)\dots\int_{\mathcal{D}}\frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\mathcal{L}(T)\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}}\frac{\mathcal{L}(T\|\mathbf{x}_2-\mathbf{x}_3\|)}{\mathcal{L}(T)\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}}\cdots\frac{\mathcal{L}(T\|\mathbf{x}_m-\mathbf{x}_1\|)}{\mathcal{L}(T)\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}d\mathbf{x}_1d\mathbf{x}_2\cdots d\mathbf{x}_m\leq\\ &\leq C\int_{\mathcal{D}}\dots(m)\dots\int_{\mathcal{D}}\frac{d\mathbf{x}_1d\mathbf{x}_2\cdots d\mathbf{x}_m}{\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}\cdots\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}. \qquad (30)\end{aligned}$$

Condition A3 is satisfied by slowly varying functions such that

$$\sup_{T,\mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}}\frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\mathcal{L}(T)}\leq C_0, \qquad (31)$$

for $0<C_0\leq 1$. For instance,

$$B(\|\mathbf{z}\|)=\frac{1}{(1+\|\mathbf{z}\|^{\beta})^{\gamma}},\quad 0<\beta\leq 2,\quad \gamma>0,$$

constitutes a particular case of the family of covariance functions (20) studied here, satisfying **Condition A3**, in the case where $\mathcal{D} \subseteq \mathcal{B}(\mathbf{0})$, with $\mathcal{B}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d, \ \|\mathbf{x}\| \leq 1\}$. Specifically, in that example, we consider $\alpha = \beta\gamma$, and $\mathcal{L}(\|\mathbf{z}\|) = \|\mathbf{z}\|^{\beta\gamma}/(1 + \|\mathbf{z}\|^{\beta})^{\gamma}$.

The next result provides the characteristic function of the limit in distribution of functional (27) in Theorem 1, when F has Laguerre rank $k = 1$. In the derivation of such a result, the limit characteristic function (see equation (34) below) is obtained from the application of the Fredholm determinant formula of a trace operator. The following definition provides this formula.

Definition 2 *(see [37], Chapter 5, pp.47-48, equation (5.12)) Let A be a trace operator on a separable Hilbert space H. The Fredholm determinant of A is defined as*

$$\mathcal{D}(\omega) = det(I - \omega A) = \exp\left(-\sum_{k=1}^{\infty} \frac{TrA^k}{k}\omega^k\right) = \exp\left(-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[\lambda_l(A)]^k \frac{\omega^k}{k}\right), \tag{32}$$

for $\omega \in \mathbb{C}$, and $|\omega|\|A\|_1 < 1$, with $\|A\|_1$ denoting the trace of operator A. Note that $\|A^m\|_1 \leq \|A\|_1^m$, for A being a trace operator.

Theorem 2 *Let $\{\chi_r^2(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^d\}$ be the chi-squared random field introduced in (17). Consider the functional*

$$S_T^{\chi_r^2} = \frac{1}{a_{d,1}^{\chi_r^2}(\mathcal{D})\mathcal{L}(T)T^{d-\alpha}}\left[\int_{\mathcal{D}(T)} F(\chi_r^2(\mathbf{x}))d\mathbf{x} - C_0^L T^d|\mathcal{D}|\right], \tag{33}$$

*where function F has Laguerre rank $k = 1$, and $a_{d,1}^{\chi_r^2}(\mathcal{D})$ is given in (26) for $k = 1$. Under **Conditions A2 and A3**, for $0 < \alpha < d/2$, the limit $S_\infty^{\chi_r^2}$, in distribution sense, as $T \to \infty$, of (33) has characteristic function of the form*

$$\phi(\theta) = E\left[\exp\left(i\theta S_\infty^{\chi_r^2}\right)\right] = \exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\frac{(-2i\theta C_1^L/a_{d,1}^{\chi_r^2}(\mathcal{D})\sqrt{2r})^m}{m}c_m\right), \quad \theta \in \mathbb{R}, \tag{34}$$

where c_m, $m \geq 2$, are defined as follows:

$$c_m = \int_{\mathcal{D}} \underset{(m)}{\dots} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^{\alpha}} \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_3\|^{\alpha}} \cdots \frac{1}{\|\mathbf{x}_m - \mathbf{x}_1\|^{\alpha}} d\mathbf{x}_1 \dots d\mathbf{x}_m. \tag{35}$$

Remark 2 *From Theorem 1, the limit in distribution sense, as $T \to \infty$, of the functional defined from F in equation (33) coincides with the limit in distribution of*

$$\frac{C_1^L}{a_{d,1}^{\chi_r^2}(\mathcal{D})\mathcal{L}(T)T^{d-\alpha}} \int_{\mathcal{D}(T)} e_1^{(r/2)}(\chi_r^2(\mathbf{x}))d\mathbf{x}. \tag{36}$$

Equivalently, the following limit holds in distribution sense:

$$S_\infty^{\chi_r^2} = \lim_{T \longrightarrow \infty} \frac{C_1^L}{a_{d,1}^{\chi_r^2}(\mathcal{D})\mathcal{L}(T)T^{d-\alpha}} \int_{\mathcal{D}(T)} e_1^{(r/2)}(\chi_r^2(\mathbf{x})) \ d\mathbf{x}, \tag{37}$$

where $S_{\infty}^{\chi_r^2}$ has characteristic function given in (34).

Proof. In view of Remark 2, we restrict our attention to the functional in (36). The first Laguerre polynomial of the chi-squared random field $\{\chi_r^2(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\}$ is the sum of r independent copies of the second Hermite polynomial of the Gaussian random field $\{Y(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\}$ appearing in (17), satisfying **Condition A2**, that is, for $\mathbf{x}\in\mathbb{R}^d$, we have by (11),

$$e_1^{(r/2)}(\chi_r^2(\mathbf{x}))=\sqrt{\frac{2}{r}}\left(\frac{r}{2}-\sum_{j=1}^{r}Y_j^2(\mathbf{x})\right)=-\frac{1}{\sqrt{2r}}\sum_{j=1}^{r}(Y_j^2(\mathbf{x})-1)=-\frac{1}{\sqrt{2r}}\sum_{j=1}^{r}H_2(Y_j(\mathbf{x})). \tag{38}$$

From equation (49) of Theorem 3.2(i) in [26], for every $\theta\in\mathbb{R}$,

$$\begin{aligned}\psi_T(\theta) &= E\left[\exp\left(\frac{\mathrm{i}\theta}{T^{d-\alpha}\mathcal{L}(T)}\int_{\mathcal{D}(T)}(Y^2(\mathbf{x})-1)d\mathbf{x}\right)\right]\\ &= \exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}\theta}{T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R_{Y,\mathcal{D}(T)}^m\right)\right),\end{aligned} \tag{39}$$

where, as before, Y is a zero-mean Gaussian random field satisfying **Condition A2**, and

$$\begin{aligned}Tr(R_{Y,\mathcal{D}(T)}^m) &= \sum_{j=1}^{\infty}\lambda_{j,T}^m(R_{Y,\mathcal{D}(T)})=\int_{\mathcal{D}(T)}B_{0,T}^{*(m)}(\mathbf{x}_m,\mathbf{x}_m)d\mathbf{x}_m\\ &= \int_{\mathcal{D}(T)}\overset{\dots}{\scriptstyle (m)}\int_{\mathcal{D}(T)}\left[\prod_{j=1}^{m-1}B_{0,T}(\mathbf{x}_{j+1}-\mathbf{x}_j)\right]B_{0,T}(\mathbf{x}_1-\mathbf{x}_m)d\mathbf{x}_1\dots d\mathbf{x}_m,\\ R_{Y,\mathcal{D}(T)}^m(f)(\mathbf{z}) &= \int_{\mathcal{D}(T)}B_{0,T}^{*(m)}(\mathbf{z},\mathbf{y})f(\mathbf{y})d\mathbf{y},\quad \forall f\in L^2(\mathcal{D}(T)),\end{aligned} \tag{40}$$

with

$$B_{0,T}(\mathbf{x},\mathbf{z})=B_{0,T}(\mathbf{x}-\mathbf{z})=E[Y(\mathbf{x})Y(\mathbf{z})],$$

for all $\mathbf{x},\mathbf{z}\in\mathcal{D}(T)$.

From equations (38) and (39), for every $\theta \in \mathbb{R}$,

$$\begin{aligned}
\phi_T(\theta) &= E\left[\exp\left(\frac{iC_1^L\theta}{T^{d-\alpha}\mathcal{L}(T)a_{d,1}^{\chi_r^2}(\mathcal{D})}\int_{\mathcal{D}(T)} e_1^{(r/2)}\left(\chi_r^2(\mathbf{x})\right)d\mathbf{x}\right)\right] \\
&= E\left[\exp\left(\frac{iC_1^L\theta}{T^{d-\alpha}\mathcal{L}(T)a_{d,1}^{\chi_r^2}(\mathcal{D})}\int_{\mathcal{D}(T)}\left(-\frac{1}{\sqrt{2r}}\sum_{j=1}^{r}H_2(Y_j(\mathbf{x}))\right)d\mathbf{x}\right)\right] \\
&= \prod_{j=1}^{r}E\left[\exp\left(\frac{-\mathrm{i}C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\int_{D(T)}(Y^2(\mathbf{x})-1)d\mathbf{x}\right)\right] \\
&= \prod_{j=1}^{r}\psi_T\left(\frac{-\theta C_1^L}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right) \qquad (41)
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^{r}\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2i\theta C_1^L}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R^m_{Y,\mathcal{D}(T)}\right)\right) \\
&= \exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2i\theta C_1^L}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R^m_{Y,\mathcal{D}(T)}\right)\right). \qquad (42)
\end{aligned}$$

Note that under **Condition A2**, since $EY^2(\mathbf{x}) = 1$,

$$\begin{aligned}
\int_{\mathcal{D}(T)} d\mathbf{x} &= \int_{\mathcal{D}(T)} E\left[Y^2(\mathbf{x})\right]d\mathbf{x} = E\left[\int_{\mathcal{D}(T)} Y^2(\mathbf{x})d\mathbf{x}\right] = \sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,\mathcal{D}(T)})E\eta_j^2 \\
&= \sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,\mathcal{D}(T)}).
\end{aligned}$$

In the following, we will apply, for $0 < \alpha < d/2$, the trace property of the square, $\mathcal{K}_\alpha^2$, of the operator

$$\mathcal{K}_\alpha(f)(\mathbf{x}) = \int_{\mathcal{D}}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^\alpha}f(\mathbf{y})d\mathbf{y}, \quad \forall f \in \mathrm{Supp}(\mathcal{K}_\alpha) \qquad (43)$$

(see Theorem 3.1 in [26]). In particular, the trace of $\mathcal{K}_\alpha^2$ is given by

$$\mathrm{Tr}\left(\mathcal{K}_\alpha^2\right) = \int_{\mathcal{D}}\int_{\mathcal{D}}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}}d\mathbf{x}d\mathbf{y} = [a_{d,1}^{\chi_r^2}(\mathcal{D})]^2 < \infty. \qquad (44)$$

