CROSSED MODULE ACTIONS ON CONTINUOUS TRACE C*-ALGEBRAS

RALF MEYER AND ULRICH PENNIG

Abstract. We lift an action of a torus $\mathbb{T}^n$ on the spectrum of a continuous trace algebra to an action of a certain crossed module of Lie groups that is an extension of $\mathbb{R}^n$. We compute equivariant Brauer and Picard groups for this crossed module and describe the obstruction to the existence of an action of $\mathbb{R}^n$ in our framework.

1. Introduction

The low energy approximation of string theory is described by a spacetime M together with a B-field over M, whose field strength gives rise to a class $\delta \in H^3(M, \mathbb{Z})$. To such a pair, string theory associates a conformal field theory. It was observed in [7] that certain transformations, now known as Buscher rules, preserve the theory. While the Buscher rules only apply in local charts, a global topological description for principal circle bundles was found in [2]. This generalization of the Buscher rules is now called topological T-duality. Its effect on Chern and Dixmier–Douady classes was worked out in [3].

Topological T-duality was generalized to the case where $M = P \to X$ is a principal torus bundle over a base space X. This setup has a very elegant description in terms of non-commutative geometry [16, 17]: Let P be a principal $\mathbb{T}^n$-bundle over X and let A be a continuous trace C*-algebra with spectrum P. The action of $\mathbb{T}^n$ does not always lift to a $\mathbb{T}^n$-action on A: this may fail already for $n = 1$. Lifting the action to an action of $\mathbb{R}^n$ on A works much more often: for this, we only need to know that A restricts to trivial bundles over the orbits of the action (see [12]). If an $\mathbb{R}^n$-action on A exists and if $\hat{A} = A \rtimes \mathbb{R}^n$ is again a continuous trace C*-algebra with spectrum $\hat{P}$, then $\hat{A}$ is considered T-dual to A, and the Connes–Thom isomorphism $\mathrm{K}_*(A) \cong \mathrm{K}_{*+n}(\hat{A})$ is the T-duality isomorphism between the twisted K-theory groups of P and $\hat{P}$ given by A and $\hat{A}$, respectively, see [16, 17]. The C*-algebra $\hat{A}$ may fail to have continuous trace, for instance, by being a principal bundle of noncommutative tori. In this case, one may still consider the C*-algebra $\hat{A}$ to be a T-dual for A.

If there is no action of $\mathbb{R}^n$ on A, then [4] suggests non-associative algebras that may play the role of the T-dual. Another situation where non-associative algebras appear naturally are Fell bundles over crossed modules or, equivalently, actions of crossed modules on C*-algebras (see [9, 10]). We relate these two appearances of non-associativity: the action of $\mathbb{T}^n$ always lifts to an action of a certain crossed module on A. Its actions are equivalent to a certain type of non-associative Fell bundle over $\mathbb{R}^n$. Such a Fell bundle may be viewed as a continuous spectral decomposition for the dual $\mathbb{R}^n$-action on a non-associative algebra of the type studied in [4]. Thus crossed module actions give an alternative framework for understanding the non-associative algebras proposed in [4].

T-duality is most often formulated for principal circle bundles. The case of non-free circle actions with finite stabilizers has been studied by Bunke and Schick in [6]. It corresponds to the study of U(1)-bundles over orbispaces, which are modeled by topological stacks in [6]. The authors also find a sufficient condition for the T-duality transformation for a generalized cohomology theory to be an isomorphism. T-duality is generalized to arbitrary circle actions in [18], using the Borel construction. We treat general $\mathbb{T}^n$-actions right away, that is, we allow P to be an arbitrary second countable, locally compact $\mathbb{T}^n$-space. Any $\mathbb{T}^n$-action lifts to an action of our crossed module.

A crossed module of topological groups $\mathcal{H} = (H^1, H^2, \partial)$ is given by topological groups H^1 and H^2 with continuous group homomorphisms $\partial\colon H^2 \to H^1$ and $c\colon H^1 \to \mathrm{Aut}(H^2)$ with two conditions that mimic the properties of a normal subgroup and the conjugation action on the subgroup. We shall mostly use the following crossed module:

$$\begin{aligned} H^1 &= \mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n, & (t_1,\eta_1)\cdot(t_2,\eta_2) &= (t_1+t_2, \eta_1+\eta_2+t_1\wedge t_2),\\ H^2 &= \Lambda^2\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n, & (\theta_1,\xi_1)\cdot(\theta_2,\xi_2) &= (\theta_1+\theta_2, \xi_1+\xi_2),\\ \partial\colon H^2 &\to H^1, & \partial(\theta,\xi) &= (0,\theta),\\ c\colon H^1 &\to \mathrm{Aut}(H^2), & c_{(t,\eta)}(\theta,\xi) &= (\theta, \xi + t\wedge\theta), \end{aligned} \tag{1}$$

where $t, t_1, t_2 \in \mathbb{R}^n$, $\eta, \eta_1, \eta_2, \theta, \theta_1, \theta_2 \in \Lambda^2\mathbb{R}^n$ and $\xi, \xi_1, \xi_2 \in \Lambda^3\mathbb{R}^n$. Here $\Lambda^k\mathbb{R}^n$ denotes the kth exterior power of $\mathbb{R}^n$. An action of this crossed module on a C*-algebra A consists of two continuous group homomorphisms $\alpha\colon H^1 \to \mathrm{Aut}(A)$ and $u\colon H^2 \to U(M(A))$ with $\alpha_{\partial(h)} = \mathrm{Ad}_{u(h)}$ for all $h \in H^2$ and $\alpha_h(u_k) = u_{c_h(k)}$ for all $h \in H^1$, $k \in H^2$; in addition, we want α to lift the given action of $\mathbb{R}^n$ on the spectrum of A.

An action (α, u) as above is determined uniquely by $\alpha_t = \alpha_{(t,0)}$ for $t \in \mathbb{R}^n$ and $u_{t_1\wedge t_2} = u_{(t_1\wedge t_2,0)}$ for $t_1, t_2 \in \mathbb{R}^n$ because $\alpha_{t_1}\alpha_{t_2}\alpha_{t_1+t_2}^{-1} = \alpha_{(0,t_1\wedge t_2)}$ and $\alpha_{t_1}(u_{t_2\wedge t_3}) = u_{(0,t_1\wedge t_2\wedge t_3)}$. The unitaries $u_{t_1\wedge t_2}$ are such that $\mathrm{Ad}_{u_{t_1\wedge t_2}} = \alpha_{(0,t_1\wedge t_2)}$. So $t \mapsto \alpha_t$ is an action of $\mathbb{R}^n$ up to inner automorphisms given by $u_{t_1\wedge t_2}$. These unitaries are, however, not α_t-invariant. We only know that the unitaries $\alpha_{t_1}(u_{t_2\wedge t_3}) = u_{(0,t_1\wedge t_2\wedge t_3)}$ are central.

If we divide out the Λ^3-part in H^2, then we get a crossed module that is equivalent to $\mathbb{R}^n$; its actions are Green twisted actions of $\mathbb{R}^n$, so they can be turned into ordinary actions of $\mathbb{R}^n$ on a C*-stabilisation. The assumptions above imply that $\Lambda^3\mathbb{R}^n$ acts through a map to central, $\mathbb{R}^n$-invariant unitaries in A, that is, by a map to $C(P/\mathbb{R}^n, \mathbb{T})$. A homomorphism $\Lambda^3\mathbb{R}^n \to C(P/\mathbb{R}^n, \mathbb{T})$ lifts uniquely to $\Lambda^3\mathbb{R}^n \to C(P/\mathbb{R}^n, \mathbb{R})$, and such a homomorphism appears as the obstruction to finding an action of $\mathbb{R}^n$ on A that lifts the given $\mathbb{T}^n$-action on P. Actions of the crossed module $\mathcal{H}$ may contain such a lifting obstruction, so that there is no longer any obstruction to lifting the action to one of $\mathcal{H}$.

Besides proving the existence of liftings, we also classify them up to equivalence; that is, we compute the equivariant Brauer group with respect to the crossed module $\mathcal{H}$: There is an exact sequence of Abelian groups

$$H^2(P,\mathbb{Z}) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \to \mathrm{Br}_{\mathcal{H}}(P) \twoheadrightarrow \mathrm{Br}(P).$$

The surjection $\mathrm{Br}_{\mathcal{H}}(P) \twoheadrightarrow \mathrm{Br}(P)$ says that any continuous trace C*-algebra over P carries an action of $\mathcal{H}$ lifting the given action of $\mathbb{T}^n$ on the spectrum. The description of the kernel is the same as for the $\mathbb{R}^n$-equivariant Brauer group, so our result says that whenever we may lift the action on P to an $\mathbb{R}^n$-action on A, then there is a bijection between actions of $\mathcal{H}$ and $\mathbb{R}^n$ on A.

We also compute the analogue of the equivariant Picard group for our crossed module actions, and get the same result as in the case of $\mathbb{R}^n$-actions. Thus the only effect of replacing $\mathbb{R}^n$ by the crossed module $\mathcal{H}$ is to remove the obstruction to the existence of actions.

Our proof uses a smaller weak 2-group $\mathcal{G}$ that is equivalent to $\mathcal{H}$. It consists of the Abelian groups $G^1 = \operatorname{coker}(\partial_{\mathcal{H}}) \cong \mathbb{R}^n$ and $G^2 = \ker(\partial_{\mathcal{H}}) \cong \Lambda^3\mathbb{R}^n$, which are linked by the non-trivial associator

$$a(t_1, t_2, t_3) = -t_1 \wedge t_2 \wedge t_3 \qquad \text{for } t_1, t_2, t_3 \in \mathbb{R}^n.$$

We describe morphisms that give an equivalence between $\mathcal{G}$ and the weak 2-group associated to $\mathcal{H}$. The crossed module actions above are strict, but there is a more flexible notion of weak action that makes sense for weak 2-groups and 2-groupoids as well, see [11]. The Packer–Raeburn Stabilisation Trick extends to crossed modules and shows that any weak action of a crossed module is equivalent to a strict action. Since $\mathcal{G}$ and $\mathcal{H}$ are equivalent, they have equivalent weak actions on C*-algebras. Thus we get the desired strict action of $\mathcal{H}$ once we construct a weak action of $\mathcal{G}$.

The weak actions we have in mind are equivalent to saturated Fell bundles in the case of a group action. For actions of a weak 2-group such as $\mathcal{G}$, they are non-associative Fell bundles over $G^1 = \mathbb{R}^n$ where the associator is given by unitaries coming from the action of $G^2 = \Lambda^3\mathbb{R}^n$. Allowing weak actions simplifies the study of equivariant Brauer groups, already in the group case. We reprove the obstruction theory for $\mathbb{R}^n$-actions on continuous trace C*-algebras along the way. The bigroup version of the result is only notationally more difficult.

The crossed module $\mathcal{H}$ described above can be made slightly smaller: we may divide out the lattice $\Lambda^3\mathbb{Z}^n$ inside $\Lambda^3\mathbb{R}^n$, resulting in a compact group. This is so because the lifting obstructions that appear for $\mathbb{T}^n$-actions on continuous trace algebras always vanish on the lattice $\Lambda^3\mathbb{Z}^n$, and hence so do all the actions of $\mathcal{H}$ that appear. This feature, however, is special to actions of $\mathbb{R}^n$ that factor through the standard torus $\mathbb{R}^n/\mathbb{Z}^n$. The existence result for actions of $\mathcal{H}$ still works for actions of $\mathbb{R}^n$ if the stabiliser lattices are allowed to vary continuously for different orbits.

To compute the lifting obstruction of a given continuous trace C*-algebra, it suffices to consider a single free $\mathbb{T}^n$-orbit, that is, the case of the standard translation action of $\mathbb{T}^n$ on itself. Since $\mathbb{R}^n$ acts transitively on $\mathbb{T}^n$, the transformation crossed module $\mathcal{H} \ltimes \mathbb{T}^n$ is equivalent in a suitable sense to the "stabiliser" of a point in $\mathbb{T}^n$; this gives the subcrossed module $\tilde{\mathcal{H}}$ of $\mathcal{H}$ with

$$\tilde{H}^1 = \mathbb{Z}^n \times \Lambda^2\mathbb{R}^n, \qquad \tilde{H}^2 = H^2.$$

We do not develop the full theory of induction for crossed modules because the relevant special case is easy to do by hand. It turns out that actions of $\mathcal{H}$ on continuous trace C*-algebras over $\mathbb{T}^n$ are equivalent to actions of $\tilde{\mathcal{H}}$ on continuous trace C*-algebras over the point. Since our whole theory is up to equivalence, this is the same as weak actions of $\tilde{\mathcal{H}}$ on the complex numbers $\mathbb{C}$. Replacing $\tilde{\mathcal{H}}$ by the corresponding sub-2-group $\tilde{\mathcal{G}} \subseteq \mathcal{G}$, it is straightforward to classify these. This also gives the equivariant Brauer group for a single $\mathbb{T}^n$-orbit, and then allows to identify the lifting obstruction, up to a sign maybe, with the family of Dixmier–Douady invariants of the restrictions of A to the orbits of the $\mathbb{R}^n$-action.

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2. A crossed module extension of $\mathbb{R}^n$

We construct a crossed module extension $\mathcal{H}$ of $\mathbb{R}^n$ that circumvents the obstruction to lifting $\mathbb{R}^n$-actions from spaces to continuous trace C*-algebras described in [12]. We also describe a smaller weak topological 2-group $\mathcal{G}$ equivalent to $\mathcal{H}$.

Definition 2.1. A *crossed module* of topological groups $\mathcal{H} = (H^1, H^2, \partial)$ is given by topological groups H^1 and H^2 with continuous group homomorphisms $\partial\colon H^2 \to H^1$ and $c\colon H^1 \to \mathrm{Aut}(H^2)$, such that

$$\partial(c_{h_1}(h_2)) = h_1\,\partial(h_2)\,h_1^{-1}, \qquad c_{\partial(h_2)}(h_2') = h_2\,h_2'\,h_2^{-1}$$

for all $h_1 \in H^1$, $h_2, h_2' \in H^2$; continuity of c means that the map $H^1 \times H^2 \to H^2$, $(h_1, h_2) \mapsto c_{h_1}(h_2)$, is continuous.

A weak 2-group is a bicategory with only one object and with invertible arrows and 2-arrows. A 2-arrow or 2-cell in a bicategory is an "arrow" between two arrows. We call 2-arrows "bigons" because they should be drawn like this:

$$\begin{tikzcd} y & x; \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow a"] \end{tikzcd}$$

here x and y are objects, f and g are arrows $x \to y$, and a is a bigon from f to g. The standard reference for bicategories is [1]; we mainly follow [11], which is concerned with their C*-algebraic applications.