In addition, under **Condition A3**, there exists a positive constant C such that

$$\begin{aligned}
\frac{1}{T^{d-\alpha}\mathcal{L}(T)}\mathrm{Tr}\left(R^2_{Y,\mathcal{D}(T)}\right) &= \int_{\mathcal{D}}\int_{\mathcal{D}}\frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\mathcal{L}(T)}\frac{\mathcal{L}(T\|\mathbf{x}_2-\mathbf{x}_1\|)}{\mathcal{L}(T)}\frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{2\alpha}}d\mathbf{x}_1d\mathbf{x}_2 \\
&\leq C\int_{\mathcal{D}}\int_{\mathcal{D}}\frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{2\alpha}}d\mathbf{x}_1d\mathbf{x}_2 = C\mathrm{Tr}\left(\mathcal{K}_\alpha^2\right) < \infty, \qquad (45)
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{T^{dm-\alpha m}[\mathcal{L}(T)]^m}\mathrm{Tr}\left(R^m_{Y,\mathcal{D}(T)}\right) = \\
&= \frac{1}{[\mathcal{L}(T)]^m}\int_{\mathcal{D}} \underset{(m)}{\dots} \int_{\mathcal{D}} \frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}}\frac{\mathcal{L}(T\|\mathbf{x}_2-\mathbf{x}_3\|)}{\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}}\cdots\frac{\mathcal{L}(T\|\mathbf{x}_m-\mathbf{x}_1\|)}{\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}d\mathbf{x}_1\dots d\mathbf{x}_m \\
&\leq C\int_{\mathcal{D}} \underset{(m)}{\dots} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}}\frac{1}{\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}}\cdots\frac{1}{\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}d\mathbf{x}_1\dots d\mathbf{x}_m \\
&= C\mathrm{Tr}\left(\mathcal{K}^m_{\alpha}\right) < \infty, \quad m>2, \qquad (46)
\end{aligned}
$$

since $\|\mathcal{K}^m_{\alpha}\|_1 \leq M\|\mathcal{K}^2_{\alpha}\|_1$, for certain $M>0$, and for $m>2$.

From equations (40), (42), (45) and (46), considering the upper bounds derived in equations (54)–(57) for function ψ_T in [26], we obtain, for every $\theta \in \mathbb{R}$,

$$
\begin{aligned}
|\psi_T(\theta)| &= \left|\exp\left(\frac{-1}{2}\sum_{m=2}^{\infty}\frac{(-1)^m}{2m-2}\left(\frac{2\theta}{d_T}\right)^{2m-2}\mathrm{Tr}\left(R^{2m-2}_{Y,D(T)}\right)\right)\right| \\
&\leq \left|\exp\left(\frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{4n}(2\theta)^{4n}\,\mathrm{Tr}\left(\mathcal{K}^{4n}_{\alpha}\right)\right)\right| \\
&= \left|\exp\left(\frac{C}{8}\sum_{n=1}^{\infty}\frac{1}{n}\left(16\theta^4\right)^n\mathrm{Tr}\left((\mathcal{K}^4_{\alpha})^n\right)\right)\right| = \left[\mathcal{D}_{\mathcal{K}^4_{\alpha}}(16\theta^4)\right]^{-C/8}, \qquad (47)
\end{aligned}
$$

where, in the last identity we have applied Definition 2, for $A=\mathcal{K}^4_{\alpha}$, and $\omega = 16\theta^4$. Hence, equation (47) is finite for $|16\theta^4| = 16\theta^4 < 1/\|\mathcal{K}^4_{\alpha}\|_1$, or equivalently, for $|\theta| < 1/2[\|\mathcal{K}^4_{\alpha}\|_1]^{1/4}$, since, as indicated in Definition 2, the trace property of $\mathcal{K}^2_{\alpha}$ implies the trace property of $\mathcal{K}^4_{\alpha}$.

From equations (41) and (47), we then have, for every $\theta \in \mathbb{R}$,

$$
|\phi_T(\theta)| \leq \left[\mathcal{D}_{\mathcal{K}^4_{\alpha}}\left(16\left(\frac{C^L_1\theta}{\sqrt{2r}a^{\chi^2_r}_{d,1}(\mathcal{D})}\right)^4\right)\right]^{-rC/8} < \infty, \qquad (48)
$$

where, as before, $\mathcal{D}_{\mathcal{K}^4_{\alpha}}$ denotes the Fredholm determinant of $\mathcal{K}^4_{\alpha}$. Note that, in this case, we have also applied Definition 2, for $A=\mathcal{K}^4_{\alpha}$, and

$$
\omega = 16\left(\frac{C^L_1\theta}{\sqrt{2r}a^{\chi^2_r}_{d,1}(\mathcal{D})}\right)^4,
$$

which is finite for

$$
|\omega| = \left|16\left(\frac{C^L_1\theta}{\sqrt{2r}a^{\chi^2_r}_{d,1}(\mathcal{D})}\right)^4\right| = 16\left(\frac{C^L_1\theta}{\sqrt{2r}a^{\chi^2_r}_{d,1}(\mathcal{D})}\right)^4 < \frac{1}{\|\mathcal{K}^4_{\alpha}\|_1},
$$

or equivalently, for

$$|\theta| < \frac{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}{2C_1^L[\|\mathcal{K}_\alpha^4\|_1]^{1/4}}.$$

From equation (48), for such $|\theta|$ the limit, as $T \to \infty$, of $\phi_T(\theta)$ exists and it is finite. We are now going to compute such a limit. Since

$$\lim_{T\to\infty} \mathcal{L}(T\|\mathbf{x}\|)/\mathcal{L}(T) = 1,$$

by slowly varying property of function $\mathcal{L}$, we have, for every $m \geq 2$,

$$\lim_{T\to\infty} \frac{\mathrm{Tr}\left(R^m_{Y,D(T)}\right)}{[T^{d-\alpha}\mathcal{L}(T)]^m} = \mathrm{Tr}\left(\mathcal{K}_\alpha^m\right). \tag{49}$$

Under **Condition A3**, from equations (45) and (46), for each $m \geq 2$, and for every $\theta \in \mathbb{R}$,

$$\frac{1}{m}\left|\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^m \left|\frac{\mathrm{Tr}\left(R^m_{Y,D(T)}\right)}{[T^{d-\alpha}\mathcal{L}(T)]^m}\right| \leq C\frac{1}{m}\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^m \mathrm{Tr}\left(\mathcal{K}_\alpha^m\right). \tag{50}$$

The upper bound in equation (50) satisfies, for $\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right| < 1$, i.e., for $|\theta| < \sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})/2C_1^L$,

$$\begin{aligned}
\sum_{m=2}^{\infty}\frac{1}{m}\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^m \mathrm{Tr}\left(\mathcal{K}_\alpha^m\right) &= \sum_{m=1}^{\infty}\frac{1}{2m}\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{2m} \mathrm{Tr}\left(\mathcal{K}_\alpha^{2m}\right)\\
&+ \sum_{m=1}^{\infty}\frac{1}{2m+1}\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{2m+1} \mathrm{Tr}\left(\mathcal{K}_\alpha^{2m+1}\right)\\
&\leq \widetilde{M}\sum_{m=1}^{\infty}\frac{1}{m}\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{m} \mathrm{Tr}\left(\mathcal{K}_\alpha^{2m}\right)\\
&= -\widetilde{M}\ln\left(\mathcal{D}_{\mathcal{K}_\alpha^2}\left(\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|\right)\right),
\end{aligned} \tag{51}$$

for certain positive constant $\widetilde{M}$, where, in the last identity, we have applied Definition 2, for $A = \mathcal{K}_\alpha^2$, and

$$\omega = \left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|.$$

Hence, equation (51) is finite for

$$\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right| < \frac{1}{\mathrm{Tr}(\mathcal{K}_\alpha^2)} \Leftrightarrow |\theta| < \frac{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}{2C_1^L\mathrm{Tr}(\mathcal{K}_\alpha^2)}.$$

In addition, in the derivation of equation (51), the following straightforward inequalities have been applied: For $m \geq 1$,

$$\frac{1}{2m+1} \leq \frac{1}{m}, \quad \frac{1}{2m} \leq \frac{1}{m}, \quad \left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{2m+1} \leq \left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{m},$$

$$\left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{2m} \leq \left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right|^{m}, \quad \text{since} \left|\frac{2C_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right| < 1,$$

$$\mathrm{Tr}\left(\mathcal{K}_\alpha^{2m+1}\right) \leq \widetilde{\widetilde{M}}\mathrm{Tr}\left(\mathcal{K}_\alpha^{2m}\right),$$

where $\widetilde{\widetilde{M}} = \widetilde{M} - 1$, with $\widetilde{M}$ given in (51).

From equations (42), and (48)–(51), we can apply Dominated Convergence Theorem, considering integration with respect to a counting measure, for

$$|\theta| < \frac{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}{2C_1^L\|\mathcal{K}_\alpha^2\|_1} \wedge \frac{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}{2C_1^L} \wedge \frac{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}{2C_1^L[\|\mathcal{K}_\alpha^4\|_1]^{1/4}},$$

leading to

$$\begin{aligned}
\lim_{T\to\infty}\phi_T(\theta) &= \lim_{T\to\infty}\exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R_{Y,\mathcal{D}(T)}^m\right)\right) \\
&= \exp\left(\frac{r}{2}\lim_{T\to\infty}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R_{Y,\mathcal{D}(T)}^m\right)\right) \\
&= \exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\lim_{T\to\infty}\frac{1}{m}\left(\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})T^{d-\alpha}\mathcal{L}(T)}\right)^m \mathrm{Tr}\left(R_{Y,\mathcal{D}(T)}^m\right)\right) \\
&= \exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right)^m \mathrm{Tr}\left(\mathcal{K}_\alpha^m\right)\right) \\
&= \exp\left(\frac{r}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-2iC_1^L\theta}{\sqrt{2r}a_{d,1}^{\chi_r^2}(\mathcal{D})}\right)^m c_m\right) = \phi(\theta). \qquad (52)
\end{aligned}$$

An analytic continuation argument (see [29], Th. 7.1.1) guarantees that ϕ defines the unique limit characteristic function for all real values of θ.

5. Multiple Wiener-Itô stochastic integral representation of the limit for functionals with Laguerre rank equal to two

Consider the chi-squared field defined (17). The multiple Wiener-Itô stochastic integral representation of the limit in distribution of

$$S_{2,T} = \frac{1}{a_{d,2}^{\chi_r^2}(\mathcal{D})\mathcal{L}^2(T)T^{d-2\alpha}} \int_{\mathcal{D}(T)} e_2^{(r/2)}(\chi_r^2(\mathbf{x}))\, d\mathbf{x} \tag{53}$$

is derived in Theorem 3 below. Here, $a_{d,2}^{\chi_r^2}$ is defined as in (26) for $k = 2$, and $\mathcal{L}(T)$ is the slowly varying function introduced in (20). The function $e_2^{r/2}(u)$ is given in equations (9)–(10).