Crossed modules model strict 2-groups, that is, 2-groups with trivial unitors and associator. Let $\mathcal{H} = (H^1, H^2, \partial)$ be a crossed module. The associated strict 2-group $\mathcal{C}_\mathcal{H}$ has a single object and H^1 as its space of arrows. The space of bigons is $H^2 \times H^1$ with source map $(h_2, h_1) \mapsto h_1$ and target map $(h_2, h_1) \mapsto \partial(h_2)h_1$. We denote this by $h_2\colon h_1 \Rightarrow \partial(h_2)h_1$ or

$$\begin{tikzcd} * & *. \arrow[l, bend right, "h_1"'] \arrow[l, bend left, "\partial(h_2)h_1"] \arrow[l, phantom, "\Downarrow h_2"] \end{tikzcd}$$

The composition of arrows is the group multiplication in H^1. The vertical composition of bigons is the multiplication in H^2, and the horizontal composite of $h_2\colon h_1 \Rightarrow \partial(h_2)h_1$ and $k_2\colon k_1 \Rightarrow \partial(k_2)k_1$ is $h_2 \bullet k_2 = h_2 c_{h_1}(k_2)\colon h_1k_1 \Rightarrow \partial(h_2)h_1\partial(k_2)k_1 = \partial(h_2 c_{h_1}(k_2))h_1k_1$.

We shall mostly use the crossed module $\mathcal{H}$ defined in (1). It is routine to check that $\mathcal{H}$ is a crossed module of Lie groups.

Since the arrow (t, η) in the Lie 2-group $\mathcal{C}_\mathcal{H}$ is equivalent to $(t, 0)$ by a bigon, we should be able to shrink $\mathcal{C}_\mathcal{H}$, replacing the space of arrows by $\mathbb{R}^n$. This should reduce the space of bigons to $\mathbb{R}^n \times \Lambda^3\mathbb{R}^n$ because bigons $(t_1, 0) \Rightarrow (t_2, 0)$ in $\mathcal{C}_\mathcal{H}$ are of the form $(0, \xi)$. The result of this shrinking is a Lie 2-group $\mathcal{G}$ equivalent to $\mathcal{C}_\mathcal{H}$. This is no longer strict, however, so it does not come directly from a crossed module. The starting point is the smooth map Φ sending an arrow (t, η) to the bigon

$$\begin{tikzcd} * & *. \arrow[l, bend right, "(t,\eta)"'] \arrow[l, bend left, "(t,0)"] \arrow[l, phantom, "\Downarrow (-\eta,0)"] \end{tikzcd}$$

This generates a morphism $F = \mathrm{Ad}_\Phi\colon \mathcal{C}_\mathcal{H} \to \mathcal{C}_\mathcal{H}$ of Lie 2-groups that is equivalent to the identity functor (morphisms between weak 2-categories are described in [11, Definition 4.1]). We now describe F. It maps $(t, \eta) \in H^1$ to the range $(t, 0) \in H^1$ of the bigon $\Phi(t, \eta)$. It maps a bigon $(\theta, \xi)\colon (t, \eta) \Rightarrow (t, \eta + \theta)$ to the vertical composite

$$(t, 0) \xLeftarrow{\Phi(t,\eta)} (t, \eta) \xRightarrow{(\theta,\xi)} (t, \eta + \theta) \xRightarrow{\Phi(t,\eta+\theta)} (t, 0),$$

which gives $(0,\xi)\colon (t,0) \Rightarrow (t,0)$. Since $\Phi(t,0) = (0,0)$, our morphism is strictly unital, so the bigon $u_*\colon 1_* \Rightarrow F(1_*)$ in the definition of a morphism is trivial. A morphism F also needs natural bigons

$$\omega_F((t_1,\eta_1),(t_2,\eta_2))\colon F(t_1,\eta_1)\cdot F(t_2,\eta_2) \Rightarrow F((t_1,\eta_1)\cdot(t_2,\eta_2))$$

describing its compatibility with the multiplication. We get

$$\omega_F((t_1,\eta_1),(t_2,\eta_2))\colon (t_1,0)\cdot(t_2,0) = (t_1+t_2, t_1\wedge t_2) \Rightarrow (t_1+t_2,0)$$

by vertically composing the following 2-arrows:

$$\begin{aligned}(t_1,0)\cdot(t_2,0) &\xRightarrow{\Phi(t_1,\eta_1)^{-1}\bullet\Phi(t_2,\eta_2)^{-1}} (t_1,\eta_1)\cdot(t_2,\eta_2)\\ &= (t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2) \xRightarrow{\Phi(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2)} (t_1+t_2,0);\end{aligned}$$

this gives $c_{(t_1,0)}(\eta_2,0)+(\eta_1,0)-(\eta_1+\eta_2+t_1\wedge t_2,0) = (-t_1\wedge t_2, t_1\wedge\eta_2)$, so

$$\omega_F((t_1,\eta_1),(t_2,\eta_2)) = (-t_1\wedge t_2, t_1\wedge\eta_2).$$

By construction, Φ is a transformation from the identity functor to the functor F. Since all bigons in $\mathcal{C}_{\mathcal{H}}$ are invertible, this is even an equivalence. We are going to describe the range of F, which is a weak Lie 2-group $\mathcal{G}$. Its arrows and bigons are

$$\begin{aligned}G^1 &= \mathbb{R}^n = \{(t,0) \mid t\in\mathbb{R}^n\} \subseteq H^1,\\ G^2 &= \mathbb{R}^n\times\Lambda^3\mathbb{R}^n = \{((t,0),(0,\xi)) \mid t\in\mathbb{R}^n, \xi\in\Lambda^3\mathbb{R}^n\} \subseteq H^1\times H^2;\end{aligned}$$

so $(t,\xi)\in\mathbb{R}^n\times\Lambda^3\mathbb{R}^n$ corresponds to the bigon $(0,\xi)\colon (t,0)\Rightarrow \partial(0,\xi)(t,0) = (t,0)$ and thus has range and source t. The vertical composition of bigons adds the ξ-components as in $\mathcal{H}$. The composition of arrows gives

$$(t_1,0)\cdot_{\mathcal{G}}(t_2,0) = F((t_1,0)\cdot(t_2,0)) = F(t_1+t_2,t_1\wedge t_2) = (t_1+t_2,0),$$

so it is simply the addition in $\mathbb{R}^n$. The horizontal composition of bigons in $\mathcal{G}$ is defined by applying F to the horizontal composition in $\mathcal{C}_{\mathcal{H}}$. Since $c_{(t,0)}(0,\xi) = (0,\xi)$ for all $t\in\mathbb{R}^n$, $\xi\in\Lambda^3\mathbb{R}^n$, the horizontal composition in $\mathcal{G}$ simply adds the ξ-components. The unit arrow in $\mathcal{G}$ is $(0,0)$ and the left and right unitors are the identity bigons $(0,0)$ because the morphism F is strictly unital. The associator $a(t_1,t_2,t_3)$ for $t_1,t_2,t_3\in\mathbb{R}^n$ is defined so that the following diagram of 2-arrows commutes:

$$\begin{tikzcd}
F((t_1,0)\cdot F((t_2,0)\cdot(t_3,0))) & F(F((t_1,0)\cdot(t_2,0))\cdot(t_3,0)) \arrow[l, Rightarrow, "a(t_1,t_2,t_3)"'] \\
(t_1,0)\cdot F((t_2,0)\cdot(t_3,0)) \arrow[u, Rightarrow, "\Phi(t_1+t_2+t_3,t_1\wedge(t_2+t_3))"'] & F((t_1,0)\cdot(t_2,0))\cdot(t_3,0) \arrow[u, Rightarrow, "\Phi(t_1+t_2+t_3,(t_1+t_2)\wedge t_3)"] \\
(t_1,0)\cdot((t_2,0)\cdot(t_3,0)) \arrow[u, Rightarrow, "1_{(t_1,0)}\bullet\Phi(t_2+t_3,t_2\wedge t_3)"'] \arrow[r, equal] & ((t_1,0)\cdot(t_2,0))\cdot(t_3,0) \arrow[u, Rightarrow, "\Phi(t_1+t_2,t_1\wedge t_2)\bullet 1_{(t_3,0)}"]
\end{tikzcd}$$

Since this involves $c_{(t_1,0)}(-t_2\wedge t_3,0) = (-t_2\wedge t_3, -t_1\wedge t_2\wedge t_3)$, we get

$$\begin{aligned}a(t_1,t_2,t_3) &= (-t_1\wedge(t_2+t_3),0) - (-(t_1+t_2)\wedge t_3,0)\\ &\qquad + (-t_2\wedge t_3, -t_1\wedge t_2\wedge t_3) - (-t_1\wedge t_2,0) = (0,-t_1\wedge t_2\wedge t_3).\end{aligned}$$

These computations lead us to the following definition:

Definition 2.2. Let $\mathcal{G}(*,*)$ be the Lie groupoid given by the constant bundle of Abelian Lie groups $\Lambda^3\mathbb{R}^n$ over $\mathbb{R}^n$. Let $\mathcal{G}$ be the weak topological 2-group with one object $*$ and $\mathcal{G}(*,*)$ as its groupoid of arrows and bigons. The composition functor $\cdot\colon \mathcal{G}(*,*)\times\mathcal{G}(*,*)\to\mathcal{G}(*,*)$ is addition in both components. The unitors

l and r (called identity transformations in [1]) are trivial. The associator (called associativity isomorphism in [1]) is

$$\alpha\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \Lambda^3\mathbb{R}^n, \quad (t_1, t_2, t_3) \mapsto (t_1 + t_2 + t_3, -t_1 \wedge t_2 \wedge t_3).$$

It is routine to check that $\mathcal{G}$ is a weak 2-category (see [11, Definition 2.7]) or a bicategory in the notation of [1, (1.1)]. All arrows and bigons are invertible and $\mathcal{G}$ has only one object, so it is a weak Lie 2-group.

The computations above suggest that $\mathcal{G}$ should be equivalent to the strict 2-group $\mathcal{C}_\mathcal{H}$ associated to the crossed module $\mathcal{H}$. More precisely, we are going to construct morphisms $\iota\colon \mathcal{G} \to \mathcal{C}_\mathcal{H}$ and $\pi\colon \mathcal{C}_\mathcal{H} \to \mathcal{G}$ such that $\pi \circ \iota$ is the identity morphism on $\mathcal{G}$ and $\iota \circ \pi = \mathrm{Ad}_\Phi$ is equivalent to the identity morphism on $\mathcal{C}_\mathcal{H}$ by the transformation Φ. The morphism ι maps the arrow t to $(t, 0)$ and $\xi\colon t \Rightarrow t$ to $(0, \xi)\colon (t, 0) \Rightarrow (t, 0)$. This is strictly unital, so the bigon $\iota(1_*) \Rightarrow 1_*$ is $(0, 0)$. The compatibility with multiplication is given by

$$\begin{aligned}\omega_\iota(t_1, t_2) = (-t_1 \wedge t_2, 0)\colon (t_1 + t_2, t_1 \wedge t_2) = \iota(t_1) \cdot \iota(t_2)&\\ \Rightarrow \iota(t_1 + t_2) = (t_1 + t_2, 0).&\end{aligned}$$

Routine computations show that ι is a morphism; in particular, the cocycle condition [11, (4.2)] for ω_ι becomes

$$\begin{aligned}c_{(t_1,0)}(\omega_\iota(t_2, t_3)) - \omega_\iota(t_1 + t_2, t_3) + \omega_\iota(t_1, t_2 + t_3) - \omega_\iota(t_1, t_2)&\\ - (0, -t_1 \wedge t_2 \wedge t_3) = 0,&\end{aligned}$$

which holds because of our choice of the associator in $\mathcal{G}$.

The projection $\pi\colon \mathcal{C}_\mathcal{H} \to \mathcal{G}$ is defined so that $\iota \circ \pi = \mathrm{Ad}_\Phi$. Since ι is faithful on arrows and bigons, this determines π uniquely, and it implies that π is a morphism because Ad_Φ is one. Our Ansatz dictates $\pi(t, \eta) = t$ and

$$\pi((\theta, \xi)\colon (t, \eta) \Rightarrow (t, \eta + \theta)) = (\xi\colon t \Rightarrow t),$$

so that $\iota \circ \pi$ and Ad_Φ involve the same maps on arrows and bigons. The morphism π must be strictly unital because ι and Ad_Φ are so. The canonical bigon

$$\omega_{\iota\pi}((t_1, \eta_1), (t_2, \eta_2))\colon \iota\pi(t_1, \eta_1) \cdot \iota\pi(t_2, \eta_2) \Rightarrow \iota\pi((t_1, \eta_1) \cdot (t_2, \eta_2))$$

is the vertical composite of

$$\begin{aligned}&\omega_\iota(\pi(t_1, \eta_1), \pi(t_2, \eta_2))\colon \iota\pi(t_1, \eta_1) \cdot \iota\pi(t_2, \eta_2) \Rightarrow \iota(\pi(t_1, \eta_1) \cdot \pi(t_2, \eta_2)),\\ &\iota(\omega_\pi((t_1, \eta_1), (t_2, \eta_2)))\colon \iota(\pi(t_1, \eta_1) \cdot \pi(t_2, \eta_2)) \Rightarrow \iota(\pi((t_1, \eta_1) \cdot (t_2, \eta_2))).\end{aligned}$$

So we must put $\omega_\pi((t_1, \eta_1), (t_2, \eta_2)) = t_1 \wedge \eta_2\colon t_1 + t_2 \Rightarrow t_1 + t_2$ to get $\omega_{\iota\pi} = \omega_{\mathrm{Ad}_\Phi}$. This finishes the construction of π.

Since $\iota\pi = \mathrm{Ad}_\Phi$, Φ is a transformation from the identity on $\mathcal{C}_\mathcal{H}$ to $\iota\pi$. The composite $\pi\iota\colon \mathcal{G} \to \mathcal{G}$ is the identity on objects and arrows, strictly unital, and involves the identical transformation $\omega_{\pi\iota}$, so it is equal to the identity functor. Thus ι and π are equivalences of weak 2-groups inverse to each other. Both are given by smooth maps, so we have an equivalence of Lie 2-groups.

Remark 2.3. In the arguments above, we may replace $\Lambda^3\mathbb{R}^n$ everywhere by $\Lambda^3\mathbb{R}^n/\Gamma$ for any closed subgroup Γ. We shall be interested, in particular, in the case where we use $\Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n$ because the latter group is compact and because the actions of $\mathcal{G}$ or $\mathcal{H}$ that we need factor through this quotient in the case of $\mathbb{T}^n$-bundles.

3. Equivariant Brauer groups for bigroupoids

The equivariant Brauer group $\mathrm{Br}_G(P)$ of a transformation group $G \ltimes P$ is defined in [12]. It classifies continuous trace C*-algebras with spectrum P with a G-action that lifts the given action on P. This definition is extended from transformation groups to locally compact groupoids in [15]. Here we extend it further to locally compact bigroupoids. "Bigroupoids" are called "weak 2-groupoids" in [11]. The bigroupoids we need combine the groupoid $\mathcal{G}$ defined in Section 2 with an action $\alpha\colon \mathbb{R}^n \to \mathrm{Diff}(P)$ of $\mathbb{R}^n$ by diffeomorphisms on a manifold P.

Explicitly, we consider the following locally compact bigroupoid $\mathcal{C}$. It has object space $\mathcal{C}^0 = P$, arrow space $\mathcal{C}^1 = \mathbb{R}^n \times P$, and space of bigons $\mathcal{C}^2 = \Lambda^3\mathbb{R}^n \times \mathbb{R}^n \times P$. Here an arrow $(t,p) \in \mathbb{R}^n \times P$ has source p and range $\alpha_t(p)$, and the multiplication is the usual one: $(t_1, \alpha_{t_2}(p_2)) \cdot (t_2,p_2) = (t_1+t_2,p_2)$; a bigon $(\xi,t,p) \in \Lambda^3\mathbb{R}^n \times \mathbb{R}^n \times P$ has source and range (t,p); the vertical composition adds the ξ-components: $(\xi_1,t,p)\cdot(\xi_2,t,p) = (\xi_1+\xi_2,t,p)$; and the horizontal composition is

$$(\xi_1, t_1, \alpha_{t_2}(p_2)) \bullet (\xi_2,t_2,p_2) = (\xi_1+\xi_2, t_1+t_2, p_2).$$

The unit arrow on an object p is $(0,p)$, the unit bigon on an arrow (t,p) is $(0,t,p)$. Units are strict, that is, the left and right unitors are trivial. The associator is

$$a((t_1,p_1),(t_2,p_2),(t_3,p_3)) = (-t_1\wedge t_2 \wedge t_3, t_1+t_2+t_3, p_3)$$

for a triple of composable arrows; that is, for $p_i \in P$, $t_i \in \mathbb{R}^n$ for $i = 1,2,3$ with $p_{i-1} = \alpha_{t_i}(p_i)$ for $i = 2,3$. It is routine to check that this is a bigroupoid. All the spaces are smooth manifolds, hence locally compact, all the operations are smooth maps, hence continuous, and range and source maps are surjective submersions, hence open. Thus $\mathcal{C} = \mathcal{G} \ltimes P$ is a Lie bigroupoid and a locally compact bigroupoid.

Actions of locally compact bigroupoids on C*-algebras are defined in [11]. We shall use [11, Definition 4.1] for the correspondence bicategory as target bicategory ("bicategories" are called "weak 2-categories" in [11]), but with some changes. First, we require functors to be strictly unital, that is, the bigons u_x in [11, Definition 4.1] are identities; this is no restriction of generality because any functor is equivalent to one with this property, as remarked in [11]. Secondly, since $\mathcal{C}$ has invertible arrows, all correspondences appearing in an action of $\mathcal{C}$ are equivalences, that is, imprimitivity bimodules. Third, we need continuous actions; continuity is explained at the end of Section 4.1 in [11]. Fourth, we work with the opposite bicategory of imprimitivity bimodules, that is, an A,B-imprimitivity bimodule is viewed as an arrow from B to A; otherwise the multiplication maps below would go from $E_g \otimes E_h$ to E_{hg} instead of E_{gh}.

Now we define a *continuous action of a bigroupoid $\mathcal{C}$ by correspondences* or, equivalently, by imprimitivity bimodules, with the modifications mentioned above. Such an action requires the following data:

- a $C_0(\mathcal{C}^0)$-C*-algebra A; we denote its fibres by A_x for $x \in \mathcal{C}^0$;
- a $C_0(\mathcal{C}^1)$-linear imprimitivity bimodule E between the pull-backs $r^*(A)$ and $s^*(A)$ of A along the range and source maps; thus E is a bundle over $\mathcal{C}^1$ where the fibre at $g \in \mathcal{C}^1$ is an $A_{r(g)}, A_{s(g)}$-imprimitivity bimodule E_g;
- isomorphisms $\omega_{g,h}\colon E_g \otimes_{A_{s(g)}} E_h \to E_{gh}$ of imprimitivity bimodules for all $g,h \in \mathcal{C}^1$ with $s(g) = r(h)$, which are continuous in the sense that pointwise application of $\omega_{g,h}$ gives an isomorphism of imprimitivity bimodules

  $$\omega\colon \mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E) \to \mu^*(E),$$

  where $\mathrm{pr}_1, \mathrm{pr}_2, \mu$ are the continuous maps that map a pair (g,h) of composable arrows to g, h and gh, respectively; so $s\,\mathrm{pr}_1 = r\,\mathrm{pr}_2$ maps (g,h) to $s(g) = r(h)$;

- isomorphisms of imprimitivity bimodules $U_b\colon E_{s_2(b)} \to E_{r_2(b)}$ for all bigons $b \in \mathcal{C}^2$, which are continuous in the sense that they give an isomorphism of imprimitivity bimodules $U\colon s_2^*(E) \to r_2^*(E)$; here $r_2, s_2\colon \mathcal{C}^2 \rightrightarrows \mathcal{C}^1$ map a bigon to its range and source arrow;

this data is subject to the following conditions:

(A1) $E_{1_x} = A_x$ for all $x \in \mathcal{C}^0$, and the restriction of E to units is A;
(A2) $\omega_{g,1}\colon E_g \otimes_{A_{s(g)}} A_{s(g)} \to E_g$ and $\omega_{1,h}\colon A_{r(h)} \otimes_{A_{r(h)}} E_h \to E_h$ are the canonical isomorphisms for all $g, h \in \mathcal{C}^1$;
(A3) U is multiplicative for vertical products: $U_{b_1} \circ U_{b_2} = U_{b_1 \cdot b_2}$ for vertically composable bigons $b_1, b_2 \in \mathcal{C}^2$;
(A4) if $b_1\colon f_1 \Rightarrow g_1$ and $b_2\colon f_2 \Rightarrow g_2$ are horizontally composable bigons, that is, $s(f_1) = s(g_1) = r(f_2) = r(g_2)$, then the following diagram commutes:

$$\begin{tikzcd}
E_{f_1} \otimes_{A_{s(f_1)}} E_{f_2} \arrow[r, "\omega_{f_1,f_2}"] \arrow[d, "U_{b_1} \otimes U_{b_2}"'] & E_{f_1 f_2} \arrow[d, "U_{b_1 \bullet b_2}"] \\
E_{g_1} \otimes_{A_{s(g_1)}} E_{g_2} \arrow[r, "\omega_{g_1,g_2}"] & E_{g_1 g_2}
\end{tikzcd}$$

(A5) if g_1, g_2, g_3 are composable arrows in $\mathcal{C}^1$, then the following diagram commutes:

$$\begin{tikzcd}
(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \arrow[rr, leftrightarrow] \arrow[d, "\omega_{g_1,g_2} \otimes 1"] & & E_{g_1} \otimes (E_{g_2} \otimes E_{g_3}) \arrow[d, "1 \otimes \omega_{g_2,g_3}"] \\
E_{g_1 g_2} \otimes E_{g_3} \arrow[d, "\omega_{g_1 g_2, g_3}"] & & E_{g_1} \otimes E_{g_2 g_3} \arrow[d, "\omega_{g_1, g_2 g_3}"] \\
E_{(g_1 g_2) g_3} \arrow[rr, "U_{a(g_1,g_2,g_3)}"] & & E_{g_1 (g_2 g_3)}
\end{tikzcd}$$

here $a(g_1, g_2, g_3)\colon (g_1 g_2) g_3 \to g_1(g_2 g_3)$ is the associator of $\mathcal{C}^2$, and we dropped the subscripts $A_?$ on $\otimes$ to avoid clutter.

Condition (A3) says that the maps $g \mapsto E_g$ and $b \mapsto U_b$ form a functor; (A4) says that the maps ω_{g_1,g_2} are natural with respect to bigons; (A1) says that our functor is strictly unital; (A2) is equivalent to the coherence conditions [11, (4.3)] for unitors; (A5) is [11, (4.2)].

To define the equivariant Brauer group of $\mathcal{C}$, we also need equivalences between such actions. Let $(A^1, E^1, \omega^1, U^1)$ and $(A^2, E^2, \omega^2, U^2)$ be continuous actions of $\mathcal{C}$. A *transformation* between them consists of the following data:

- a $C_0(\mathcal{C}^0)$-linear correspondence F between A^1 and A^2, with fibres F_x for $x \in \mathcal{C}^0$;
- isomorphisms of correspondences

$$V_g\colon E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \to F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g$$

for all $g \in \mathcal{C}^1$, which are continuous in the sense that they give an isomorphism $V\colon E^1 \otimes_{s^*(A^1)} s^*(F) \to r^*(F) \otimes_{r^*(A^2)} E^2$;

this must satisfy the following conditions:

(T1) for each bigon $b\colon g \Rightarrow h$, the following diagram commutes:

$$\begin{tikzcd} E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \arrow[r, "V_g"] \arrow[d, "U^1_b\otimes 1"'] & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \arrow[d, "1\otimes U^2_b"] \\ E^1_h \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[r, "V_h"] & F_{r(h)} \otimes_{A^2_{r(h)}} E^2_h \end{tikzcd}$$

(T2) $V_{1_x}\colon A^1_x \otimes_{A^1_x} F_x \to F_x \otimes_{A^2_x} A^2_x$ is the canonical isomorphism;

(T3) the following diagram commutes for $g, h \in \mathcal{C}^1$ with $s(g) = r(h)$:

$$\begin{tikzcd} E^1_g \otimes_{A^1_{s(g)}} E^1_h \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[r, "\omega^1_{g,h}\otimes 1"] \arrow[dd, "1\otimes V_h"'] & E^1_{gh} \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[d, "V_{gh}"] \\ & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_{gh} \\ E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \otimes_{A^2_{s(g)}} E^2_h \arrow[r, "V_g\otimes 1"] & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \otimes_{A^2_{s(g)}} E^2_h \arrow[u, "1\otimes\omega^2_{g,h}"'] \end{tikzcd}$$

Condition (T1) says that V is natural with respect to bigons, (T2) is the coherence condition for units and (T3) is the coherence condition for multiplication.

An *equivalence* between two actions is a transformation where each F_x or, equivalently, F is an imprimitivity bimodule; then the maps V_g and V are automatically isomorphisms of imprimitivity bimodules, that is, compatibility with the left inner product comes for free.

A *modification* between two transformations

$$(F^1, V^1), (F^2, V^2)\colon (A^1, E^1, \omega^1, U^1) \rightrightarrows (A^2, E^2, \omega^2, U^2)$$

is given by the following data:

- isomorphisms of correspondences $W_x\colon F^1_x \to F^2_x$ for all $x \in \mathcal{C}^0$ that give an isomorphism $W\colon F^1 \to F^2$;

this must satisfy the following condition:

(M1) for each $g \in \mathcal{C}^1$ the following diagram commutes:

$$\begin{tikzcd} E^1_g \otimes_{A^1_{s(g)}} F^1_{s(g)} \arrow[r, "V^1_g"] \arrow[d, "1\otimes W_{s(g)}"'] & F^1_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \arrow[d, "W_{r(g)}\otimes 1"] \\ E^1_g \otimes_{A^1_{s(g)}} F^2_{s(g)} \arrow[r, "V^2_g"] & F^2_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \end{tikzcd}$$

This is the obvious notion of isomorphism between two transformations.

Two actions of $\mathcal{C}$ may be tensored together in the obvious way, using the fibrewise maximal tensor product as in [14].

Definition 3.1. The *equivariant Brauer group* $\mathrm{Br}(\mathcal{C})$ of the locally compact bigroupoid $\mathcal{C}$ is the set of equivalence classes of continuous actions of $\mathcal{C}$ on continuous trace C*-algebras with spectrum $\mathcal{C}^0$; the group structure is the tensor product over $\mathcal{C}^0$. We also write $\mathrm{Br}_{\mathcal{G}}(P) = \mathrm{Br}(\mathcal{G} \ltimes P)$.

The usual proof in [12] that this defines an Abelian group carries over to our case. The formula for the multiplication is particularly easy:

$$(A_1, E_1, \omega_1, U_1) \otimes_P (A_2, E_2, \omega_2, U_2) = (A_1 \otimes_P A_2, E_1 \otimes_P E_2, \omega_1 \otimes_P \omega_2, U_1 \otimes_P U_2),$$

where $\otimes_P$ means the maximal fibrewise tensor product of C*-algebras over P, the corresponding external tensor product of imprimitivity bimodules, or the external tensor product of operators. Similar formulas work for equivalences between actions, so that the operation $\otimes_P$ descends to equivalence classes. The usual symmetry of $\otimes_P$ gives that this multiplication is commutative. Further details are left to the reader.

Definition 3.2. The *equivariant Picard group* $\mathrm{Pic}_{\mathcal{C}}(A, E, \omega, U)$ of an action of $\mathcal{C}$ is the group of all equivalence classes of self-equivalences on (A, E, ω, U), where two self-equivalences are considered equivalent if there is a modification between them. The group structure is the (vertical) composition of transformations.

In the special case of actions on continuous trace C*-algebras, the definitions above simplify because of the following lemma:

Lemma 3.3. *Let A and B be continuous trace* C*-*algebras with spectrum X. Let E_1 and E_2 be two $C_0(X)$-linear equivalences from A to B. There is a Hermitian complex line bundle L over X with $E_2 \cong E_1 \otimes_X L$, and conversely $E_1 \otimes_X L$ is another equivalence from A to B for any Hermitian complex line bundle L.*

Let $U_1, U_2 \colon E_1 \rightrightarrows E_2$ be two isomorphisms of equivalences. There is a continuous map $\varphi\colon X \to \mathbb{T}$ with $U_2(x) = \varphi(x) \cdot U_1(x)$ for all $x \in X$, and conversely $U_1 \cdot \varphi$ for a continuous map $\varphi\colon X \to \mathbb{T}$ is another isomorphism $E_1 \to E_2$.

We always identify a Hermitian complex line bundle with its space of C_0-sections, which is a $C_0(P), C_0(P)$-imprimitivity bimodule.

Proof. Let E_2^* be the inverse equivalence. Then $E = E_1 \otimes_B E_2^*$ is a $C_0(X)$-linear self-equivalence of A. We are going to prove that E is of the form $A \otimes_X L$ for a Hermitian complex line bundle L over X. The opposite algebra A^{op} is an inverse for A in the Brauer group, that is, $A \otimes_X A^{\mathrm{op}} \cong C_0(X, \mathbb{K})$; this is Morita equivalent to $C_0(X)$. It is well-known that a $C_0(X)$-linear self-equivalence of $C_0(X)$ is the same as a Hermitian complex line bundle over X. Since $C_0(X)$ and $C_0(X, \mathbb{K})$ are $C_0(X)$-linearly equivalent, they have isomorphic groups of $C_0(X)$-linear self-equivalences. Thus $E \otimes_X A^{\mathrm{op}}$ is the space $L \otimes \mathbb{K}$, where L is the space of sections of a Hermitian complex line bundle over X. On the one hand, $E \otimes_X (A^{\mathrm{op}} \otimes_X A) \cong E \otimes_X C_0(X, \mathbb{K})$ is just the stabilisation of E; on the other hand, it is $(E \otimes_X A^{\mathrm{op}}) \otimes_X A \cong L \otimes \mathbb{K} \otimes_X A$. Due to the $C_0(X)$-linear equivalence between A and $A \otimes \mathbb{K}$, we may remove the stabilisations again to see that $E \cong A \otimes_X L$. It is clear, conversely, that $E \otimes_X L$ is again a $C_0(X)$-linear self-equivalence of A.