Let us denote by

$$\nu(\beta) = \frac{\pi^{d/2}2^{\beta}\Gamma(\beta/2)}{\Gamma\left(\frac{d-\beta}{2}\right)}, \quad 0 < \beta < d, \tag{54}$$

and by

$$\mathcal{F}(\psi)(\mathbf{z}) = \int_{\mathbb{R}^d} \exp\left(-\mathrm{i}\langle \mathbf{x}, \mathbf{z}\rangle\right)\psi(\mathbf{x})d\mathbf{x},$$

the Fourier transform of ψ.

Before deriving the main results of this section, Proposition 1 and Theorem 3 below, the following lemma provides the Fourier transforms and convolution formulae that will be needed in the subsequent development. In particular, these formulae are applied in the space $C_0^{\infty}(\mathcal{D}) \subset \mathcal{S}(\mathbb{R}^d)$, of infinitely differentiable functions with compact support contained in $\mathcal{D}$, with $\mathcal{S}(\mathbb{R}^d)$ denoting the space of infinitely differentiable functions on $\mathbb{R}^d$, whose derivatives remain bounded when multiplied by polynomials, i.e., whose derivatives are rapidly decreasing (see Lemma 1 of [38], p.117).

Lemma 1 *(i) The Fourier transform of the function $\|\mathbf{z}\|^{-d+\beta}$ is $\nu(\beta)\|\mathbf{z}\|^{-\beta}$, in the sense that*

$$\int_{\mathbb{R}^d} \|\mathbf{z}\|^{-d+\beta}\overline{\psi(\mathbf{z})}d\mathbf{z} = \int_{\mathbb{R}^d} \nu(\beta)\|\mathbf{z}\|^{-\beta}\overline{\mathcal{F}(\psi)(\mathbf{z})}d\mathbf{z}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{55}$$

(ii) The identity $\mathcal{F}\left((-\Delta)^{-\beta/2}(f)\right)(\mathbf{z}) = \|\mathbf{z}\|^{-\beta}\mathcal{F}(f)(\mathbf{z})$ holds in the sense that

$$\int_{\mathbb{R}^d} (-\Delta)^{-\beta/2}(f)(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x})\|\mathbf{x}\|^{-\beta}\overline{\mathcal{F}(g)(\mathbf{x})}d\mathbf{x}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \tag{56}$$

for $0 < \beta < d$.

(iii) The following convolution formula is obtained by iteration of (55)–(56)

$$\int_{\mathbb{R}^d} \frac{1}{\nu(4\beta)} \|\mathbf{z}\|^{-d+4\beta}\overline{f(\mathbf{z})}d\mathbf{z} = \int_{\mathbb{R}^d} \|\mathbf{z}\|^{-4\beta}\overline{\mathcal{F}(f)(\mathbf{z})}d\mathbf{z}$$
$$= \int_{\mathbb{R}^d} \frac{1}{[\nu(\beta)]^4} \left[\int_{\mathbb{R}^{3d}} \|\mathbf{z}-\mathbf{x}_1\|^{-d+\beta}\|\mathbf{x}_1-\mathbf{x}_2\|^{-d+\beta}\|\mathbf{x}_2-\mathbf{y}\|^{-d+\beta}\|\mathbf{y}\|^{-d+\beta}d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{y}\right]$$
$$\times\overline{f(\mathbf{z})}d\mathbf{z}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad 0<\beta<d/4. \tag{57}$$

The proof of this lemma can be seen in [38], p.117, and equations (26)–(27) in [26].

Let K be the characteristic function of the uniform distribution over set $\mathcal{D}$, given by

$$K(\boldsymbol{\lambda}, \mathcal{D}) = \int_{\mathcal{D}} e^{-\mathrm{i}\langle \boldsymbol{\lambda}, \mathbf{x}\rangle} p_{\mathcal{D}}(\mathbf{x})\, d\mathbf{x} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} e^{-\mathrm{i}\langle \boldsymbol{\lambda}, \mathbf{x}\rangle} d\mathbf{x} = \frac{\vartheta(\boldsymbol{\lambda})}{|\mathcal{D}|}, \tag{58}$$

with associated probability density function $p_{\mathcal{D}}(\mathbf{x}) = 1/|\mathcal{D}|$ if $\mathbf{x}\in\mathcal{D}$, and 0 otherwise.

Theorem 4.1(i) in [26] provides, for $0<\alpha<d/2$, the Hilbert-Schmidt property of the integral operator with kernel $H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = K(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, \mathcal{D})$, where K is given in (58), as an operator acting on the space of square integrable functions with respect to the measure

$$\mu(d\boldsymbol{\lambda}) = \frac{d\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|^{d-\alpha}}.$$

Remark 3 *Note that for $\mathcal{D} = \mathcal{B}(\mathbf{0}) = \{\mathbf{x}\in\mathbb{R}^d;\ \|\mathbf{x}\|\leq 1\}$, the function $\vartheta(\boldsymbol{\lambda})$ in (58) is of the form:*

$$\int_{\mathcal{B}(\mathbf{0})} \exp\left(\mathrm{i}\,\langle \mathbf{x}, \boldsymbol{\lambda}\rangle\right) d\mathbf{x} = (2\pi)^{d/2}\frac{\mathcal{J}_{d/2}\left(\|\boldsymbol{\lambda}\|\right)}{\|\boldsymbol{\lambda}\|^{d/2}}, \quad d\geq 2,$$

where $\mathcal{J}_\nu(\mathbf{z})$ is the Bessel function of the first kind and order $\nu>-1/2$. For a rectangle, $\mathcal{D} = \prod = \{a_i\leq x_i\leq b_i,\ i=1,\dots,d\}, \quad \mathbf{0}\in\prod$,

$$\vartheta(\boldsymbol{\lambda}) = \prod_{j=1}^{d}\left(\exp\left(\mathrm{i}\lambda_j b_j\right) - \exp\left(\mathrm{i}\lambda_j a_j\right)\right)/\mathrm{i}\lambda_j, \quad d\geq 1$$

(see, for example, [25]).

Let us consider the following additional condition:

Condition A4. Suppose that **Condition A2** holds, and there exists a spectral density $f_0(\|\boldsymbol{\lambda}\|)$, $\boldsymbol{\lambda}\in\mathbb{R}^d$, being decreasing function for $\|\boldsymbol{\lambda}\|\in(0,\varepsilon]$, with $\varepsilon>0$.

Under **Condition A4**, from equation (20), applying a Tauberian Theorem (see [14], and Theorems 4 and 11 in [25]),

$$f_0(\|\boldsymbol{\lambda}\|) \sim c(d,\alpha)\mathcal{L}\left(\frac{1}{\|\boldsymbol{\lambda}\|}\right)\|\boldsymbol{\lambda}\|^{\alpha-d}, \quad 0<\alpha<d, \quad \|\boldsymbol{\lambda}\|\to 0, \tag{59}$$

where

$$c(d,\alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^{\alpha}\pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)} = \frac{1}{\nu(\alpha)},$$

with $\nu(\alpha)$ being given in (54).
As indicated in [26], **Condition A4** holds, in particular, for the correlation functions of the form

$$B(\|\mathbf{z}\|) = \frac{1}{(1+\|\mathbf{z}\|^{\beta})^{\gamma}}, \quad 0 < \beta \leq 2, \quad \gamma > 0, \tag{60}$$

with the isotropic spectral density

$$f_0(\|\boldsymbol{\lambda}\|) = \frac{\|\boldsymbol{\lambda}\|^{1-\frac{d}{2}}}{2^{\frac{d}{2}-1}\pi^{\frac{d}{2}+1}} \int_0^{\infty} K_{\frac{d}{2}-1}(\|\boldsymbol{\lambda}\|u) \frac{\sin\left(\gamma \arg\left(1+u^{\beta}\exp\left(\frac{i\pi\beta}{2}\right)\right)\right)}{\left|1+u^{\beta}\exp\left(\frac{i\pi\beta}{2}\right)\right|^{\gamma}} u^{\frac{d}{2}} du, \tag{61}$$

where $K_{\nu}(z)$ is the modified Bessel function of the second kind. By Corollary 3.10 in [28], the spectral density (61) satisfies (59), with $\alpha = \beta\gamma < d$.

Note that, from Theorem 4.1(ii) in [26], under **Conditions A2** and **A4**, considering the Hilbert-Schmidt property of the integral operator with kernel constructed from (58), for $0 < \alpha < d/2$, the limiting distribution $S_{\infty}^{\chi_r^2}$, with characteristic function (34), of the functional $S_T^{\chi_r^2}$ in (27), for Laguerre rank equal to one, admits the following double Wiener-Itô stochastic integral representation:

$$S_{\infty}^{\chi_r^2} = -\frac{|\mathcal{D}|C_1^L}{\nu(\alpha)\sqrt{2r}} \sum_{j=1}^{r} \int_{\mathbb{R}^{2d}}' H(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) \frac{Z_j\left(d\boldsymbol{\lambda}_1\right) Z_j\left(d\boldsymbol{\lambda}_2\right)}{\|\boldsymbol{\lambda}_1\|^{\frac{d-\alpha}{2}} \|\boldsymbol{\lambda}_2\|^{\frac{d-\alpha}{2}}}, \tag{62}$$

where Z_j, $j = 1,\dots,r$, are independent Gaussian white noise measures, ν is defined in (54), and the notation $\int_{\mathbb{R}^{2d}}'$ means that one does not integrate on the hyperdiagonals $\boldsymbol{\lambda}_1 = \pm\boldsymbol{\lambda}_2$. As before,

$$H\left(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2\right) = K\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2,\mathcal{D}\right), \tag{63}$$

where $K\left(\boldsymbol{\lambda},\mathcal{D}\right)$ is given in (58). In particular, equation (62) holds for the limit, in distribution sense, as $T \to \infty$, of (see (37))

$$\begin{aligned} S_{1,T} &= \frac{C_1^L}{a_{d,1}^{\chi_r^2}(\mathcal{D})\mathcal{L}(T)T^{d-\alpha}} \int_{\mathcal{D}(T)} e_1^{(r/2)}(\chi_r^2(\mathbf{x}))\, d\mathbf{x} \\ &= -\left[\frac{C_1^L}{a_{d,1}^{\chi_r^2}(\mathcal{D})\mathcal{L}(T)T^{d-\alpha}}\right] \left[\frac{1}{\sqrt{2r}} \sum_{j=1}^{r} \int_{\mathcal{D}(T)} H_2(Y_j(\mathbf{x}))d\mathbf{x}\right]. \end{aligned} \tag{64}$$

On the other hand, the zero-mean Gaussian random field Y with an absolutely continuous spectrum has the isonormal representation

$$Y(\mathbf{x}) = \int_{\mathbb{R}^d} \exp\left(i\left\langle \boldsymbol{\lambda},\mathbf{x}\right\rangle\right) \sqrt{f_0(\|\boldsymbol{\lambda}\|)} Z(d\boldsymbol{\lambda}), \tag{65}$$

where Z is a complex white noise Gaussian random measure with Lebesgue control measure.

We now turn to the case $k = 2$.

Proposition 1 *Let $\mathcal{D}$ be a regular bounded open domain, and let $K(\boldsymbol{\lambda}, \mathcal{D})$ be defined in (58). For $0 < \alpha < d/4$, the following identities hold:*

$$\int_{\mathbb{R}^{4d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 + \boldsymbol{\lambda}_4, \mathcal{D})|^2 \frac{\prod_{i=1}^{4} d\boldsymbol{\lambda}_i}{\prod_{i=1}^{4}(\|\boldsymbol{\lambda}_i\|)^{d-\alpha}} = \frac{[a_{d,2}^{\chi_r^2}(\mathcal{D})]^2[\nu(\alpha)]^4}{|\mathcal{D}|^2} < \infty, \tag{66}$$

where $a_{d,2}^{\chi_r^2}(\mathcal{D})$ is defined as in equation (26) for $k = 2$, and $\nu(\alpha)$ is introduced in equation (54).

Proof. The proof follows from the application of Theorem 3.1 in [26], where the asymptotic spectral properties of operator $\mathcal{K}_\alpha$ in equation (43), on a regular bounded domain $\mathcal{D}$, are established.

For $0 < \alpha < d/4$, let us now consider on the space of infinitely differentiable functions with compact support contained in $\mathcal{D}$, $C_0^\infty(\mathcal{D}) \subset \mathcal{S}(\mathbb{R}^d)$, the norm

$$\begin{aligned}
&\|f\|^2_{(-\Delta)^{2\alpha-d/2}} = \left\langle (-\Delta)^{2\alpha-d/2}(f), f \right\rangle_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} (-\Delta)^{2\alpha-d/2}(f)(\mathbf{x})\overline{f(\mathbf{x})}d\mathbf{x} = \int_{\mathbb{R}^d} \frac{1}{\nu(d-4\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{4\alpha}} f(\mathbf{y})\overline{f(\mathbf{x})}d\mathbf{y}d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}(f)(\boldsymbol{\lambda})|^2 \|\boldsymbol{\lambda}\|^{-(d-4\alpha)} d\boldsymbol{\lambda}, \quad \forall f \in C_0^\infty(\mathcal{D}).
\end{aligned} \tag{67}$$

The associated inner product is given by ($0 < \alpha < d/4$)

$$\langle f, g\rangle_{(-\Delta)^{2\alpha-d/2}} = \int_{\mathbb{R}^d} \frac{1}{\nu(d-4\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{4\alpha}} f(\mathbf{y})\overline{g(\mathbf{x})}d\mathbf{y}d\mathbf{x}, \quad \forall f, g \in C_0^\infty(\mathcal{D}). \tag{68}$$

The closure of $C_0^\infty(\mathcal{D})$ with the norm $\|\cdot\|_{(-\Delta)^{2\alpha-d/2}}$, introduced in (67), defines a Hilbert space, which will be denoted as

$$\mathcal{H}_{4\alpha-d} = \overline{C_0^\infty(\mathcal{D})}^{\|\cdot\|_{(-\Delta)^{2\alpha-d/2}}}.$$

For a bounded open domain $\mathcal{D}$, from Proposition 2.2 in [10], with $D = n-1$, $p = q = 2$, and $s = 0$ (hence, $A_{pq}^s(\mathcal{D}) = A_{22}^0(\mathcal{D}) = L^2(\mathcal{D})$, where, as usual, $L^2(\mathcal{D})$ denotes the space of square integrable functions on $\mathcal{D}$), we have

$$\overline{C_0^\infty(\mathcal{D})}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} = L^2(\mathcal{D}), \tag{69}$$

(see also [42], for the case of regular bounded open domains with $C^\infty-$boundaries). In addition, for all $f \in C_0^\infty(\mathcal{D})$, by definition of the norm (67),

$$\|f\|_{(-\Delta)^{2\alpha-d/2}} \leq C\|f\|_{L^2(\mathbb{R}^d)},$$

that is, all convergent sequences of $C_0^\infty(\mathcal{D})$ in the $L^2(\mathbb{R}^d)$ norm are also convergent in the $\mathcal{H}_{4\alpha-d}$ norm. Hence, the closure of $C_0^\infty(\mathcal{D})$, with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}^d)}$, is included in the closure of $C_0^\infty(\mathcal{D})$, with respect to the norm $\|\cdot\|_{(-\Delta)^{2\alpha-d/2}}$. Therefore, from equation (69),

$$L^2(\mathcal{D}) = \overline{C_0^\infty(\mathcal{D})}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} \subseteq \overline{C_0^\infty(\mathcal{D})}^{\|\cdot\|_{(-\Delta)^{2\alpha-d/2}}} = \mathcal{H}_{4\alpha-d}. \tag{70}$$

In particular, let us compute

$$\|1_{\mathcal{D}}\|^2_{\mathcal{H}_{4\alpha-d}} = \int_{\mathcal{D}} \frac{1}{\nu(d-4\alpha)} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{4\alpha}} d\mathbf{y}d\mathbf{x} = \frac{[a_{d,2}^{\chi_r^2}(\mathcal{D})]^2}{\nu(d-4\alpha)}. \tag{71}$$

As noted before, from Theorem 3.1 in [26],

$$\mathrm{Tr}(\mathcal{K}_\alpha^2) = \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y}d\mathbf{x} < \infty, \quad 0 < \alpha < d/2. \tag{72}$$

Thus, for $\alpha = 2\beta$,

$$\int_{\mathcal{D}} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{4\beta}} d\mathbf{y}d\mathbf{x} < \infty, \quad 0 < \beta < d/4.$$

Therefore,

$$[a_{d,2}^{\chi_r^2}(\mathcal{D})]^2 = \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{4\alpha}} d\mathbf{y}d\mathbf{x} = \nu(d-4\alpha)\|1_{\mathcal{D}}\|^2_{\mathcal{H}_{4\alpha-d}} < \infty, \quad 0 < \alpha < d/4.$$

Equivalently, $1_{\mathcal{D}}$ belongs to the Hilbert space $\mathcal{H}_{4\alpha-d}$, for $0 < \alpha < d/4$.

Applying the convolution formula (57) in Lemma 1, we then obtain

$$\begin{aligned}
\frac{[a_{d,2}^{\chi_r^2}(\mathcal{D})]^2}{\nu(d-4\alpha)} &= \|1_{\mathcal{D}}\|^2_{\mathcal{H}_{4\alpha-d}} = \frac{|\mathcal{D}|^2}{(2\pi)^d} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, \mathcal{D})|^2 \|\boldsymbol{\omega}_1\|^{-d+4\alpha} d\boldsymbol{\omega}_1 \\
&= \frac{|\mathcal{D}|^2}{(2\pi)^d} \frac{\nu(4\alpha)}{[\nu(\alpha)]^4} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, \mathcal{D})|^2 \left[\int_{\mathbb{R}^{3d}} \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^{-d+\alpha} \|\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3\|^{-d+\alpha} \right. \\
&\qquad \left. \times \|\boldsymbol{\omega}_3 - \boldsymbol{\omega}_4\|^{-d+\alpha} \|\boldsymbol{\omega}_4\|^{-d+\alpha} \prod_{i=2}^4 d\boldsymbol{\omega}_i \right] d\boldsymbol{\omega}_1 \\
&= \frac{|\mathcal{D}|^2 \nu(4\alpha)}{(2\pi)^d [\nu(\alpha)]^4} \int_{\mathbb{R}^{4d}} \left| K\left(\sum_{i=1}^4 \boldsymbol{\lambda}_i, \mathcal{D} \right) \right|^2 \frac{\prod_{i=1}^4 d\boldsymbol{\lambda}_i}{\prod_{i=1}^4 \|\boldsymbol{\lambda}_i\|^{d-\alpha}}.
\end{aligned}$$

Hence,

$$[a_{d,2}^{\chi_r^2}(\mathcal{D})]^2 = \frac{|\mathcal{D}|^2}{[\nu(\alpha)]^4} \int_{\mathbb{R}^{4d}} \left| K\left(\sum_{i=1}^4 \boldsymbol{\lambda}_i, \mathcal{D} \right) \right|^2 \frac{\prod_{i=1}^4 d\boldsymbol{\lambda}_i}{\prod_{i=1}^4 \|\boldsymbol{\lambda}_i\|^{d-\alpha}},$$

since $\frac{\nu(4\alpha)\nu(d-4\alpha)}{(2\pi)^d} = 1$. Equation (66) then holds.

Note that by definition of the norm in $\mathcal{H}_{4\alpha-d}$, from equation (70),

$$1_{\mathcal{D}} \star 1_{\mathcal{D}}(\mathbf{x}) = \int_{\mathbb{R}^d} 1_{\mathcal{D}}(\mathbf{y})1_{\mathcal{D}}(\mathbf{x}+\mathbf{y})d\mathbf{y} = \int_{\mathcal{D}} 1_{\mathcal{D}}(\mathbf{x}+\mathbf{y})d\mathbf{y} \in L^2(\mathcal{D}) \subseteq \mathcal{H}_{4\alpha-d},$$

since

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} 1_{\mathcal{D}}(\mathbf{y})1_{\mathcal{D}}(\mathbf{x}+\mathbf{y})d\mathbf{y} \right|^2 d\mathbf{x} \leq \left|\mathcal{B}_{R(\mathcal{D})}(\mathbf{0})\right|^3,$$

where $|\mathcal{B}_{R(\mathcal{D})}(\mathbf{0})|$ denotes the Lebesgue measure of the ball of center $\mathbf{0}$ and radius $R(\mathcal{D})$, with $R(\mathcal{D})$ being equal to two times the diameter of the regular bounded open set $\mathcal{D}$ containing the point $\mathbf{0}$. Hence,

$$\mathcal{F}(1_{\mathcal{D}} \star 1_{\mathcal{D}})(\boldsymbol{\lambda}) = |\mathcal{D}|^2 |K(\boldsymbol{\lambda}, \mathcal{D})|^2$$

belongs to the space of Fourier transforms of functions in $\mathcal{H}_{4\alpha-d}$ (see also Remark 3.1 in [26]).