If $f_1, f_2 \colon E_1 \to E_2$ are two isomorphisms of imprimitivity bimodules, then $f_2^{-1} f_1 \colon E_1 \to E_1$ is an isomorphism. In each fibre, $(E_1)_x$ is an imprimitivity bimodule from $\mathbb{K}$ to $\mathbb{K}$. The isomorphism $f_2^{-1} f_1$ gives a unitary bimodule map, and any such map is just multiplication with a scalar of absolute value 1. This gives the function $\varphi\colon X \to \mathbb{T}$, which is continuous because U_1 and U_2 are continuous. □

Proposition 3.4. *The groups $\mathrm{Pic}_{\mathcal{C}}(A, E, \omega, U)$ are canonically isomorphic for all actions (A, E, ω, U) of $\mathcal{C}$ on continuous-trace* C*-*algebras over $\mathcal{C}^0$.*

Proof. Let $(A_i, E_i, \omega_i, U_i)$ for $i = 1, 2$ be two actions of $\mathcal{C}$ on continuous-trace C*-algebras over $\mathcal{C}^0$. Since the Brauer group has inverses, there are other actions $(A_i, E_i, \omega_i, U_i)$, $i = 3, 4$, of $\mathcal{C}$ on continuous-trace C*-algebras over $\mathcal{C}^0$ such that

$$(A_1, E_1, \omega_1, U_1) \otimes_{\mathcal{C}^0} (A_3, E_3, \omega_3, U_3) \cong (A_2, E_2, \omega_2, U_2),$$
$$(A_2, E_2, \omega_2, U_2) \otimes_{\mathcal{C}^0} (A_4, E_4, \omega_4, U_4) \cong (A_1, E_1, \omega_1, U_1).$$

Tensoring a self-equivalence of $(A_1, E_1, \omega_1, U_1)$ with the identity equivalence of $(A_3, E_3, \omega_3, U_3)$ gives a self-equivalence of $(A_2, E_2, \omega_2, U_2)$. Tensoring a self-equivalence of $(A_2, E_2, \omega_2, U_2)$ with the identity equivalence of $(A_4, E_4, \omega_4, U_4)$ gives a self-equivalence of $(A_1, E_1, \omega_1, U_1)$. This defines group homomorphisms between the two Picard groups that are inverse to each other because $(A_3, E_3, \omega_3, U_3)$ and $(A_4, E_4, \omega_4, U_4)$ are inverse to each other in the Brauer group. $\square$

We denote the common Picard group of all actions of $\mathcal{C}$ on continuous-trace C*-algebras over $\mathcal{C}^0$ by $\mathrm{Pic}(\mathcal{C})$ and also write $\mathrm{Pic}_{\mathcal{G}}(P) = \mathrm{Pic}(\mathcal{G} \ltimes P)$.

Lemma 3.3 becomes even more powerful when we use that the functors that send a paracompact space X to the set of equivalence classes of Hermitian complex line bundles or continuous trace C*-algebras with spectrum X are both homotopy invariant; actually, they are $H^2(X, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$.

Lemma 3.5. *Assume that the spaces $\mathcal{C}^i$ are paracompact. Assume that there is a continuous homotopy $H\colon \mathcal{C}^1 \times [0,1] \to \mathcal{C}^1$ with $H_0 = \mathrm{id}_{\mathcal{C}^1}$, $H_1 = u \circ r$ for the unit and range maps u, r between $\mathcal{C}^1$ and $\mathcal{C}^0$, and $r \circ H_t = r$ for all $t \in [0,1]$. Assume further that $r_2\colon \mathcal{C}^2 \to \mathcal{C}^1$ is a homotopy equivalence. Let A be any continuous trace* C*-*algebra with spectrum $\mathcal{C}^0$. Then:*

(a) *there is an imprimitivity bimodule E between $r^*(A)$ and $s^*(A)$ that restricts to the identity on units;*
(b) *any two such E are isomorphic with an isomorphism that is the identity on units;*
(c) *there are isomorphisms*
$$\omega\colon \mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E) \to \mu^*(E) \quad \text{and} \quad U\colon s_2^*(E) \to r_2^*(E)$$
such that $\omega_{1_{r(g)},g}$ and $\omega_{g,1_{s(g)}}$ are the canonical isomorphisms and U_{1_g} is the identity for all $g \in \mathcal{C}^1$;
(d) *any two choices for ω and U as above differ by pointwise multiplication with* $\exp(2\pi \mathrm{i}\varphi)$ *for a continuous map $\varphi\colon \mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1 \sqcup \mathcal{C}^2 \to \mathbb{R}$ with $\varphi(1_{r(g)}, g) = 0$, $\varphi(g, 1_{s(g)}) = 0$ and $\varphi(1_g) = 0$ for all $g \in \mathcal{C}^1$.*

Let $(A^i, E^i, \omega^i, U^i)$ for $i = 1, 2$ be actions of $\mathcal{C}$ and let F be an A^1, A^2-imprimitivity bimodule. Then:

(e) *there is an isomorphism of imprimitivity bimodules $V\colon E^1 \otimes_{s^*(A^1)} s^*(F) \cong r^*(F) \otimes_{r^*(A^2)} E^2$ that restricts to the canonical isomorphism on units;*
(f) *any two choices V^1 and V^2 as in (e) differ by pointwise multiplication with* $\exp(2\pi \mathrm{i}\varphi)$ *for a continuous map $\varphi\colon \mathcal{C}^1 \to \mathbb{R}$ with $\varphi(1_x) = 0$ for all $x \in \mathcal{C}^0$.*

Proof. The pull-backs $s^*(A)$ and $r^*(A)$ are continuous trace C*-algebras over $\mathcal{C}^1$ that restrict to the same continuous trace C*-algebra on $u(\mathcal{C}^0) \subseteq \mathcal{C}^1$. By assumption, $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$, and the functor $X \mapsto \mathrm{Br}(X)$ is homotopy invariant on paracompact spaces. Since $u^*s^*(A) \cong u^*r^*(A)$, there must be an imprimitivity bimodule E between $s^*(A)$ and $r^*(A)$. The restriction of E to units differs from the identity imprimitivity bimodule by some line bundle by Lemma 3.3. This line bundle extends to a line bundle over $\mathcal{C}^1$ because $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$. Tensoring E with the opposite of that line bundle, we can arrange that $u^*(E)$ is isomorphic to the identity equivalence on A; then we may replace E by an isomorphic equivalence such that $u^*(E)$ is *equal* and not just isomorphic to the identity equivalence on A. This proves (a).

Let E^1 and E^2 be two equivalences between $s^*(A)$ and $r^*(A)$ that restrict to the identity on units. Then $E^2 \cong E^1 \otimes_{\mathcal{C}^1} L$ for some line bundle L by Lemma 3.3. Since $u^*(E^2) = u^*(E^1)$, $u^*(L)$ is trivialisable. Since $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$, this implies that L is trivialisable over $\mathcal{C}^1$, so $E^1 \cong E^2$. This isomorphism restricted to units differs from the identity isomorphism by pointwise multiplication with some function $\psi\colon \mathcal{C}^0 \to \mathbb{T}$ by Lemma 3.3. Since the unit map is a homotopy equivalence, this function extends to $\bar{\psi}\colon \mathcal{C}^1 \to \mathbb{T}$. Multiplying pointwise with $\bar{\psi}^{-1}$ gives another isomorphism $E^1 \cong E^2$ that restricts to the identity on units. This proves (b).

We claim that the inclusion of the subspace

$$X = \{(g, 1_{s(g)}) \mid g \in \mathcal{C}^1\} \cup \{(1_{r(h)}, h) \mid h \in \mathcal{C}^1\}$$

into $\mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1$ is a homotopy equivalence. Indeed, the maps $(g,h) \mapsto (g, H_t(h))$ deformation-retract $\mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1$ to the subspace of pairs $(g, 1_{s(g)})$, and they restrict to a deformation retraction from the subspace X onto the same space.

The $C_0(\mathcal{C}^1 \times_{s,r} \mathcal{C}^1)$-linear imprimitivity bimodules $\mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E)$ and $\mu^*(E)$ differ by some line bundle by Lemma 3.3. Since the two imprimitivity bimodules are canonically isomorphic on the subspace X, the line bundle is trivial on X. Since the inclusion of X is a homotopy equivalence, the line bundle is trivial everywhere. Hence there is an isomorphism ω between $\mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E)$ and $\mu^*(E)$. On X, we also have the canonical isomorphism, which differs from ω by some continuous function $\psi\colon X \to \mathbb{T}$ by Lemma 3.3. As above, we may correct the isomorphism ω so that it restricts to the identity on X because ψ extends continuously to $\mathcal{C}^1 \times_{s,r} \mathcal{C}^1$.

A similar argument gives an isomorphism $U\colon s_2^*(E) \to r_2^*(E)$ over the space of bigons $\mathcal{C}^2$ with $U(1_g) = \mathrm{id}_{E_g}$ for all $g \in \mathcal{C}^1$ because r_2 is a homotopy equivalence. This proves (c).

Any two choices for ω and U differ through pointwise multiplication with some functions $\mathcal{C}^1 \times_{s,r} \mathcal{C}^1 \to \mathbb{T}$ and $\mathcal{C}^2 \to \mathbb{T}$ by Lemma 3.3; these functions are 1 on X or on unit bigons by our normalisations. Since the inclusions of X and unit bigons are homotopy equivalences, covering space theory allows to lift such a function to $\mathbb{R}$ as required in (d).

The proofs of (e) and (f) use the same ideas. An isomorphism V exists because the two imprimitivity bimodules are isomorphic on units and the inclusion of units is a homotopy equivalence, and it may be arranged to be the canonical isomorphism on units because any continuous function $u(\mathcal{C}^0) \to \mathbb{T}$ extends to a continuous function $\mathcal{C}^1 \to \mathbb{T}$. Lemma 3.3 shows that two isomorphisms V_1 and V_2 differ by a function $\mathcal{C}^1 \to \mathbb{T}$, which is constant equal to 1 on units. Any such function lifts to $\mathbb{R}$, giving (f). □

Let (A, E, ω, U) be a continuous action of $\mathcal{C}$. If F is a $C_0(\mathcal{C}^0)$-linear Morita equivalence from A to some other C*-algebra A', then we may transfer the action from A to A' along F: let $E' = F \otimes_A E \otimes_A F^*$ and translate ω and U accordingly. Hence up to equivalence of actions, only the equivalence class of A matters. Similarly, if A is fixed and E' is another equivalence $s^*(A) \cong r^*(A)$ with $E \cong E'$, then we may use the isomorphism $E \cong E'$ to transfer ω and U to E'. So in the definition of the Brauer group, only the Morita equivalence class of A and, for fixed A, the isomorphism class of E matter.

4. Lifting actions to continuous trace algebras

Let P be a second countable, locally compact space with a continuous action of $\mathbb{R}^n$; then P is paracompact. Lemma 3.5 applies to the transformation groupoid $\mathbb{R}^n \ltimes P$ and the transformation bigroupoid $\mathcal{G} \ltimes P$ because the Lie groups $\mathbb{R}^n$ and $\Lambda^3\mathbb{R}^n$ are contractible. We use this to analyse the obstruction to lifting an $\mathbb{R}^n$-action from P to a continuous trace C*-algebra over P. Our results for $\mathbb{R}^n \ltimes P$ are already contained in [12].

Consider the case $\mathbb{R}^n$ first. Here there are no bigons, so the datum U is not there and the conditions (A3)–(A4) in the definition of an action are empty. Let A be a continuous trace C*-algebra over P. Lemma 3.5 provides the data E and ω for an action, satisfying (A1) and (A2), but not yet satisfying the cocycle condition (A5).

Since E is unique up to isomorphism, its choice does not affect whether or not there is ω satisfying (A5), nor the resulting element in the equivariant Brauer group. By Lemma 3.5, any two choices for ω differ through pointwise multiplication with a function of the form $\exp(2\pi \mathrm{i}\varphi)$ for a continuous function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \times P \to \mathbb{R}$; here we have identified the space of composable arrows in $\mathbb{R}^n \ltimes P$ with $\mathbb{R}^n \times \mathbb{R}^n \times P$.

Similarly, the space of composable k-tuples of arrows in $\mathbb{R}^n \ltimes P$ is $(\mathbb{R}^n)^k \times P$. By Lemma 3.3, the two isomorphisms $(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \to E_{g_1 g_2 g_3}$ in (A5) differ by pointwise multiplication with a function $(\mathbb{R}^n)^3 \times P \to \mathbb{T}$. We can also view this as the difference between the isomorphism $(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \to E_{g_1} \otimes (E_{g_2} \otimes E_{g_3})$ induced by the two isomorphisms above and the canonical one, which makes it clear that this function plays the role of an associator. It is 1 if one of the $\mathbb{R}^n$-entries is 0 by our normalisations. Hence arguments as in the proof of Lemma 3.5 show that it lifts uniquely to an $\mathbb{R}$-valued function $\psi\colon (\mathbb{R}^n)^3 \times P \to \mathbb{R}$ that is 0 if one of the $\mathbb{R}^n$-entries is 0. When we multiply ω pointwise by $\exp(2\pi\mathrm{i}\varphi)$, then this adds the following function to ψ:

$$\partial\varphi(t_1,t_2,t_3,p) = \varphi(t_2,t_3,p) - \varphi(t_1+t_2,t_3,p) + \varphi(t_1,t_2+t_3,p) - \varphi(t_1,t_2,\alpha_{t_3}(p)).$$

Since the associator diagram

$$\begin{tikzcd}
((E_{g_1} \otimes E_{g_2}) \otimes E_{g_3}) \otimes E_{g_4} \arrow[r] \arrow[dd] & (E_{g_1} \otimes (E_{g_2} \otimes E_{g_3})) \otimes E_{g_4} \arrow[d] \\
& E_{g_1} \otimes ((E_{g_2} \otimes E_{g_3}) \otimes E_{g_4}) \arrow[d] \\
(E_{g_1} \otimes E_{g_2}) \otimes (E_{g_3} \otimes E_{g_4}) \arrow[r] & E_{g_1} \otimes (E_{g_2} \otimes (E_{g_3} \otimes E_{g_4}))
\end{tikzcd}$$

commutes, we deduce that ψ automatically satisfies the cocycle condition

$$\begin{aligned} 0 = \partial\psi(t_1,t_2,t_3,t_4,p) &= \psi(t_2,t_3,t_4,p) - \psi(t_1+t_2,t_3,t_4,p) \\ &\quad + \psi(t_1,t_2+t_3,t_4,p) - \psi(t_1,t_2,t_3+t_4,p) + \psi(t_1,t_2,t_3,\alpha_{t_4}(p)). \end{aligned}$$

More precisely, the function $\exp(2\pi\mathrm{i}\partial\psi)$ is automatically the constant function 1, and this implies the above because $\partial\psi$ vanishes if one of the t_i is 0 and the lifting of $\mathbb{T}$-valued functions to normalised $\mathbb{R}$-valued functions is unique if it exists.