Theorem 2 establishes the limit of (36) involving $e_1^{(r/2)}$. The next theorem provides the limit of (53), involving $e_2^{(r/2)}$. Note that $e_2^{r/2}$ is defined in (11), but also satisfies (75) below. As before, $K(\cdot, \mathcal{D})$ will denote the characteristic function of the uniform distribution over the set $\mathcal{D}$. The random measures $Z_j(\cdot)$, $j = 1, 2, 3, 4$, appearing in Theorem 3 below are independent Wiener measures. The symbol $\int'_{\mathbb{R}^{2d}}$ means that, in the integration in the mean-square sense, the hyperdiagonals $\boldsymbol{\lambda}_1 = \pm\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3 = \pm\boldsymbol{\lambda}_4$, related to each component Z_j and Z_k, are excluded (see [17]). Furthermore, $\int''_{\mathbb{R}^{4d}}$ means that one can not integrate on the hyperdiagonals $\boldsymbol{\lambda}_i = \pm\boldsymbol{\lambda}_j$, $i \neq j$, $i, j = 1, 2, 3, 4$.

Theorem 3 *Assume that* ***Conditions A2*** *and* ***A4*** *hold. Then, for* $0 < \alpha < d/4$, *the functional* $S_{2,T}$ *defined in (53) converges in distribution sense to the random variable* S_∞ *admitting the following multiple Wiener-Itô stochastic integral representation:*

$$\begin{aligned} S_\infty \underset{d}{=} & \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2} \left[r\left(\frac{r}{2}+1\right)\right]^{-1/2} \\ & \times \left[\sum_{k,j;k\neq j}^{r} \int'_{\mathbb{R}^{2d}} \int'_{\mathbb{R}^{2d}} K\left(\sum_{i=1}^4 \boldsymbol{\lambda}_i, \mathcal{D}\right) \frac{Z_j(d\boldsymbol{\lambda}_1)Z_j(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)}{\prod_{i=1}^4 \|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}} \right. \\ & \left. - \sum_{k=1}^{r} \int''_{\mathbb{R}^{4d}} K\left(\sum_{i=1}^4 \boldsymbol{\lambda}_i, \mathcal{D}\right) \frac{\prod_{i=1}^4 Z_k(d\boldsymbol{\lambda}_i)}{\prod_{i=1}^4 \|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}} \right]. \end{aligned} \tag{73}$$

Proof.

The restriction to $\mathcal{D}(T)$ of the independent copies Y_j, $j = 1, \dots, r$, of Gaussian random field Y, i.e., $\{Y_j(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}(T),\ j = 1, \dots, r\}$, satisfying **Conditions A2** and **A4**, admit the following stochastic integral representation (see equation (65)):

$$Y_j(\mathbf{x}) = \frac{|\mathcal{D}(T)|}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(\mathrm{i}\left\langle \mathbf{x}, \boldsymbol{\lambda}\right\rangle\right) K\left(\boldsymbol{\lambda}, \mathcal{D}(T)\right) f_0^{1/2}(\boldsymbol{\lambda}) Z_j(d\boldsymbol{\lambda}), \quad \mathbf{x} \in \mathcal{D}(T), \quad j = 1, \dots, r. \tag{74}$$

It is well-known (see, for example, [2]) that

$$
\begin{aligned}
e_2^{(r/2)}(\chi_r^2(\mathbf{x})) &= \frac{1}{4}\left(r\left(\frac{r}{2}+1\right)\right)^{-1/2} \\
&\times \left[\sum_{k,j=1,\ k\neq j}^{r} H_2(Y_k(\mathbf{x}))H_2(Y_j(\mathbf{x})) - \sum_{k=1}^{r} H_4(Y_k(\mathbf{x}))\right],
\end{aligned}
\tag{75}
$$

where, as before, $\chi_r^2(\mathbf{x})$ is the chi-squared random field introduced in (17), and $e_2^{(r/2)}$ denotes the second Laguerre polynomial with index $r/2$ (see [5], Chapter 10). Here, $H_2(u) = u^2 - 1$ is the second Chebyshev-Hermite polynomial, and $H_4(u) = u^4 - 6u^2 + 3$ is the fourth Chebyshev-Hermite polynomial.

From equation (75), the functional (53) admits the following representation:

$$
\begin{aligned}
S_{2,T} &= \frac{1}{a_{d,2}^{\chi_r^2}(\mathcal{D})\mathcal{L}^2(T)T^{d-2\alpha}} \int_{\mathcal{D}(T)} e_2^{r/2}(\chi_r^2(\mathbf{x}))\, d\mathbf{x} \\
&= \frac{1}{4}\left(r\left(\frac{r}{2}+1\right)\right)^{-1/2} \frac{1}{d_T}\left[\sum_{k,j=1,\ k\neq j}^{r} \int_{\mathcal{D}(T)} H_2(Y_k(\mathbf{x}))H_2(Y_j(\mathbf{x}))d\mathbf{x}\right. \\
&\quad \left. -\sum_{k=1}^{r}\int_{\mathcal{D}(T)} H_4(Y_k(\mathbf{x}))d\mathbf{x}\right],
\end{aligned}
\tag{76}
$$

where

$$
d_T = a_{d,2}^{\chi_r^2}(\mathcal{D})\mathcal{L}^2(T)T^{d-2\alpha}.
$$

Using Itô's formula (see, for example, [13]; [30]), we obtain from equation (76)

$$
\begin{aligned}
S_{2,T} &= \frac{1}{4}\left(r\left(\frac{r}{2}+1\right)\right)^{-1/2} \frac{1}{d_T}\left[\sum_{k,j=1,\ k\neq j}^{r} \int_{\mathcal{D}(T)} \int_{\mathbb{R}^{2d}}' \int_{\mathbb{R}^{2d}}' \exp\left(\left\langle \mathbf{x}, \sum_{i=1}^{4}\boldsymbol{\lambda}_i\right\rangle\right)\right. \\
&\quad \times \prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|)} \prod_{i=1}^{2} Z_j(d\boldsymbol{\lambda}_i) \prod_{i=3}^{4} Z_k(d\boldsymbol{\lambda}_i) d\mathbf{x} \\
&\quad \left. -\sum_{k=1}^{r}\int_{\mathcal{D}(T)}\int_{\mathbb{R}^{4d}}'' \exp\left(\left\langle \mathbf{x}, \sum_{i=1}^{4}\boldsymbol{\lambda}_i\right\rangle\right) \prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|)} Z_k(d\boldsymbol{\lambda}_i) d\mathbf{x}\right]
\end{aligned}
$$

$$= \frac{1}{4}\left(r\left(\frac{r}{2}+1\right)\right)^{-1/2} \frac{|\mathcal{D}|}{[\nu(\alpha)]^2 d_T}\left[\sum_{k,j=1,\ k\neq j}^{r} \int_{\mathbb{R}^{2d}}' \int_{\mathbb{R}^{2d}}' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\right.$$
$$\times\left([\nu(\alpha)]^2 \prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|/T)}\right)\prod_{i=1}^{2} Z_j(d\boldsymbol{\lambda}_i)\prod_{i=3}^{4} Z_k(d\boldsymbol{\lambda}_i)$$
$$\left.-\sum_{k=1}^{r}\int_{\mathbb{R}^{4d}}'' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\left([\nu(\alpha)]^2\prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|/T)}\right) Z_k(d\boldsymbol{\lambda}_i)\right]. \tag{77}$$

Hence, applying Minkowski inequality,

$$E\left[S_{2,T} - \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\left[\sum_{k,j;k\neq j}^{r}\int_{\mathbb{R}^{2d}}'\int_{\mathbb{R}^{2d}}' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\right.\right.$$
$$\times\frac{Z_j(d\boldsymbol{\lambda}_1)Z_j(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}$$
$$\left.\left.-\sum_{k=1}^{r}\int_{\mathbb{R}^{4d}}'' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\frac{\prod_{i=1}^{4} Z_k(d\boldsymbol{\lambda}_i)}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right]\right]^2$$
$$= \left[\frac{1}{4}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\frac{|\mathcal{D}|}{[\nu(\alpha)]^2}\right]^2 E\left[Y_{1T} - Y_1 + Y_2 - Y_{2T}\right]^2$$
$$\leq \left[\frac{1}{4}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\frac{|\mathcal{D}|}{[\nu(\alpha)]^2}\right]^2\left[\left(E\left[Y_{1T}-Y_1\right]^2\right)^{1/2} + \left(E\left[Y_2 - Y_{2T}\right]^2\right)^{1/2}\right]^2, \tag{78}$$

where

$$E\left[Y_{1T}-Y_1\right]^2 = \sum_{k,j;k\neq j}^{r} E\left|\int_{\mathbb{R}^{2d}}'\int_{\mathbb{R}^{2d}}' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\left[\frac{[\nu(\alpha)]^2}{d_T}\prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|/T)}\right.\right.$$
$$\left.\left.-\frac{1}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right] Z_j(d\boldsymbol{\lambda}_1)Z_j(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)\right|^2$$
$$= \sum_{k,j;k\neq j}^{r}\int_{\mathbb{R}^{4d}}\left|K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i, \mathcal{D}\right)\right|^2 Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)\prod_{i=1}^{4}\frac{d\boldsymbol{\lambda}_i}{\|\boldsymbol{\lambda}_i\|^{d-\alpha}}$$

$$
\begin{aligned}
E\left[Y_2-Y_{2T}\right]^2 &= \sum_{k=1}^{r} E\left|\int_{\mathbb{R}^{4d}}'' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)\left[\frac{[\nu(\alpha)]^2}{d_T}\prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|/T)}\right.\right.\\
&\quad \left.\left. -\frac{1}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right] Z_k(d\boldsymbol{\lambda}_1)Z_k(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)\right|^2\\
&= \sum_{k=1}^{r}\int_{\mathbb{R}^{4d}}\left|K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)\right|^2 Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)\prod_{i=1}^{4}\frac{d\boldsymbol{\lambda}_i}{\|\boldsymbol{\lambda}_i\|^{d-\alpha}},
\end{aligned}
\tag{79}
$$

with, for every $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)\in\mathbb{R}^{4d}$,

$$
Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)=\left(\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}\frac{[\nu(\alpha)]^2}{d_T}\prod_{i=1}^{4}\sqrt{f_0(\|\boldsymbol{\lambda}_i\|/T)}-1\right)^2.
$$

The convergence to zero, as $T\to\infty$, of $Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)$, for every $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)\in\mathbb{R}^{4d}$, can be proved applying Proposition 1, following the methodological approach given in pp. 21–22 of [25] (see also Theorem 4.1(ii) in [26]). The suitable application of Dominated Convergence Theorem to equation (79) then leads to the convergence to zero of

$$
\begin{aligned}
E\Bigg[S_{2,T}-\frac{1}{4}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\frac{|\mathcal{D}|}{[\nu(\alpha)]^2}&\Bigg[\sum_{k,j;k\neq j}^{r}\int_{\mathbb{R}^{4d}}'' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)\\
&\times\frac{Z_j(d\boldsymbol{\lambda}_1)Z_j(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\\
-\sum_{k=1}^{r}\int_{\mathbb{R}^{4d}}'' K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)&\frac{\prod_{i=1}^{4}Z_k(d\boldsymbol{\lambda}_i)}{\prod_{i=1}^{4}\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\Bigg]\Bigg]^2
\end{aligned}
$$

as $T\to\infty$, from equation (78), which implies the convergence in probability and hence, in distribution sense, as we wanted to prove.

6. Series representation in terms of independent random variables

The main result of this section provides the series representation, in terms of independent random variables, of the limit random variable S_∞ of functional $S_{2,T}$ in (53).

Note that, in the case where the Laguerre rank is equal to one, from Theorem 4.1 and Corollary 4.1 in [26], the limit random variable $S_\infty^{\chi_r^2}$ admits the following series representation:

$$S_{\infty}^{\chi_r^2} \underset{d}{=} -\frac{1}{\sqrt{2r}\nu(\alpha)}|\mathcal{D}|\sum_{j=1}^{r}\sum_{n=1}^{\infty}\mu_n(\widetilde{\mathcal{H}})(\varepsilon_{jn}^2-1) = \sum_{j=1}^{r}\sum_{n=1}^{\infty}\lambda_n(S_{\infty}^{\chi_r^2})(\varepsilon_{jn}^2-1), \tag{80}$$

where $\nu(\alpha)$ is given in (54), $\{\varepsilon_{jn},\ n\geq 1,\ j=1,\dots,r\}$ are independent and identically distributed standard Gaussian random variables, and

$$\lambda_n(S_{\infty}^{\chi_r^2}) = -\frac{1}{\sqrt{2r}\nu(\alpha)}|\mathcal{D}|\mu_n(\widetilde{\mathcal{H}}), \tag{81}$$

with $\mu_n(\widetilde{\mathcal{H}})$, $n\geq 1$, being a decreasing sequence of non-negative real numbers, which are the eigenvalues of the self-adjoint Hilbert-Schmidt operator

$$\widetilde{\mathcal{H}}(f)(\boldsymbol{\lambda}_1) = \int_{\mathbb{R}^d} h_1(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)\, f(\boldsymbol{\lambda}_2)\, G_{\alpha}(d\boldsymbol{\lambda}_2) : L^2_{G_\alpha}\left(\mathbb{R}^d\right) \longrightarrow L^2_{G_\alpha}\left(\mathbb{R}^d\right), \tag{82}$$

for all $f\in L^2_{G_\alpha}\left(\mathbb{R}^d\right)$. Here,

$$G_{\alpha}(d\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^{d-\alpha}}d\mathbf{x}, \tag{83}$$

and the symmetric kernel $h_1(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2) = H(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = K\left(\sum_{i=1}^{2}\boldsymbol{\lambda}_i,\mathcal{D}\right)$, with K being the characteristic function of the uniform distribution over the set $\mathcal{D}$.

In the following result we denote by

$$\mathcal{H} : L^2_{G_\alpha\otimes G_\alpha}\left(\mathbb{R}^{2d}\right) \longrightarrow L^2_{G_\alpha\otimes G_\alpha}\left(\mathbb{R}^{2d}\right),$$

the integral operator defined by:

$$\mathcal{H}(h)(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = \int_{\mathbb{R}^{2d}} K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right) h(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)\, G_{\alpha}(d\boldsymbol{\lambda}_3)G_{\alpha}(d\boldsymbol{\lambda}_4), \tag{84}$$

for all $h\in L^2_{G_\alpha\otimes G_\alpha}\left(\mathbb{R}^{2d}\right)$.

Theorem 4 *Assume that the conditions of Proposition 1 and Theorem 3 hold. For the case of Laguerre rank being equal to two, the limit random variable S_∞ in Theorem 3 admits the following series representation:*

$$\frac{S_\infty}{\frac{1}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}|\mathcal{D}|} \underset{d}{=} \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn} \times\left[\sum_{k,j:k\neq j}^{r}(\varepsilon_{j,p,n}^2-1)(\varepsilon_{k,q,n}^2-1) - \sum_{k=1}^{r}(\varepsilon_{k,p,n}^2-1)(\varepsilon_{k,q,n}^2-1)\right], \tag{85}$$

where $\{\varepsilon_{j,p,n},\ j=1,\dots,r,\ p\geq 1,\ n\geq 1\}$ are independent standard Gaussian random variables, in particular, $E[\varepsilon_{j,p,n}\varepsilon_{k,q,m}] = \delta_{n,m}\delta_{p,q}\delta_{j,k}$, for every $j,k=1,\dots,r$, and

$n, m, q, p \geq 1$. Here, $\mu_n(\mathcal{H})$, $n \geq 1$, are the eigenvalues, arranged in decreasing order of their modulus magnitude, associated with the eigenvectors φ_n, $n \geq 1$, of the integral operator (84). Moreover, for each $n \geq 1$, γ_{jn}, $j \geq 1$, are the eigenvalues associated with the integral operator on $L^2_{G_\alpha}(\mathbb{R}^d)$, with kernel $\varphi_n(\cdot,\cdot)$, the n–th eigenvector of integral operator (84).

Proof. From Proposition 1, the operator $\mathcal{H}$ defined in (84) is a Hilbert-Schmidt operator. Equivalently, its kernel

$$H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4) = K\left(\sum_{i=1}^{4} \boldsymbol{\lambda}_i, \mathcal{D}\right)$$

belongs to the space $L^2_{G_\alpha \otimes G_\alpha \otimes G_\alpha \otimes G_\alpha}\left(\mathbb{R}^{4d}\right)$. Thus, $\mathcal{H} \in \mathcal{S}\left(L^2_{G_\alpha \otimes G_\alpha}\left(\mathbb{R}^{2d}\right)\right)$, where as usual $\mathcal{S}(H)$ denotes the Hilbert space of Hilbert Schmidt operators on the Hilbert space H. Hence, it admits a kernel diagonal spectral representation in terms of a sequence of eigenfunctions $\{\varphi_n,\ n \geq 1\} \subset L^2_{G_\alpha \otimes G_\alpha}\left(\mathbb{R}^{2d}\right)$, and a sequence of associated eigenvalues $\{\mu_n(\mathcal{H}),\ n \geq 1\}$. That is,

$$H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4) = \sum_{n=1}^{\infty} \mu_n(\mathcal{H}) \varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \varphi_n(\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4). \tag{86}$$

In particular, since, for every $n \geq 1$, $\varphi_n \in L^2_{G_\alpha \otimes G_\alpha}\left(\mathbb{R}^{2d}\right)$, then,

$$\int_{\mathbb{R}^{2d}} |\varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{\|\boldsymbol{\lambda}_1\|^{d-\alpha} \|\boldsymbol{\lambda}_2\|^{d-\alpha}} < \infty,$$

which means that $\varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ defines an integral Hilbert-Schmidt operator Υ on $L^2_{G_\alpha}(\mathbb{R}^d)$, given by

$$\Upsilon(f)(\boldsymbol{\lambda}_1) = \int_{\mathbb{R}^d} \varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) f(\boldsymbol{\lambda}_2) G_\alpha(d\boldsymbol{\lambda}_2), \quad \boldsymbol{\lambda}_1 \in \mathbb{R}^d,\ \forall f \in L^2_{G_\alpha}(\mathbb{R}^d).$$

Therefore, it admits a kernel diagonal spectral representation in $L^2_{G_\alpha}(\mathbb{R}^d)$, in terms of a sequence of eigenvalues $\{\gamma_{pn},\ p \geq 1\}$, and an orthonormal system of eigenfunctions $\{\phi_{pn},\ p \geq 1\}$ of $L^2_{G_\alpha}(\mathbb{R}^d)$, of the form

$$\varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \underset{L^2_{G_\alpha \otimes G_\alpha}(\mathbb{R}^{2d})}{=} \sum_{p=1}^{\infty} \gamma_{pn} \phi_{pn}(\boldsymbol{\lambda}_1) \phi_{pn}(\boldsymbol{\lambda}_2), \tag{87}$$

for each $n \geq 1$, where convergence holds in the norm of the space $L^2_{G_\alpha \otimes G_\alpha}(\mathbb{R}^{2d})$. Replacing in equation (86) the functions $\{\varphi_n\}_{n=1}^{\infty}$ by their respective series representations as given in equation (87), we obtain

$$H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4) \underset{L^2_{\otimes^4 G_\alpha}(\mathbb{R}^{4d})}{=} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \mu_n(\mathcal{H}) \gamma_{pn} \gamma_{qn} \phi_{pn}(\boldsymbol{\lambda}_1) \phi_{pn}(\boldsymbol{\lambda}_2) \phi_{qn}(\boldsymbol{\lambda}_3) \phi_{qn}(\boldsymbol{\lambda}_4), \tag{88}$$

where convergence holds in the norm of the space $L^2_{\otimes^4 G_\alpha}(\mathbb{R}^{4d}) := L^2_{G_\alpha\otimes G_\alpha\otimes G_\alpha\otimes G_\alpha}\left(\mathbb{R}^{4d}\right)$, since, from equations (86)–(87), considering Minkowski inequality, we have

$$\begin{aligned}
&\left\| H(\cdot,\cdot,\cdot,\cdot) - \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}\phi_{pn}(\cdot)\otimes\phi_{pn}(\cdot)\otimes\phi_{qn}(\cdot)\otimes\phi_{qn}(\cdot)\right\|^2_{L^2_{\otimes^4 G_\alpha}(\mathbb{R}^{4d})} \\
&= \left\| \sum_{n=1}^{\infty}\mu_n(\mathcal{H})\varphi_n(\cdot,\cdot)\otimes\varphi_n(\cdot,\cdot)\right. \\
&\qquad \left. - \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}\phi_{pn}(\cdot)\otimes\phi_{pn}(\cdot)\otimes\phi_{qn}(\cdot)\otimes\phi_{qn}(\cdot)\right\|^2_{L^2_{\otimes^4 G_\alpha}(\mathbb{R}^{4d})} \\
&\leq \left[\sum_{n=1}^{\infty}\mu_n(\mathcal{H})\left[\int_{\mathbb{R}^{2d}}|\varphi_n(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)|^2 G_\alpha(d\boldsymbol{\lambda}_3)G_\alpha(d\boldsymbol{\lambda}_4)\right]^{1/2}\right. \\
&\quad \left.\times\left[\int_{\mathbb{R}^{2d}}\left|\varphi_n(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) - \sum_{p=1}^{\infty}\gamma_{pn}\phi_{pn}(\boldsymbol{\lambda}_1)\phi_{pn}(\boldsymbol{\lambda}_2)\right|^2 G_\alpha(d\boldsymbol{\lambda}_1)G_\alpha(d\boldsymbol{\lambda}_2)\right]^{1/2}\right]^2 = 0,
\end{aligned} \tag{89}$$

where we have applied convergence in $L^2_{G_\alpha\otimes G_\alpha}(\mathbb{R}^{2d})$ of the series $\sum_{q=1}^{\infty}\gamma_{qn}\phi_{qn}(\cdot)\otimes\phi_{qn}(\cdot)$ to the function $\varphi_n(\cdot,\cdot)$, for each $n\geq 1$, which, in particular, implies that such a series differs from $\varphi_n(\cdot,\cdot)$ in a set of null $G_\alpha\otimes G_\alpha$-measure.