As a result, when we view ψ as a function from $(\mathbb{R}^n)^3$ to the Fréchet space $C(P,\mathbb{R}^n)$ of continuous functions $P \to \mathbb{R}^n$ with the action of $\mathbb{R}^n$ induced from the action on P, then ψ is a continuous 3-cocycle; and we may choose ω to satisfy the cocycle condition (A5) if and only if this 3-cocycle is a coboundary. Thus the action of $\mathbb{R}^n$ on P lifts to an action on A if and only if the class of ψ in the continuous group cohomology $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R}))$ vanishes. We call this class the *lifting obstruction* of the continuous trace C*-algebra A.

The Packer–Raeburn Stabilisation Trick shows that any action of $\mathbb{R}^n$ by equivalences as above is equivalent to a strict action by automorphisms on the stabilisation of A (this is contained in [11, Theorem 5.3]). Since stabilisation does not change the class of A in the Brauer group, we may assume that A is stable. Then the lifting obstruction is the obstruction to the existence of a strict action of $\mathbb{R}^n$ by automorphisms. An obstruction for this is also constructed in [12], but in the measurable group cohomology $H^3_{\mathrm{M}}(\mathbb{R}^n, C(P,\mathbb{R}))$. These two cohomology groups coincide by [21, Theorem 3].

The continuous cohomology group $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R}))$ may be simplified if the action of $\mathbb{R}^n$ factors through a torus $\mathbb{T}^n$. Let $\Omega^k\mathbb{R}^n = (\Lambda^k\mathbb{R}^n)^*$ denote the vector space of antisymmetric k-linear maps $(\mathbb{R}^n)^k \to \mathbb{R}$.

Proposition 4.1. *Let P be a second countable, locally compact $\mathbb{T}^n$-space, viewed as an $\mathbb{R}^n$-space, with orbit space $P/\mathbb{R}^n$. Then*

$$H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R})) \cong C(P/\mathbb{R}^n, \Omega^k\mathbb{R}^n)$$

for all $k \geq 0$. The isomorphism maps $\chi\colon P/\mathbb{R}^n \to \Omega^k\mathbb{R}^n$ to the cocycle given by

$$(\mathbb{R}^n)^k \times P \to \mathbb{R}, \qquad (t_1,\dots,t_k,p) \mapsto \chi([p])(t_1 \wedge \dots \wedge t_k).$$

Proof. If the action of $\mathbb{T}^n$ on P is free, then this is [16, Lemma 2.1]. We explain why the result remains true for non-free actions of $\mathbb{T}^n$. Throughout this proof, group cohomology is understood to be continuous group cohomology.

Continuous representations of $\mathbb{R}^n$ on Fréchet spaces over $\mathbb{R}$ are equivalent to non-degenerate modules over the Banach algebra $L^1(\mathbb{R}^n)$ of integrable functions $\mathbb{R}^n \to \mathbb{R}$. Let $\mathbb{R}$ denote the trivial representation of $\mathbb{R}^n$. The continuous group cohomology for $\mathbb{R}^n$ with coefficients in a Fréchet $\mathbb{R}^n$-module W is the same as $\mathrm{Ext}^k_{L^1(\mathbb{R}^n)}(\mathbb{R}, W)$. Since $L^1(\mathbb{R}^n)$ is Abelian, the module structures on W and $\mathbb{R}$ induce an $L^1(\mathbb{R}^n)$-module structure on $\mathrm{Ext}^k_{L^1(\mathbb{R}^n)}(\mathbb{R}, W)$ as well, and both module structures are the same. Since one is trivial, so is the other.

Let V be a Fréchet space with a continuous action of $\mathbb{T}^n$, which we view as an action of $\mathbb{R}^n$. Split $V \cong V^1 \oplus V^2$ where $V^1 \subseteq V$ is the space of $\mathbb{T}^n$-invariant elements and V^2 is the closed linear span of the other homogeneous components. There is an element $f \in L^1(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ satisfies $\hat{f}(0) = 1$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}^n \setminus \{0\}$; for instance, we may take f to be the inverse Fourier transform of a smooth bump function around 0 supported in $(-1,1)^n$. The function f acts on V by the projection onto V^1. The induced action by f on the cohomology $H^k(\mathbb{R}^n, V)$ is the same as the action of $\hat{f}(0)$ because the latter is how f acts on the trivial representation. Hence $H^k(\mathbb{R}^n, V^2) = 0$, so $H^k(\mathbb{R}^n, V) = H^k(\mathbb{R}^n, V^1)$. In our case, the $\mathbb{T}^n$-invariant elements in $C(P,\mathbb{R})$ are exactly the functions in $C(P/\mathbb{R}^n, \mathbb{R})$.

This proves $H^k(\mathbb{R}^n, C(P,\mathbb{R})) \cong H^k(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R}))$, where the action of $\mathbb{R}^n$ on the Fréchet space $C(P/\mathbb{R}^n, \mathbb{R})$ is trivial. Hence we get $H^k(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R})) \cong H^k(\mathbb{R}^n, \mathbb{R}) \otimes_{\mathbb{R}} C(P/\mathbb{R}^n, \mathbb{R})$, and $H^k(\mathbb{R}^n, \mathbb{R}) \cong \Omega^k\mathbb{R}^n$ as in the proof of [16, Lemma 2.1]. □

Remark 4.2. Proposition 4.1 still works if the stabiliser lattice $\mathbb{Z}^n$ of the action varies over P, that is, if there is a continuous function $\gamma\colon P/\mathbb{R}^n \to \mathrm{Gl}_n(\mathbb{R})/\mathrm{Gl}_n(\mathbb{Z})$ such that, for each $p \in P$, the lattice $\gamma_{[p]}(\mathbb{Z}^n) \subseteq \mathbb{R}^n$ fixes p.

Now we replace $\mathbb{R}^n$ by the crossed module $\mathcal{G}$. We assume that the canonical map

$$C(P/\mathbb{R}^n, \Omega^k\mathbb{R}) \cong H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R})) \to H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R}))$$

is an isomorphism for $k = 2, 3$. This holds, in particular, in the situation of Proposition 4.1 or Remark 4.2. Theorems 4.3–4.4 extend to this case, although we only state them in the situation of Proposition 4.1.

Under our assumption, the lifting obstruction in $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R}))$ is cohomologous to a *unique* function $\psi\colon P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$. We also call this function the *lifting obstruction* of A. The action of $\mathbb{R}^n$ on P lifts to an action on A if and only if this function $P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ vanishes.

Theorem 4.3. *Let P be a second countable, locally compact $\mathbb{T}^n$-space and let A be a continuous trace* C*-*algebra with spectrum P. Then the action of $\mathbb{T}^n$ on P lifts to an action of the bigroupoid $\mathcal{G} \ltimes P$ on A.*

Proof. Since $\mathcal{G} \ltimes P$ and $\mathbb{R}^n \ltimes P$ have the same objects and arrows, we may construct E and ω as above. For an action of $\mathcal{G} \ltimes P$, we also need the datum U, and this modifies the cocycle condition for ω.

The condition (A3) for vertical products says that $U(\xi, t, p)$ for fixed $t \in \mathbb{R}^n$ and $p \in P$ is a continuous homomorphism $\Lambda^3\mathbb{R}^n \to \mathbb{T}$. In particular, $U(0, t, p) = 1$ for all t, p. The condition (A4) for horizontal products gives

$$U(\xi_1, t_1, \alpha_{t_2}(p_2)) \cdot U(\xi_2, t_2, p_2) = U(\xi_1 + \xi_2, t_1 + t_2, p_2)$$

for all $\xi_1, \xi_2 \in \Lambda^3\mathbb{R}^n$, $t_1, t_2 \in \mathbb{R}^n$ and $p \in P$. For $t_2 = 0$ and $\xi_1 = 0$, this says that $U(\xi_2, 0, p_2) = U(\xi_2, t_1, p_2)$, so $U(\xi, t, p)$ does not depend on t and we may write $U(\xi, t, p)$ as $U(\xi, p)$; for $t_1 = 0$ and $\xi_2 = 0$, (A4) says $U(\xi_1, \alpha_{t_2}(p_2)) = U(\xi_1, p_2)$, that is, $U(\xi, p)$ only depends on the $\mathbb{R}^n$-orbit $[p]$ of p. For a function depending only on ξ and $[p]$, the two multiplicativity conditions (A3) and (A4) are equivalent. Thus (A3) and (A4) say exactly that U is a continuous function from $P/\mathbb{R}^n$ to the space of group homomorphisms $\Lambda^3\mathbb{R}^n \to \mathbb{T}$. Such homomorphisms lift uniquely to homomorphisms $\Lambda^3\mathbb{R}^n \to \mathbb{R}$. Thus $U(\xi, t, p) = \exp(2\pi \mathrm{i} \upsilon(\xi, [p]))$ for a continuous function $\upsilon\colon P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ that is uniquely determined by U.

Proposition 4.1 shows that there is a unique continuous function $P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ that represents the lifting obstruction for an $\mathbb{R}^n$-action; that is, this function measures the failure of the cocycle condition (A5), without the associator U, for a particular choice of ω. When we put in the associator, then (A5) holds if and only if υ above is equal to the lifting obstruction. Hence there is an action (E, ω, U) of $\mathcal{G} \ltimes P$ on A. $\square$

Our next goal is a long exact sequence containing the Brauer and Picard groups of $\mathcal{C}$ and some known cohomology groups of P and $P/\mathbb{R}^n$. Let $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$ and $\mathrm{For}\colon \mathrm{Pic}_{\mathcal{G}}(P) \to \mathrm{Pic}(P)$ denote the forgetful maps.

Theorem 4.4. *Let P be a second countable, locally compact $\mathbb{T}^n$-space. There is a natural long exact sequence*

$$\begin{aligned} 0 \leftarrow \mathrm{Br}(P) \xleftarrow{\mathrm{For}} \mathrm{Br}_{\mathcal{G}}(P) \leftarrow C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \leftarrow \mathrm{Pic}(P) \xleftarrow{\mathrm{For}} \mathrm{Pic}_{\mathcal{G}}(P) \\ \leftarrow C(P/\mathbb{R}^n, \mathbb{R}^n) \leftarrow H^1(P, \mathbb{Z}) \leftarrow H^1(P/\mathbb{R}^n, \mathbb{Z}) \leftarrow 0. \end{aligned}$$

Here $H^1(-, \mathbb{Z})$ denotes Čech cohomology. Furthermore, there are natural isomorphisms $\mathrm{Br}(P) \cong H^3(P, \mathbb{Z})$ and $\mathrm{Pic}(P) \cong H^2(P, \mathbb{Z})$.

Proof. The surjectivity of $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$ is asserted in Theorem 4.3. Choose an action (E, ω, U) on A in $\mathrm{Br}_{\mathcal{G}}(P)$. Any other action (E', ω', U') on the same A has $E' \cong E$ by (b) in Lemma 3.5; this gives an equivalence to an action with $E' = E$. The proof of Theorem 4.3 shows that we cannot modify U at all, that is, $U = U'$, because it must be the exponential of the lifting obstruction of A. The freedom in the choice of ω is to multiply it with $\exp(2\pi \mathrm{i}\varphi)$ for a continuous function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\varphi(t, 0, p) = \varphi(0, t, p) = 0$; it must satisfy $\partial\varphi = 0$ so as not to violate (A5). That is, φ is a cocycle for $H^2_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$.

An equivalence of actions allows, among other things, to conjugate ω by an isomorphism $V\colon E \to E$ that restricts to the identity on units. By Lemma 3.5, this function V differs from the identity by a continuous function $\mathbb{R}^n \times P \to \mathbb{T}$, which lifts uniquely to a continuous function $\kappa\colon \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\kappa(0, p) = 0$ for all $p \in P$. Conjugating ω by the equivalence of actions defined by κ multiplies it by $\exp(2\pi \mathrm{i}\partial\kappa)$ with

$$\partial\kappa(t_1, t_2, p) = \kappa(t_2, p) - \kappa(t_1 + t_2, p) + \kappa(t_1, \alpha_{t_2}(p)).$$

Thus only the class of φ in $H^2_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ matters for the equivalence class of the action. Proposition 4.1 identifies $H^2_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ with $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; that is, any cocycle φ is cohomologous to a unique one of the form $(t_1, t_2, p) \mapsto \chi[p](t_1 \wedge t_2)$ for a continuous function $\chi\colon P/\mathbb{R}^n \to \Omega^2\mathbb{R}^n$. Summing up, any action of $\mathcal{C}$ on A is equivalent to one having the same E and U and $\omega \cdot \exp(2\pi \mathrm{i}\chi)$ for some

$\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$. Twisting the unit element of $\mathrm{Br}_{\mathcal{G}}(P)$ with $\exp(2\pi\mathrm{i}\chi)$ as above defines a group homomorphism $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \to \mathrm{Br}_{\mathcal{G}}(P)$, which is one of the maps in our exact sequence. Our argument shows that its range is the kernel of $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$, that is, our sequence is exact at $\mathrm{Br}_{\mathcal{G}}(P)$.

So far, we have only used equivalences of a special form. In general, an equivalence between the two actions of $\mathcal{C}$ on A also involves a self-equivalence of A, that is, an element $F \in \mathrm{Pic}(P)$. This is given by a line bundle over P by Lemma 3.3. An equivalence $A \sim A'$ allows to transport the given action (E, ω, U) on A to an action (E', ω', U') on A'; now we apply this to the self-equivalence F. This gives $E' = r^*(L) \otimes_P E \otimes_P s^*(L)^*$, with ω' and U' defined by first cancelling pull-backs of $L \otimes_P L^*$ in the middle and then applying ω and U. By Lemma 3.5 (b), there is an isomorphism $E' \cong E$ that restricts to the identity on units. Using this, we transfer ω' and U' to E. The argument above shows that, choosing the part V in the equivalence suitably, we may arrange that $\omega' = \exp(2\pi\mathrm{i}\chi)\omega$ and $U' = U$ for some $\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; furthermore, χ is independent of choices.