From Theorem 3,

$$\begin{aligned}
S_\infty \underset{d}{=} \frac{1}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}|\mathcal{D}|&\left[\sum_{k,j;k\neq j}^{r}\int'_{\mathbb{R}^{2d}}\int'_{\mathbb{R}^{2d}}K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)\right. \\
&\qquad\times\frac{Z_j(d\boldsymbol{\lambda}_1)Z_j(d\boldsymbol{\lambda}_2)Z_k(d\boldsymbol{\lambda}_3)Z_k(d\boldsymbol{\lambda}_4)}{\prod_{i=1}^4\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}} \\
&\left.-\sum_{k=1}^{r}\int''_{\mathbb{R}^{4d}}K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)\frac{\prod_{i=1}^4 Z_k(d\boldsymbol{\lambda}_i)}{\prod_{i=1}^4\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right].
\end{aligned}$$

Replacing

$$H(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4) = K\left(\sum_{i=1}^{4}\boldsymbol{\lambda}_i,\mathcal{D}\right)$$

by its series representation (88) in the above equation, one can get

$$S_\infty \underset{d}{=} \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\left[\sum_{k,j:k\neq j}^{r}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}\right.$$
$$\times\int_{\mathbb{R}^{2d}}'\prod_{i=1}^{2}\phi_{pn}(\boldsymbol{\lambda}_i)\frac{Z_j(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}$$
$$\times\int_{\mathbb{R}^{2d}}'\prod_{i=3}^{4}\phi_{qn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}-\sum_{k=1}^{r}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}$$
$$\left.\times\int_{\mathbb{R}^{2d}}'\prod_{i=1}^{2}\phi_{pn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\int_{\mathbb{R}^{2d}}'\prod_{i=3}^{4}\phi_{qn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right]$$

$$\underset{d}{=} \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}$$
$$\times\left[\sum_{k,j:k\neq j}^{r}\int_{\mathbb{R}^{2d}}'\prod_{i=1}^{2}\phi_{pn}(\boldsymbol{\lambda}_i)\frac{Z_j(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\int_{\mathbb{R}^{2d}}'\prod_{i=3}^{4}\phi_{qn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right.$$
$$\left.-\sum_{k=1}^{r}\int_{\mathbb{R}^{2d}}'\prod_{i=1}^{2}\phi_{pn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\int_{\mathbb{R}^{2d}}'\prod_{i=3}^{4}\phi_{qn}(\boldsymbol{\lambda}_i)\frac{Z_k(d\boldsymbol{\lambda}_i)}{\|\boldsymbol{\lambda}_i\|^{(d-\alpha)/2}}\right].$$

Applying Itô's formula (see, for example, [13]; [30]),

$$S_\infty \underset{d}{=} \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}$$
$$\times\left[\sum_{k,j:k\neq j}^{r}H_2\left(\int_{\mathbb{R}^d}'\phi_{pn}(\boldsymbol{\lambda})\frac{Z_j(d\boldsymbol{\lambda})}{\|\boldsymbol{\lambda}\|^{(d-\alpha)/2}}\right)H_2\left(\int_{\mathbb{R}^d}'\phi_{qn}(\boldsymbol{\lambda})\frac{Z_k(d\boldsymbol{\lambda})}{\|\boldsymbol{\lambda}\|^{(d-\alpha)/2}}\right)\right.$$
$$\left.-\sum_{k=1}^{r}H_2\left(\int_{\mathbb{R}^d}'\phi_{pn}(\boldsymbol{\lambda})\frac{Z_k(d\boldsymbol{\lambda})}{\|\boldsymbol{\lambda}\|^{(d-\alpha)/2}}\right)H_2\left(\int_{\mathbb{R}^d}'\phi_{qn}(\boldsymbol{\lambda})\frac{Z_k(d\boldsymbol{\lambda})}{\|\boldsymbol{\lambda}\|^{(d-\alpha)/2}}\right)\right]$$
$$\underset{d}{=} \frac{|\mathcal{D}|\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}}{4[\nu(\alpha)]^2}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mu_n(\mathcal{H})\gamma_{pn}\gamma_{qn}$$
$$\times\left[\sum_{k,j:k\neq j}^{r}(\varepsilon_{j,p,n}^2-1)(\varepsilon_{k,q,n}^2-1)-\sum_{k=1}^{r}(\varepsilon_{k,p,n}^2-1)(\varepsilon_{k,q,n}^2-1)\right], \tag{90}$$

as we wanted to prove.

In addition, the orthonormality of the systems of eigenfunctions $\{\varphi_n,\ n \geq 1\}$, in the Hilbert space $L^2_{G_\alpha \otimes G_\alpha}(\mathbb{R}^{2d})$, means that

$$\langle \varphi_n, \varphi_k \rangle_{L^2_{G_\alpha \otimes G_\alpha}(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}} \varphi_n(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \varphi_k(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) G_\alpha(d\boldsymbol{\lambda}_1) G_\alpha(d\boldsymbol{\lambda}_2) = \delta_{n,k}, \tag{91}$$

where, as before $\delta_{n,k}$ denotes the Kronecker delta function. Replacing φ_n and φ_k in (91) by its series representation (87) on $L^2_{G_\alpha}(\mathbb{R}^d)$, we obtain

$$\begin{aligned} & \langle \varphi_n, \varphi_k \rangle_{L^2_{G_\alpha \otimes G_\alpha}(\mathbb{R}^{2d})} = \delta_{n,k} \\ &= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \gamma_{pn} \gamma_{qk} \left[\int_{\mathbb{R}^d} \phi_{pn}(\boldsymbol{\lambda}_1) \phi_{qk}(\boldsymbol{\lambda}_1) G_\alpha(d\boldsymbol{\lambda}_1) \right] \left[\int_{\mathbb{R}^d} \phi_{pn}(\boldsymbol{\lambda}_2) \phi_{qk}(\boldsymbol{\lambda}_2) G_\alpha(d\boldsymbol{\lambda}_2) \right] \\ &= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \gamma_{pn} \gamma_{qk} \left[\langle \phi_{pn}, \phi_{qk} \rangle_{L^2_{G_\alpha}(\mathbb{R}^d)} \right]^2, \end{aligned} \tag{92}$$

which implies that

$$\langle \phi_{pn}, \phi_{qk} \rangle_{L^2_{G_\alpha}(\mathbb{R}^d)} = 0, \quad n \neq k, \quad \forall p, q \geq 1, \tag{93}$$

and we know that

$$\langle \phi_{pn}, \phi_{qk} \rangle_{L^2_{G_\alpha}(\mathbb{R}^d)} = \delta_{p,q}, \quad n = k, \tag{94}$$

from the orthonormality of the system of eigenfunctions providing the diagonal spectral representation (87) of φ_n, for each $n \geq 1$. Hence, in addition, from (92) and (94), we have

$$\sum_{p=1}^{\infty} \gamma_{pn}^2 = 1, \quad \forall n \geq 1.$$

Thus, from equations (90)–(94), and from the independence of the Gaussian copies $Y_i(\cdot)$, $i = 1, \ldots, r$, of random field $Y(\cdot)$, we obtain $E[\varepsilon_{j,q,n} \varepsilon_{k,p,m}] = \delta_{n,m} \delta_{p,q} \delta_{k,j}$, for every $j, k = 1, \ldots, r$, and $n, m, q, p \geq 1$.

In the following, let us denote, by $\mathbf{1}^T$ and $\mathbf{1}$, $1 \times r$ and $r \times 1$ vectors with entries equal to one, respectively. For $p, n \geq 1$, $\boldsymbol{\varepsilon}_{p,n}$ denotes a $r \times 1$ random vector with entries $\varepsilon^2_{i,p,n} - 1$, $i = 1, \ldots, r$, and $\otimes$ denotes the tensorial product of vectors. Finally, Trace($\mathbf{A}$) represents the trace of matrix $\mathbf{A}$.

Corollary 1 *Under the conditions of Theorem 4, for the case where Laguerre rank is* *equal to two, the limit random variable (73) admits the series expansion*

$$\begin{aligned} S_{\infty} &\underset{d}{=} \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2}\sum_{n=1}^{\infty}\mu_n(\mathcal{H})\boldsymbol{\eta}_n \\ &\underset{d}{=} \frac{|\mathcal{D}|}{4[\nu(\alpha)]^2}\left[r\left(\frac{r}{2}+1\right)\right]^{-1/2} \\ &\times \sum_{n=1}^{\infty}\mu_n(\mathcal{H})\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\gamma_{pn}\gamma_{qn}\mathbf{1}^T\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n} - \mathit{Trace}\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n}\right)\right)\mathbf{1}\right. \\ &\qquad\qquad \left. - \mathit{Trace}\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n}\right)\right]. \end{aligned} \tag{95}$$

Note that the random variable system $\{\boldsymbol{\eta}_n\}_{n\geq 1}$, with

$$\boldsymbol{\eta}_n = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\gamma_{pn}\gamma_{qn}\mathbf{1}^T\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n} - \mathrm{Trace}\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n}\right)\right)\mathbf{1} - \mathrm{Trace}\left(\boldsymbol{\varepsilon}_{p,n}\otimes\boldsymbol{\varepsilon}_{q,n}\right), \quad n\geq 1,$$

is constituted by independent random variables. Specifically, for each $n \geq 1$, $\boldsymbol{\eta}_n$ is a function of the random variables $\{(\varepsilon_{i,p,n}^2 - 1),\ i = 1,\dots,r,\ p \geq 1\}$. From equation (90), for $n \neq k$, with $n, k \geq 1$,

$$\{(\varepsilon_{i,p,n}^2 - 1),\ i = 1,\dots,r,\ p \geq 1\}$$

and

$$\{(\varepsilon_{i,q,k}^2 - 1),\ i = 1,\dots,r,\ q \geq 1\}$$

are mutually independent, since the function sequences $\{\phi_{pn}\}_{p\geq 1}$ and $\{\phi_{qk}\}_{q\geq 1}$ are orthogonal in the space $L^2_{G_\alpha}(\mathbb{R}^d)$, as follows from equation (93).

7. Final Comments

In this paper, a reduction principle is established in Theorem 1 for the case of Gamma correlated random fields, whose marginal distributions are characterized in terms of an arbitrary positive shape parameter β. However, we restrict our attention to the case of chi-squared random fields, when the characteristic function of the limit distribution, and its series expansion, in terms of independent random variables, are derived. The main reason is of technical nature, since in the chi-squared random field case, we can exploit the relationship with Gaussian random fields, to obtain the limit characteristic function. Furthermore, the relationship with the Wiener chaos is also applied, to use the so-called Itô formula, for deriving the multiple Wiener-Itô stochastic integral representation of the limit random variable, as well as its series expansion. Otherwise, we would need to define a Fock space for Laguerre polynomials (see [1]; [34], among others), and to develop a kind of Itô formula for Laguerre first and second chaos. Specifically, the extension of the derived results to a more general context, including, for example, non linear functionals of LRD random fields with Gamma and Beta marginal distributions would require more complex tools, which should then be established in terms of the Laguerre and Jacobi first and second chaos. This goes beyond the aim and scope of this paper. In particular, the Gamma correlated random field case could be achieved in an easier way than the case of random fields with beta marginal distributions, which are not infinitely divisible (see [44], for bivariate expansions from Jacobi polynomials). Note that the integration theory for infinitely divisible random measures, established in [35], could be applicable for Gamma-correlated processes and fields, but not for Beta-correlated fields, since the Beta distribution is not infinitely divisible. In the Gamma case, one can then derive non-central limit results, and the properties of the limiting distributions, from a multiple stochastic integral representation of such limit random variables, with respect to some infinite divisible random measure, using a *higher order* analogue of Fredholm determinants, etc. These potential results could constitute the subject of a subsequent paper. Note that, even the chi-squared random field case is non-trivial (see the proof of Theorem 2, and the results displayed in Sections 5–6). Also, the properties of limiting distributions for the case of Hermite rank larger than or equal to three are not yet well-established. Moreover, as explained in the Introduction, non linear functionals of chi-squared random fields arise in several applied fields in Statistics, for example, in the context of Minkowski functionals defined in (1). Thus, this case is of interest by itself. Finally, we remark that the approach presented in [15], for stochastic processes with discrete time, can also be extended to the random field case studied here.

Appendix A: Appendix

This appendix provides the infinitely divisible property of $S_{\infty}^{\chi_r^2}$, the limit random variable obtained in the case where the Laguerre rank is equal to one. Specifically, from Theorem 5.1 in [26], its Lévy-Khintchine representation is derived, as given in the following result.

Theorem 5 *Under the conditions assumed in Theorem 5.1 in [26],*

$$\phi(\theta) = E\left[\exp\left(\mathrm{i}\theta S_{\infty}^{\chi_r^2}\right)\right] = \exp\left(\int_0^{\infty} (\exp(\mathrm{i}u\theta) - 1 - \mathrm{i}u\theta)\, \mu_{\alpha/d}(du)\right), \tag{96}$$

where $\mu_{\alpha/d}$ is supported on $(0,\infty)$ having density

$$q_{\alpha/d}(u) = \frac{r}{2u}\sum_{k=1}^{\infty} \exp\left(-\frac{u}{2\lambda_k(S_{\infty}^{\chi_r^2})}\right), \quad u > 0, \tag{97}$$

with $\lambda_k(S_{\infty}^{\chi_r^2})$, $k \geq 1$, being given in (81). Furthermore, $q_{\alpha/d}$ has the following asymptotics *as* $u \longrightarrow 0^+$ *and* $u \longrightarrow \infty$,

$$q_{\alpha/d}(u) \sim \frac{[\widetilde{c}(d,\alpha)|\mathcal{D}|^{(d-\alpha)/d}]^{1/(1-\alpha/d)}\Gamma\left(\frac{1}{1-\alpha/d}\right)\left(\frac{u}{2}\right)^{-1/(1-\alpha/d)}}{2u[(1-\alpha/d)]}$$

$$= \frac{2^{\frac{\alpha/d}{1-\alpha/d}}[\widetilde{c}(d,\alpha)|\mathcal{D}|^{(d-\alpha)/d}]^{1/(1-\alpha/d)}\Gamma\left(\frac{1}{1-\alpha/d}\right)u^{\frac{(\alpha/d)-2}{(1-\alpha/d)}}}{[(1-\alpha/d)]} \quad \textit{as } u \longrightarrow 0^+,$$

$$q_{\alpha/d}(u) \sim \frac{r}{2u}\exp(-u/2\lambda_1(S_{\infty}^{\chi_r^2})), \quad \textit{as } u \longrightarrow \infty, \tag{98}$$

where

$$\widetilde{c}(d,\alpha) = \pi^{\alpha/2}\left(\frac{2}{d}\right)^{(d-\alpha)/d}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\left[\Gamma\left(\frac{d}{2}\right)\right]^{(d-\alpha)/d}}.$$

Remark 4 *The derived asymptotics of the probability density* $q_{\alpha/d}$ *at the origin and at the infinity are of Gamma form with different shape parameters in the two asymptotics. Indeed, its tail behavior is not chi-squared or even gamma, and its behavior at the origin depends only on the dimension of the space, and the self-similarity parameter.*

Proof. Let us first consider a truncated version of the random series representation (80)

$$S_{\infty}^{(M)} = \sum_{l=1}^{r}\sum_{k=1}^{M}\lambda_k(S_{\infty}^{\chi_r^2})(\varepsilon_{lk}^2 - 1),$$

with $S_{\infty}^{M} \underset{d}{\longrightarrow} S_{\infty}^{\chi_r^2}$, as M tends to infinity. From the Lévy-Khintchine representation of the chi-squared distribution (see, for instance, [4], Example 1.3.22),

$$E\left[\exp(\mathrm{i}\theta S_{\infty}^{(M)})\right] = \prod_{l=1}^{r}\prod_{k=1}^{M}E\left[\exp\left(\mathrm{i}\theta\lambda_k(S_{\infty}^{\chi_r^2})(\varepsilon_{lk}^2-1)\right)\right]$$
$$= \prod_{l=1}^{r}\prod_{k=1}^{M}\exp\left(-\mathrm{i}\theta\lambda_k(S_{\infty}^{\chi_r^2}) + \int_0^{\infty}(\exp(\mathrm{i}\theta u)-1)\left[\frac{\exp\left(-u/(2\lambda_k(S_{\infty}^{\chi_r^2}))\right)}{2u}\right]du\right)$$
$$= \prod_{k=1}^{M}\exp\left(r\int_0^{\infty}(\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{\exp(-u/2\lambda_k(S_{\infty}^{\chi_r^2}))}{2u}\right]du\right)$$
$$= \exp\left(r\int_0^{\infty}(\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{1}{2u}G_{\lambda(\alpha/d)}^{(M)}(\exp(-u/2))\right]du\right). \tag{99}$$

To apply Dominated Convergence Theorem, the following upper bound is used:

$$\begin{aligned}\left|(\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{r}{2u}G^{(M)}_{\lambda(\alpha/d)}(\exp(-u/2))\right]\right| &\leq \frac{r\theta^2}{4}uG^{(M)}_{\lambda(\alpha/d)}(\exp(-u/2))\\ &\leq \frac{r\theta^2}{4}uG_{\lambda(\alpha/d)}(\exp(-u/2)),\end{aligned} \tag{100}$$

where, as indicated in [43], we have applied the inequality $|\exp(\mathrm{i}z)-1-z|\leq\frac{z^2}{2}$, for $z\in\mathbb{R}$. The right-hand side of (100) is continuous, for $0<u<\infty$, and from Lemma 4.1 of [43] with

$$G^{(M)}_{\lambda(\alpha/d)}(x)=\sum_{k=1}^{M}x^{[\lambda_k(S^{\chi_r^2}_\infty)]^{-1}},$$

keeping in mind the asymptotic order of eigenvalues of operator $\mathcal{K}_\alpha$ (see, for example, Theorem 3.1 in [26]), we obtain

$$\begin{aligned}&uG_{\lambda(\alpha/d)}(\exp(-u/2))\sim u\exp(-u/2\lambda_1(S^{\chi_r^2}_\infty)), \quad \text{as } u\longrightarrow\infty\\ &uG_{\lambda(\alpha/d)}(\exp(-u/2))\sim[\widetilde{c}(d,\alpha)|D|^{1-\alpha/d}]^{1/1-\alpha/d}\frac{u}{(1-\alpha/d)}\\ &\times\Gamma\left(\frac{1}{1-\alpha/d}\right)(1-\exp(-u/2))^{-1/(1-\alpha/d)}\sim Cu^{-\frac{\alpha/d}{1-\alpha/d}} \quad \text{as } u\longrightarrow 0,\end{aligned} \tag{101}$$

for some constant C. Since

$$0<\frac{\alpha/d}{1-\alpha/d}<1,$$

both approximations at the right-hand side of (101), which do not depend on M, lead to the integrability on $(0,\infty)$. Hence, by Dominated Convergence Theorem,

$$\begin{aligned}&E\left[\exp(\mathrm{i}\theta S^{(M)}_\infty)\right]\longrightarrow E\left[\exp(\mathrm{i}\theta S^{\chi_r^2}_\infty)\right]\\ &=\exp\left(\int_0^\infty(\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{r}{2u}G_{\lambda(\alpha/d)}(\exp(-u/2))\right]du\right),\end{aligned} \tag{102}$$

which proves that equations (96) and (97) hold. Equation (98) follows, in a similar way to the proof of Theorem 5.1(i) in [26], considering the expression obtained by the Lévy density q in equation (97).

From the above equations, in a similar way to Theorem 5.1(ii)-(iv) in [26], it can be seen that $S^{\chi_r^2}_\infty\in\mathcal{ID}(\mathbb{R})$ is selfdecomposable. Hence, it has a bounded density. It can also be showed that $S^{\chi_r^2}_\infty$ is in the Thorin class with Thorin measure

$$U(dx)=\frac{r}{2}\sum_{k=1}^{\infty}\delta_{\frac{1}{2\lambda_k(S^{\chi_r^2}_\infty)}}(dx),$$

where $\delta_a(x)$ is the Dirac delta-function at point a. Finally, $S_\infty^{\chi_r^2}$ admits the integral representation

$$S_\infty^{\chi_r^2} \underset{d}{=} \int_0^\infty \exp(-u)\, d\left(\sum_{k=1}^\infty \lambda_k(S_\infty^{\chi_r^2})A^{(k)}(u)\right) \underset{d}{=} \int_0^\infty \exp(-u)\, dZ(u), \tag{103}$$

where $Z(t) = \sum_{k=1}^\infty \lambda_k(S_\infty^{\chi_r^2})A^{(k)}(t)$, for each $t \geq 0$, with $A^{(k)}$, $k \geq 1$, being independent copies of a Lévy process.

Acknowledgements

This work has been supported in part by projects MTM2012-32674 (co-funded by European Regional Development Funds), and MTM2015–71839–P, MINECO, Spain. This research was also supported under Australian Research Council's Discovery Projects funding scheme (project number DP160101366), and under Cardiff Incoming Visiting Fellowship Scheme and International Collaboration Seedcorn Fund. Murad S. Taqqu was supported in part by the NSF grants DMS-1007616 and DMS-1309009 at Boston University.

We would like to thank the referees for their comments and suggestions, and for a number of interesting questions and ideas for future research.

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