Sending $F \in \mathrm{Pic}(P)$ to this $\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$ gives a well-defined map $\mathrm{Pic}(P) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; this is the next map in our exact sequence. It is routine to check that this map is a group homomorphism. Since we have now used the most general form of an equivalence, the actions (E, ω, U) and $(E, \exp(2\pi\mathrm{i}\chi)\omega, U)$ are equivalent if and only if χ is in the image of $\mathrm{Pic}(P) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; that is, our sequence is exact at $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$. Furthermore, if $\chi = 0$ then the equivalence (F, V) from (E, ω, U) to the twist by χ is a *self*-equivalence, so we have lifted $F \in \mathrm{Pic}(P)$ to $(F, V) \in \mathrm{Pic}_{\mathcal{G}}(P)$. Thus our sequence is exact at $\mathrm{Pic}(P)$.

Now we consider the kernel of $\mathrm{For}\colon \mathrm{Pic}_{\mathcal{G}}(P) \to \mathrm{Pic}(P)$; this consists of self-equivalences (F, V) of (A, E, ω, U) where F is isomorphic to the identity equivalence on A; we may arrange $F = A$ by a modification. Any two choices for V differ by pointwise multiplication with $\exp(2\pi\mathrm{i}\varphi)$ for some continuous function $\varphi\colon \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\varphi(0, p) = 0$ for all $p \in P$ by (f) in Lemma 3.5. Since V already satisfies the coherence condition (T3) for a cocycle, φ must be a cocycle for $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$.

Among the modifications from (F, V) to (F, V') for some V', we may consider those given by $W = \exp(2\pi\mathrm{i}\psi)$ for some $\psi\colon P \to \mathbb{R}$. This gives a modification from (F, V) to $(F, V \cdot \partial W)$ with $\partial W(t, p) = W(\alpha_t p) W(p)^{-1}$. Thus only the class of the cocycle ψ in $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ matters for the class in the Picard group. Proposition 4.1 identifies $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R})) \cong C(P/\mathbb{R}^n, \mathbb{R}^n)$. So we get a surjective group homomorphism from $C(P/\mathbb{R}^n, \mathbb{R}^n)$ onto the kernel of $\mathrm{For}\colon \mathrm{Pic}_{\mathcal{G}}(P) \to \mathrm{Pic}(P)$. This map continues our exact sequence and gives the exactness at $\mathrm{Pic}_{\mathcal{G}}(P)$.

So far, we have only used modifications that lift to a map $P \to \mathbb{R}$; general modifications involve continuous maps $\phi\colon P \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Locally, such a map lifts to $\mathbb{R}$. Choosing such a local lifting gives an open covering of P and a subordinate Čech 1-cocycle on P with values in $\mathbb{Z}$. There is a global lifting of ϕ to a function $P \to \mathbb{R}$ if and only if this 1-cocycle is a coboundary. Any Čech 1-cocycle in $H^1(P, \mathbb{Z})$ is the lifting obstruction of some continuous maps $\phi\colon P \to \mathbb{T}$ by a partition of unity argument. Thus $H^1(P, \mathbb{Z})$ is the quotient of the group of all functions $P \to \mathbb{T}$ modulo those functions of the form $\exp(2\pi\mathrm{i}\psi)$ for a continuous function $\psi\colon P \to \mathbb{R}$.

Any function $W\colon P \to \mathbb{T}$ gives a modification between a given transformation (F, V) and another transformation $(F, V \cdot \partial W)$ with ∂W as above. Since this has the same underlying equivalence F, ∂W is of the form $\partial W = \exp(2\pi i\kappa)$ for a continuous map $\kappa\colon \mathbb{R}^n \times P \to \mathbb{R}$, which represents a 1-cocycle in $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R})) \cong C(P/\mathbb{R}^n, \mathbb{R}^n)$. Thus there is a unique continuous function $\chi \in C(P/\mathbb{R}^n, \mathbb{R}^n)$ and a function $h\colon P \to \mathbb{R}$ such $\partial W = \exp(2\pi\mathrm{i}(\chi + \partial h))$. Hence we get a well-defined map $H^1(P, \mathbb{Z}) \to C(P/\mathbb{R}^n, \mathbb{R}^n)$ by sending the class of W in $H^1(P, \mathbb{Z})$ to the unique χ above. The image of this is the set of all χ for which $(F, V\exp(2\pi i\chi))$ is equivalent to (F, V). This gives the exactness of our sequence at $C(P/\mathbb{R}^n, \mathbb{R}^n)$. Furthermore, $\chi = 0$ means that $\partial(W/\exp(2\pi i h)) = 1$ for some $h \in C(P, \mathbb{R})$. Equivalently, $W/\exp(2\pi i h))$ is an $\mathbb{R}^n$-invariant function on P. This happens if and only if $[W] \in H^1(P, \mathbb{Z})$ is in the image of $H^1(P/\mathbb{R}^n, \mathbb{Z})$; here the map $H^1(P/\mathbb{R}^n, \mathbb{Z}) \to H^1(P, \mathbb{Z})$ is induced by the quotient map $P \to P/\mathbb{R}^n$. We have shown exactness of our sequence at $H^1(P, \mathbb{Z})$.

If $\varphi\colon P/\mathbb{R}^n \to \mathbb{T}$ goes to the trivial element of $H^1(P, \mathbb{Z})$, then $\varphi = \exp(2\pi i\psi)$ for some $\psi\colon P \to \mathbb{R}$. Since φ is constant on $\mathbb{R}^n$-orbits and these are connected, ψ is also constant on $\mathbb{R}^n$-orbits, so we have lifted φ to $\psi\colon P/\mathbb{R}^n \to \mathbb{R}$. Thus φ gives the trivial element of $H^1(P/\mathbb{R}^n, \mathbb{Z})$; that is, our sequence is exact also at $H^1(P/\mathbb{R}^n, \mathbb{Z})$. The natural isomorphisms $\mathrm{Br}(P) \cong H^3(P, \mathbb{Z})$ and $\mathrm{Pic}(P) \cong H^2(P, \mathbb{Z})$ are well-known. □

4.1. **Making the action strict.** So far, our actions on continuous trace C*-algebras are actions by equivalences of the Lie bigroupoid $\mathcal{G}$. We are going to turn these into strict actions of the crossed module $\mathcal{H}$. First, the equivalence between $\mathcal{H}$ and $\mathcal{G}$ shows that actions of $\mathcal{G}$ and $\mathcal{H}$ are "equivalent." In particular, $\mathrm{Br}_{\mathcal{G}}(P) \cong \mathrm{Br}_{\mathcal{H}}(P)$. Secondly, any action by correspondences of the crossed module $\mathcal{H}$ is equivalent to a strict action by automorphisms, and equivalences among such actions are equivariant Morita equivalences in an almost classical sense.

Let (A, E, ω, U) be an action of $\mathcal{G} \ltimes P$ by correspondences. This is a morphism from the bicategory $\mathcal{G} \ltimes P$ to the correspondence bicategory, satisfying some continuity conditions. We may compose this with the morphism $\mathcal{C}_{\mathcal{H}} \ltimes P \to \mathcal{G} \ltimes P$ induced by $\pi\colon \mathcal{C}_{\mathcal{H}} \to \mathcal{G}$; this gives an action of $\mathcal{C}_{\mathcal{H}} \ltimes P$ by correspondences. Conversely, an action of $\mathcal{C}_{\mathcal{H}} \ltimes P$ gives an action of $\mathcal{G} \ltimes P$ by composing with the morphism induced by $\iota\colon \mathcal{G} \to \mathcal{C}_{\mathcal{H}}$. Going back and forth induces a bijection between equivalence classes of actions because ι and π are inverse to each other up to equivalence. All this follows from general bicategory theory, which tells us how to compose morphisms between bicategories, how to compose transformations between such morphisms vertically and horizontally, and that there are related multiplication operations on modifications; in brief, bicategories with morphisms, transformations and modifications form a tricategory.

Everything above also works for topological bicategories and continuous morphisms, transformations and modifications. The correspondence bicategory, however, is not a topological bicategory in the usual sense. The continuity of a map from a locally compact space X to the set of C*-algebras is defined in an *ad hoc* way by giving a $C_0(X)$-C*-algebra as an extra datum. So continuity is not a property of a map, but extra structure. Consider a continuous map $f\colon Y \to X$ between locally compact spaces and a continuous map from X to C*-algebras, given by a map $x \mapsto A_x$ to C*-algebras and a $C_0(X)$-C*-algebra A with fibres A_x. Then the pull-back $f^*(A) = C_0(Y) \otimes_{C_0(X)} A$ is a $C_0(Y)$-C*-algebra with fibres $A_{f(y)}$; this is how we compose a continuous map to C*-algebras with the continuous map f. Similarly, we may pull back C*-correspondences and operators between them along continuous maps. This is how we compose continuous maps from locally compact spaces to the arrows and bigons in the correspondence bicategory. The general theory implies, in particular:

Theorem 4.5. *Let P be a locally compact $\mathbb{R}^n$-space. The morphisms π and ι between $\mathcal{G}$ and $\mathcal{H}$ induce an isomorphism $\mathrm{Br}_{\mathcal{G}}(P) \cong \mathrm{Br}_{\mathcal{H}}(P)$. If P is second countable and the $\mathbb{R}^n$ action factors through $\mathbb{T}^n$, then any continuous trace C*-algebra over P carries an action of $\mathcal{H}$ lifting the action of $\mathbb{R}^n$ on P.*

Let (A',E',ω',U') and (A,E,ω,U) be actions of $\mathcal{H}$ and $\mathcal{G}$ corresponding to each other. Then the equivariant Picard groups $\mathrm{Pic}_{\mathcal{G}}(A,E,\omega,U)$ and $\mathrm{Pic}_{\mathcal{H}}(A',E',\omega',U')$ are canonically isomorphic. □

We also use the notation $\mathcal{H} \ltimes P$ for the bigroupoid $\mathcal{C}_{\mathcal{H}} \ltimes P$. Now we make explicit how a continuous action (A,E,ω,U) of $\mathcal{G} \ltimes P$ gives a continuous action (A',E',ω',U') of $\mathcal{H} \ltimes P$. We put $A' = A$, and $E'_{(t,\eta)} = E_t$ with $E' = \mathrm{pr}_1^*(E)$, where $\mathrm{pr}_1\colon \mathbb{R}^n \times \Lambda^2\mathbb{R}^n \to \mathbb{R}^n$ maps $(t,\eta) \mapsto t$. Moreover, $U'_{(\theta,\xi)} = U_\xi$ for all $\theta \in \Lambda^2\mathbb{R}^n$, $\xi \in \Lambda^3\mathbb{R}^n$. The map ω' is defined by composing

$$\begin{aligned} E'_{(t_1,\eta_1,\alpha_{t_2}(p_2))} \otimes E'_{(t_2,\eta_2,p_2)} = E_{(t_1,\alpha_{t_2}(p_2))} \otimes E_{(t_2,p_2)} &\xRightarrow{\omega(t_1,t_2,p_2)} E_{(t_1+t_2,p_2)} \\ &\xRightarrow{U(\omega_\pi((t_1,\eta_1),(t_2,\eta_2)))} E_{(t_1+t_2,p_2)} = E'_{(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2,p_2)}, \end{aligned}$$

where $\omega_\pi((t_1,\eta_1),(t_2,\eta_2)) = t_1 \wedge \eta_2$ and where the tensors are over $A_{\alpha_{t_2}(p_2)}$. Thus

$$\omega'((t_1,\eta_1),(t_2,\eta_2,p_2)) = U(t_1 \wedge \eta_2,p_2) \cdot \omega(t_1,t_2,p_2).$$

It is routine to check that (A,E',ω',U') is a continuous action of $\mathcal{H} \ltimes P$.

Let (A^i,E^i,ω^i,U^i) for $i = 1,2$ be two continuous actions of $\mathcal{G} \ltimes P$ and let (F,V) be a transformation between them. Construct actions $((A^i)',(E^i)',(\omega^i)',(U^i)')$ of $\mathcal{H} \ltimes P$ for $i = 1,2$ as above. The induced transformation (F',V') between these actions of $\mathcal{H} \ltimes P$ is given by $F' = F$ and

$$\begin{aligned} V'_{(t,\eta,p)} = V_{t,p}\colon (E^1)'_{(t,\eta,p)} \otimes_{A^1_p} F_p = E^1_{(t,p)} \otimes_{A^1_p} F_p \\ \xRightarrow{V_{t,p}} F_{\alpha_t(p)} \otimes_{A^2_{\alpha_t(p)}} E^2_{(t,p)} = F_{\alpha_t(p)} \otimes_{A^2_{\alpha_t(p)}} (E^2)'_{(t,\eta,p)}. \end{aligned}$$

This is indeed a transformation, and it remains an equivalence if (F,V) is one. Thus equivalent actions of $\mathcal{G} \ltimes P$ induce equivalent actions of $\mathcal{H} \ltimes P$, as asserted by Theorem 4.5. There is nothing to do to transfer modifications between $\mathcal{G} \ltimes P$ and $\mathcal{H} \ltimes P$.

Strict actions of crossed modules of topological groupoids are defined in [10]. We make this explicit in the case we need:

Definition 4.6. Let A be a $C_0(P)$-C*-algebra. A strict action of $\mathcal{H} \ltimes P$ on A by automorphisms is given by continuous group homomorphisms $\alpha\colon H^1 \to \mathrm{Aut}(A)$ and $u\colon H^2 \to U(M(A))$ with $\alpha_{\partial(h)} = \mathrm{Ad}_{u(h)}$ for all $h \in H^2$ and $\alpha_h(u_k) = u_{c_h(k)}$ for all $h \in H^1$, $k \in H^2$, such that the homomorphism $C_0(P) \to ZM(A)$ is H^1-equivariant, where H^1 acts on P through the quotient map $H^1 \to \mathbb{R}^n$, $(t,\eta) \mapsto t$.

Such a strict action of $\mathcal{H} \ltimes P$ by automorphisms induces an action by correspondences. First, we take $E = r^*(A)$ as a left Hilbert A-module, with the right action of $s^*(A)$ through α: $(x \cdot a)(t,\eta,p) = x(t,\eta,p) \cdot \alpha_{t,\eta}(a(t,\eta,p))$ for all $x \in E$, $a \in s^*(A)$; so $a(t,\eta,p) \in A_p$ and $\alpha_{t,\eta}(a(t,\eta,p)) \in A_{\alpha_t(p)} = A_{r(t,p)} \ni x(t,\eta,p)$, as it should be. The isomorphisms ω map $x_1 \otimes x_2 \mapsto x_1 \cdot \alpha_{t,\eta}(x_2)$ for $x_1 \in E_{(t,\eta,p)}$, $x_2 \in E_{(t_2,\eta_2,p_2)}$, and $U_{((\theta,\xi),(t,\eta),p)}\colon E_{(t,\eta,p)} \to E_{(t,\eta+\theta,p)}$ multiplies on the right with the unitary $u(\theta,\xi)^*$. This is an action by correspondences as defined above.

Theorem 4.7. *Any action of $\mathcal{H} \ltimes P$ by correspondences is equivalent to an action that comes from a strict action of $\mathcal{H} \ltimes P$ by automorphisms.*

Proof. This is contained in [11, Theorem 5.3]. □

By construction, the action by correspondences coming from a strict action has $E \cong r^*(A)$ as a left Hilbert A-module and $E \cong s^*(A)$ as a right Hilbert A-module. Therefore, if F is a C*-correspondence between A^1 and A^2 for two such actions (A^1,E^1,ω^1,U^1), then $E^1 \otimes_{s^*(A^1)} s^*(F) \cong s^*(F)$ as a right Hilbert $s^*(A)$-module and $r^*(F) \otimes_{r^*(A^2)} E^2 \cong r^*(F)$ as a left Hilbert $r^*(A)$-module. In particular, these are isomorphisms of Banach spaces. The isomorphism V in a transformation is therefore given by bounded linear maps $V_g\colon F_{s(g)} \to F_{r(g)}$; the extra conditions to induce an isomorphism of correspondences $E^1 \otimes_{s^*(A^1)} s^*(F) \to r^*(F) \otimes_{r^*(A^2)} E^2$ are exactly the usual equivariance conditions for an equivariant correspondence. Thus the notion of equivalence for actions of $\mathcal{H} \ltimes P$ by correspondences amounts to a standard notion of equivariant Morita equivalence.

Theorem 4.8. *Let A be a* C*-*algebra. A strict action of $\mathcal{H}$ on A is equivalent to a pair of continuous maps $\bar{\alpha}\colon \mathbb{R}^n \to \operatorname{Aut}(A)$ and $\bar{u}\colon \Lambda^2\mathbb{R}^n \to UM(A)$ with the following properties:*

(a) $\bar{\alpha}_s\bar{\alpha}_t = \operatorname{Ad}_{\bar{u}(s\wedge t)}\bar{\alpha}_{s+t}$ *for all* $s,t \in \mathbb{R}^n$ *and* $\bar{\alpha}_0 = \mathrm{id}$*;*
(b) $\bar{u}$ *is a group homomorphism;*
(c) $\bar{\alpha}_s(\bar{u}(t\wedge v))\bar{u}(t\wedge v)^* \in UM(A)$ *is central and fixed by* $\bar{\alpha}_w$ *for all elements* $s,t,v,w \in \mathbb{R}^n$*;*
(d) $\bar{\alpha}_s(\bar{u}(t\wedge v))\,\bar{u}(t\wedge v)^* = \bar{u}(s\wedge v)\,\bar{\alpha}_t(\bar{u}(s\wedge v))^*$ *for all* $s,t,v \in \mathbb{R}^n$*.*

Proof. Let $\bar{\alpha}$ and $\bar{u}$ have the required properties. We claim that there are well-defined group homomorphisms

$$\begin{aligned} \alpha\colon H^1 &\to \operatorname{Aut}(A), & (t,\eta) &\mapsto \operatorname{Ad}_{\bar{u}(\eta)} \circ \bar{\alpha}_t,\\ u\colon H^2 &\to UM(A), & (\theta, s\wedge\theta_2) &\mapsto \bar{u}(\theta)\cdot\bar{\alpha}_s(\bar{u}(\theta_2))\bar{u}(\theta_2)^*. \end{aligned}$$

The map α is clearly well-defined. Since $\bar{\alpha}_s(\bar{u}(\eta))\bar{u}(\eta)^*$ is central, the automorphisms $\operatorname{Ad}_{\bar{u}(\eta)}$ and $\operatorname{Ad}_{\bar{\alpha}_s(\bar{u}(\eta))}$ are equal for all $s \in \mathbb{R}^n$, $\eta \in \Lambda^2\mathbb{R}^n$. Hence

$$\begin{aligned} &\alpha(t_1,\eta_1)\circ\alpha(t_2,\eta_2) = \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\bar{\alpha}_{t_1}\circ\operatorname{Ad}_{\bar{u}(\eta_2)}\circ\bar{\alpha}_{t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\operatorname{Ad}_{\bar{\alpha}_{t_1}(\bar{u}(\eta_2))}\circ\bar{\alpha}_{t_1}\circ\bar{\alpha}_{t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\operatorname{Ad}_{\bar{u}(\eta_2)}\circ\operatorname{Ad}_{\bar{u}(t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2} = \operatorname{Ad}_{\bar{u}(\eta_1)\bar{u}(\eta_2)\bar{u}(t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1+\eta_2+t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2} = \alpha(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2). \end{aligned}$$

Thus α is a homomorphism. We have $u(\theta,\xi) = u(\theta,0)\,u(0,\xi)$ and $\theta \mapsto u(\theta,0)$ is a group homomorphism. By assumption, the map $(s,t,v) \mapsto u(0, s\wedge t\wedge v)$ is antisymmetric and additive in t and v; thus it is additive also in s and hence descends to a group homomorphism on $\Lambda^3\mathbb{R}^n$. Since $u(0,s\wedge t\wedge v)$ is central, u is a well-defined group homomorphism. We have $\alpha_{(t,\eta)}(u_{\theta,\xi}) = u_{\theta,\xi+t\wedge\theta}$ and $\alpha_{(0,\theta)} = \operatorname{Ad}_{u(\theta,\xi)}$ by construction, so α and u form a strict action of the crossed module $\mathcal{H}$. Conversely, such a strict action (α,u) gives back $(\bar{\alpha},\bar{u})$ as above by taking $\bar{\alpha}_t = \alpha_{t,0}$ and $\bar{u}_\eta = u_{\eta,0}$. □

Remark 4.9. Crossed products for crossed module actions are studied in [10] in the strict case, and in [9] for actions by correspondences. This construction is, however, not very useful in our case because the crossed product is simply zero whenever the associator U is non-trivial. More precisely, the analysis in [9] shows that the crossed product factors through the quotient of A by the ideal generated by $(U_\xi - 1)a$ for all $a \in A$, $\xi \in \Lambda^3\mathbb{R}^n$. In the interesting case where the associator is needed, this quotient is zero.

To get a non-zero C*-algebra, we may first tensor A by some other action B of $\mathcal{H}$ with the opposite associator, so that the diagonal action of $\mathcal{H}$ on $A\otimes B$ has trivial associator. Thus $A\otimes B$ carries a Green twisted action of $\mathbb{R}^n$, and the crossed product with $\mathcal{H}$ on $A\otimes B$ is the appropriate $\mathbb{R}^n$-crossed product for such twisted actions. A good choice for B is the action of $\mathcal{H}$ corresponding to the non-associative compact operators defined in [4], see Theorem 4.10 for the corresponding Fell bundle over $\mathcal{G}$.

4.2. **Non-associative algebras.** We now relate actions of the bigroup $\mathcal{G}$ to non-associative Fell bundles over the group $\mathbb{R}^n$ with a trilinear associator. We interpret these as continuous spectral decompositions for non-associative C*-algebras as in [4]. In particular, non-associative C*-algebras arise as section C*-algebras for non-associative Fell bundles over $\mathbb{R}^n$.

Let (A, E, ω, U) be an action of $\mathcal{G}$ on a C*-algebra. This consists of imprimitivity A,A-bimodules E_t for $t \in \mathbb{R}^n$ with a continuity structure $E \subseteq \prod_{t\in\mathbb{R}^n} E_t$, with a continuous multiplication map $\omega\colon \bigsqcup E_t \otimes_A E_u \to \bigsqcup E_{t+u}$, and with a homomorphism $\Lambda^3\mathbb{R}^n \to ZUM(A)$, $\xi \mapsto U_\xi$, to the group $ZUM(A)$ of central unitary multipliers of A; here we use the same arguments as in the proof of Theorem 4.3 to see that the isomorphism $E_t \to E_t$ associated to the bigon $\xi\colon t \Rightarrow t$ does not depend on t, belongs to the centre, and that the map $\xi \mapsto U_\xi$ is a group homomorphism. In addition, these arguments show that the unitaries U_ξ are "invariant" under the $\mathbb{R}^n$-action; this means here that left and right multiplication by U_ξ on E_t gives the same map for each t; this is more than being in the centre of A, it means being in the centre of the whole Fell bundle.

We view the map ω as multiplication maps $E_t \times E_u \to E_{t+u}$. The cocycle condition for ω then becomes

$$x_t \cdot (x_u \cdot x_v) = U_{t\wedge u\wedge v} \cdot ((x_t \cdot x_u) \cdot x_v) = ((x_t \cdot x_u) \cdot x_v) \cdot U_{t\wedge u\wedge v} \tag{2}$$

for all $x_t \in E_t$, $x_u \in E_u$, $x_v \in E_v$, $t, u, v \in \mathbb{R}^n$. Thus our multiplication is not associative unless U is trivial, but in a controlled fashion given by the associator U. The multiplication determines the associator uniquely by (2). The existence of a continuous homomorphism $U\colon \Lambda^3\mathbb{R}^n \to ZUM(A)$ verifying (2) is a restriction on the lack of associativity of the multiplication map ω.

There are unique conjugate-linear involutions $E_t \to E_{-t}$, $x \mapsto x^*$, so that the left and right inner products on the fibres E_t are of the form xy^* and x^*y, respectively. We assumed that each E_t is an imprimitivity bimodule. This is equivalent to the surjectivity of the multiplication maps $E_t \times E_u \to E_{t+u}$. The two claims above are proved in [11, Theorem 3.3] for actions of locally compact groups; the proof carries over easily to the bigroup $\mathcal{G}$ (see also [9] for such results about actions of crossed modules). When we replace the inner products on E_t by the involutions as above, we get the data of a *saturated Fell bundle* over $\mathbb{R}^n$, except that our multiplication is non-associative with a $ZUM(A)$-valued associator. Thus we may call this a *non-associative saturated Fell bundle*.

When we interpret an associative, saturated Fell bundle over $\mathbb{R}^n$ as a group action by correspondences, then the section C*-algebra of the Fell bundle plays the role of the crossed product with $\mathbb{R}^n$ for an action by automorphisms. For a non-associative Fell bundle E as above, we may still define a convolution and an involution on the space $\Gamma_c(E)$ of compactly supported, continuous sections of E by

$$(f_1 * f_2)(t) = \int_{\mathbb{R}^n} f_1(x) f_2(t-x)\,\mathrm{d}x$$

and $f^*(t) = f(-t)^*$ for $t \in \mathbb{R}^n$. This satisfies $(f^*)^* = f$ and $(f_1 * f_2)^* = f_2^* * f_1^*$ as usual, but the convolution is not associative. Such non-associative *-algebras and their C*-completions are studied in [4, 5]. A non-associative analogue of the bounded operators on Hilbert space is defined in [5, Definition 4.2]. Non-associative C*-algebras are defined in [5, Definition B.4] as norm-closed subalgebras of the non-associative bounded operators. This implicitly defines the C*-completion of a non-associative *-algebra $C_c(E)$, by considering a supremum of norms over all *-homomorphisms to the non-associative bounded operators.

The dual action of $\mathbb{R}^n$ on a crossed product by $\mathbb{R}^n$ still works on $\Gamma_c(E)$: let $(\alpha_s f)(t) = \exp(2\pi i st) f(t)$ for $s, t \in \mathbb{R}$ and $f \in \Gamma_c(E)$. This is a *-automorphism of $\Gamma_c(E)$, which extends to the C*-completion $\mathrm{C}^*(E)$, and the map $s \mapsto \alpha_s(f)$ is continuous for each $f \in \mathrm{C}^*(E)$.

Conversely, consider a non-associative C*-algebra B as in [4], equipped with a continuous (dual) action β of $\mathbb{R}^n$. A *continuous spectral decomposition* of the $\mathbb{R}^n$-action on B is, by definition, a continuous field of Banach spaces E_t over $\mathbb{R}^n$ with a continuous multiplication map $\bigsqcup E_t \times \bigsqcup E_s \to \bigsqcup E_{t+s}$ and a continuous involution $\bigsqcup E_t \to \bigsqcup E_{-t}$, such that B is $\mathbb{R}^n$-equivariantly isomorphic to the C*-completion $\mathrm{C}^*(E)$ of the section *-algebra $C_c(E)$; thus the multiplication in E has the same associator as B. Elements of E_t give multipliers of $C_c(E)$ and hence of $\mathrm{C}^*(E)$, and this maps E_t into the space

$$M(B)_t = \{b \in M(B) \mid \beta_s(b) = \exp(2\pi i st) \cdot b \text{ for all } s \in \mathbb{R}^n\}$$

of t-homogeneous multipliers of B. Usually, however, the spaces $M(B)_t$ are too large to give a continuous spectral decomposition of B. A continuous spectral decomposition need not exist, and if it exists it is not unique.

For an associative C*-algebra B, this definition of a continuous spectral decomposition goes back to Exel [13]. Continuous spectral decompositions are related in [8] to Rieffel's theory of generalised fixed point algebras. This shows that continuous spectral decompositions need not exist and need not be unique. They always exist for dual actions, however, because the Fell bundle underlying the crossed product is a continuous spectral decomposition. Therefore, $B \otimes \mathbb{K}(L^2\mathbb{R}^n) \cong (B \rtimes \mathbb{R}^n) \rtimes \mathbb{R}^n$ always has a continuous spectral decomposition, being a dual action.

By definition, a non-associative Fell bundle over $\mathbb{R}^n$ is the same as a continuous spectral decomposition for an $\mathbb{R}^n$-action on a non-associative C*-algebra with a unitary, centre-valued associator. Thus the non-associative C*-algebras in [4] are very close to non-associative Fell bundles over $\mathbb{R}^n$, and these are the same as actions of the bigroup $\mathcal{G}$.

As an example, we now construct a Fell bundle over $\mathcal{G}$ corresponding to the twisted compact operators $\mathbb{K}_\chi(L^2(\mathbb{R}^n))$ defined in [4, Section 5]. Here $\chi\colon \Lambda^3\mathbb{R}^n \to \mathrm{U}(1)$ is a fixed tricharacter.

Let $A = C_0(\mathbb{R}^n)$. The bigroup $\mathcal{G}$ acts on A via correspondences in the following way. Let $E_t = C_0(\mathbb{R}^n)$ for $t \in \mathbb{R}^n$, and let A act on E_t on the left by a shifted pointwise multiplication,

$$(a \cdot h)(s) = a(s+t)\, h(s)$$

for $a \in A$ and $h \in E_t$, and on the right action by pointwise multiplication, $(h \cdot a)(s) = h(s)a(s)$. The inner product is the poinwise one, $\langle h_1, h_2\rangle(s) = \overline{h_1(s)}h_2(s)$. The space of C_0-sections is $E = C_0(\mathbb{R}^n \times \mathbb{R}^n)$ (we will use the first coordinate of E as the one parametrizing the field over $\mathbb{R}^n$). The multiplication maps $\omega_{t_1,t_2}\colon E_{t_1} \otimes_A E_{t_2} \to E_{t_1+t_2}$ on these fibres are

$$\omega_{t_1,t_2}(h_1 \otimes h_2)(s) = \chi(t_1 \wedge t_2 \wedge s)\, h_1(s+t_2)\, h_2(s),$$

and $\xi \in \Lambda^3\mathbb{R}^n$ acts by multiplication with the scalar $\chi(\xi)$ in each fibre E_t. The triple (A, E, ω, U) satisfies (A1)–(A5) and therefore defines a continuous action of $\mathcal{G}$ on A by correspondences.

Let $E^c = C_c(\mathbb{R}^n \times \mathbb{R}^n) \subseteq E$ be the subspace of sections of compact support. We define a (non-associative) convolution on E^c by

$$\begin{aligned}(f \bullet g)(t,s) &= \int_{\mathbb{R}^n} \omega_{r,t-r}(f(r,\cdot) \otimes g(t-r,\cdot))(s)\,\mathrm{d}r \\ &= \int_{\mathbb{R}^n} \chi(r \wedge t \wedge s)\, f(r, s+t-r)\, g(t-r,s)\,\mathrm{d}r \\ &= \int_{\mathbb{R}^n} \chi(r \wedge t \wedge s)\, f(r+t, s-r)\, g(-r,s)\,\mathrm{d}r\end{aligned}$$

Define $\Phi\colon C_c(\mathbb{R}^n \times \mathbb{R}^n) \to E^c$ by $\Phi(K)(x,y) = K(x+y,y)$. Then

$$\begin{aligned}(\Phi(K_1) \bullet \Phi(K_2))(t,s) &= \int_{\mathbb{R}^n} \chi(r \wedge t \wedge s)\, K_1(t+s, s-r)\, K_2(s-r,s)\,\mathrm{d}r \\ &= \int_{\mathbb{R}^n} \chi((t+s) \wedge r \wedge s)\, K_1(t+s, r)\, K_2(r,s)\,\mathrm{d}r\end{aligned}$$

Thus, if $K_1 \cdot K_2$ denotes the non-associative convolution operation on $C_c(\mathbb{R}^n \times \mathbb{R}^n)$ for the associator χ as in [4, Section 5], then $\Phi(K_1 \cdot K_2) = \Phi(K_1) \bullet \Phi(K_2)$.

The Fell bundle also comes with an involution given by the conjugate linear bimodule homomorphism $E_t \to E_{-t}$ with $h^*(s) = \overline{h(s+t)}$. For $f \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$ we have $f^*(t,s) = \overline{f(-t,s+t)}$. The isomorphism Φ intertwines the Fell bundle involution with the involution defined in [4] because

$$\Phi(K)^*(t,s) = \overline{\Phi(K)(-t,s+t)} = \overline{K(s,s+t)} = \Phi(K^*)(t,s).$$

The Hilbert–Schmidt norm on E^c is defined using the trace on $A \cong E_0$ as follows:

$$\|f\|_{HS}^2 = \int_{\mathbb{R}^n} (f^* \bullet f)(0,s)\,\mathrm{d}s = \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(r,s)|^2\,\mathrm{d}r\,\mathrm{d}s$$

Thus Φ preserves the norm as well. Therefore, it extends to an isometric isomorphism of the non-associative Hilbert–Schmidt operators. Let $C^*(E)$ denote the (non-associative) C^*-algebra obtained by taking the closure of the action of E^c on the space of Hilbert–Schmidt operators in the associated operator norm. The above observations imply:

Theorem 4.10. *The non-associative C^*-algebra $C^*(E)$ associated to the Fell bundle over $\mathcal{G}$ defined above is isomorphic to $\mathbb{K}_\chi(L^2(\mathbb{R}^n))$ via the unique continuous extension $\Phi\colon \mathbb{K}_\chi(L^2(\mathbb{R}^n)) \to C^*(E)$.*

5. Computing the lifting obstruction

Let P be a $\mathbb{T}^n$-space with orbit space X and let A be a continuous trace C*-algebra over P. We have seen that the $\mathbb{T}^n$-action on P lifts to an $\mathbb{R}^n$-action on A if and only if a certain tricharacter $\chi\colon \Lambda^3\mathbb{R}^n \to C(X,\mathbb{R})$ vanishes. How can we compute χ?

Let $p \in P$, let $1 \le i < j < k \le n$, and let $\mathbb{T}_{ijk} \subseteq \mathbb{T}^n$ be the three-dimensional subtorus given by the coordinates i,j,k. The value of χ at $(e_i \wedge e_j \wedge e_k, [p])$ may also be computed by the smaller system where we replace $\mathbb{T}^n$, P and A by $\mathbb{T}_{ijk}$, $\mathbb{T}_{ijk} \cdot p \subseteq P$, and the restriction of A to the orbit $\mathbb{T}_{ijk} \cdot p$; this is because the lifting obstruction χ is natural. Thus it suffices to compute the lifting obstruction in the case of a transitive action of a three-dimensional torus $\mathbb{T}^3$ on a space P, with some continuous trace C*-algebra A over P. If the stabilisers in P are not discrete, then P is two-dimensional, so $H^3(P,\mathbb{Z}) = 0$ and A is a trivial bundle of compact operators. In this case, the $\mathbb{T}^n$-action clearly lifts to a $\mathbb{T}^n$-action on A, so the lifting obstruction vanishes. So we may assume that the stabilisers of points in P are discrete.

We replace the action of $\mathbb{T}^3$ by one of $\mathbb{R}^3$, which we want to lift to A. This action is still transitive, and by assumption the stabiliser of a point is a lattice $\Gamma \subseteq \mathbb{R}^3$. We choose coordinates in $\mathbb{R}^n$ so that this lattice becomes $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. Thus we are reduced to computing the lifting obstruction in the case where $P = \mathbb{R}^3/\mathbb{Z}^3$ with the standard $\mathbb{R}^3$-action.

Our theory tells us that the continuous trace C*-algebra carries an action of the crossed module of Lie groupoids $\mathcal{H} \ltimes P$. Since $\mathbb{R}^3$ acts transitively on P, this crossed module is equivalent to the $\mathcal{H}$-stabiliser of a point in P; for general $n \in \mathbb{N}$ this is the crossed module of groups $\tilde{\mathcal{H}}$ with $\tilde{H}^1 = \mathbb{Z}^n \times \Lambda^2\mathbb{R}^n$ and $\tilde{H}^2 = \Lambda^2\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n$ with the restrictions of ∂ and c from $\mathcal{H}$. In our situation, the usual construction of induced actions of groups generalises to crossed modules. As we will see, any action of $\mathcal{H} \ltimes P$ on our continuous trace C*-algebra A is induced from an action of $\tilde{\mathcal{H}}$, namely, the restriction of the action of $\mathcal{H} \ltimes P$ on a single fibre $A_p = \mathbb{K}$.

A strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}$ consists of two group homomorphisms $\tilde{\alpha}\colon \tilde{H}^1 \to \mathrm{Aut}(\mathbb{K})$ and $\tilde{u}\colon \tilde{H}^2 \to U(M(\mathbb{K}))$ such that $\tilde{\alpha}_{\partial(h)} = \mathrm{Ad}_{\tilde{u}(h)}$ and $\tilde{u}(c_g(h)) = \tilde{\alpha}_g(\tilde{u}(h))$ for $g \in \tilde{H}^1$, $h \in \tilde{H}^2$. Let

$$A = \mathrm{Ind}_{\tilde{H}^1}^{H^1}(\mathbb{K}) = \{f \in C_b(H^1, \mathbb{K}) \mid f(\tilde{g} \cdot g) = \tilde{\alpha}_{\tilde{g}}(f(g)) \text{ for all } \tilde{g} \in \tilde{H}^1,\ g \in H^1\}$$

be the induced C*-algebra. By [20, Corollary 6.21] it has continuous trace with spectrum $H^1/\tilde{H}^1 = \mathbb{T}^n$. It carries a canonical action $\alpha\colon H^1 \to \mathrm{Aut}(A)$ given by $\alpha_h(f)(g) = f(g \cdot h)$. Let $u\colon H^2 \to U(M(A))$ be defined by $u_k(g) = \tilde{u}(c_g(k))$ for $g \in H^1$ and $k \in H^2$. The pair (α, u) defines a strict action of $\mathcal{H} \ltimes P$ on A in the sense of Definition 4.6. The proof is straightforward, using that $\partial(H^2) \subset \tilde{H}^1$ lies in the centre of H^1.

Similarly, an equivalence (F, V) between two strict actions $(\tilde{\alpha}, \tilde{u})$ and $(\tilde{\alpha}', \tilde{u}')$ induces an equivalence between the induced actions. Since any action of $\tilde{\mathcal{H}}$ on A by correspondences is equivalent to a strict one by Theorem 4.7, induction descends to a well-defined group homomorphism $\mathrm{Ind}\colon \mathrm{Br}(\tilde{\mathcal{H}}) \to \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$. Restricting an action to the fibre yields another group homomorphism $\mathrm{Res}\colon \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \to \mathrm{Br}(\tilde{\mathcal{H}})$.

Lemma 5.1. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the crossed modules described above acting on a stable continuous trace algebra A with spectrum $\mathbb{T}^n$. Restriction to the fibre of A and induction of a strict action to $\mathcal{H} \ltimes \mathbb{T}^n$ are inverse to each other and yield a group isomorphism $\mathrm{Br}(\tilde{\mathcal{H}}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$.*

Proof. If we restrict the induced action to the fibre over 0, we regain the action on $\mathbb{K}$ we started with. Thus $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}_{\mathrm{Br}(\tilde{\mathcal{H}})}$.

Let $p_0 = 0 \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let (α, u) be a strict action of $\mathcal{H} \ltimes \mathbb{T}^n$ on A. Let $(\tilde{\alpha}, \tilde{u})$ be the restricted action of $\tilde{H}^1$ on $A(p_0)$, which we identify with $\mathbb{K}$. Let $A' = \mathrm{Ind}_{\tilde{H}^1}^{H^1}(\mathbb{K})$ and denote the induced action by (α', u'). Consider the $C(\mathbb{T}^n)$-algebra homomorphism $\varphi\colon A \to A'$, which maps $a \in A$ to f_a with $f_a(g) = \alpha_g(a)(p_0)$. Since $f_{\alpha_h(a)}(g) = f_a(gh)$, it is equivariant. It is norm-preserving and therefore injective. Local triviality and a partition of unity argument show that it is also surjective, hence an isomorphism. The extension of φ to the multiplier algebra maps u to u'. Therefore, φ is an isomorphism that intertwines the two actions of $\mathcal{H} \ltimes \mathbb{T}^n$, which implies $\mathrm{Ind} \circ \mathrm{Res} = \mathrm{id}_{\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)}$. □

To compute $\mathrm{Br}(\tilde{\mathcal{H}})$ we may further simplify the situation by weakening. The equivalence between $\mathcal{H}$ and $\mathcal{G}$ restricts to one between $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{G}}$, where $\tilde{G}^1 = \mathbb{Z}^n$, $\tilde{G}^2 = \Lambda^3\mathbb{R}^n$ with the restriction of the bigroupoid structure from $\mathcal{G}$. Since $\mathbb{K}$ is equivalent to $\mathbb{C}$, strict actions of $\tilde{\mathcal{H}}$ on $\mathbb{K}$ by automorphisms are equivalent to actions of $\tilde{\mathcal{G}}$ by correspondences on $\mathbb{C}$.

Such an action has $A = \mathbb{C}$ and $E_g = \mathbb{C}$ for all $g \in \tilde{G}^1 = \mathbb{Z}^n$ because this is, up to isomorphism, the only imprimitivity bimodule from $\mathbb{C}$ to itself. The multiplication maps and the action of bigons give maps $\omega\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ and $U\colon \Lambda^3\mathbb{R}^n \to \mathbb{T}$ because an isomorphism $\mathbb{C} \to \mathbb{C}$ is simply multiplication by a scalar of modulus one. The conditions for an action require the following:

(a) $\omega(0,l) = \omega(k,0) = 1$ for all $k,l \in \mathbb{Z}^n$;
(b) U is a continuous group homomorphism;
(c) $\omega(k+l,m)\omega(k,l)U(k\wedge l\wedge m) = \omega(k,l+m)\omega(l,m)$ for all $k,l,m \in \mathbb{Z}^n$.

For a discrete group Γ and a Γ-module M, we denote the group cohomology of Γ with coefficients in M by $H^k_{\mathrm{gr}}(\Gamma, M)$.

Theorem 5.2. *Let $k = \binom{n}{3}$. The pair (ω, U) associated to an action of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ has the following properties:*

(a) *U is trivial on $\Lambda^3\mathbb{Z}^n$ and therefore yields a character on $\mathbb{T}^k = \Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n$,*
(b) *ω is a group 2-cocycle representing a central extension of $\mathbb{Z}^n$ by $\mathbb{T}$.*

Moreover, we have group isomorphisms

$$\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathrm{Br}(\tilde{\mathcal{H}}) \cong \mathrm{Br}(\tilde{\mathcal{G}}) \cong \mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}).$$

Proof. It follows from (c) in the conditions listed above that the restriction of U to $\Lambda^3\mathbb{Z}^n$ is the coboundary of ω. In particular, it represents the trivial element in cohomology. This is impossible if U is a non-trivial tricharacter on $\mathbb{Z}^n$. Thus ω is a 2-cocycle classifying a central $\mathbb{T}$-extension of $\mathbb{Z}^n$ and U is a character on $\Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n \cong \mathbb{T}^k$. The group of all such characters is the Pontrjagin dual of $\mathbb{T}^k$, which is $\mathbb{Z}^k$.

An equivalence (F, V) between two actions (ω, U) and (ω', U') has to have $F = \mathbb{C}$, and $V\colon \mathbb{Z}^n \to \mathbb{T}$ has to satisfy $V(0) = 1$ and $\omega(k,l)V(k+l) = V(l)V(k)\,\omega'(k,l)$ for all $k,l \in \mathbb{Z}^n$. Therefore (ω, U) and (ω', U') are equivalent if and only if $U = U'$ and ω differs from ω' by a coboundary. Moreover, the product of elements in the Brauer group of the bigroupoid $\tilde{\mathcal{G}}$ translates into the product of characters and cocycles. Since conditions (a)–(c) above fully characterise an action of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ by correspondences, every pair (ω, U) is associated to such an action. Altogether, we have constructed a group isomorphism $\mathrm{Br}(\tilde{\mathcal{G}}) \to \mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$. The isomorphism $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathrm{Br}(\tilde{\mathcal{H}})$ was constructed in Lemma 5.1, and $\mathrm{Br}(\tilde{\mathcal{G}}) \cong \mathrm{Br}(\tilde{\mathcal{H}})$ follows from the equivalence of the two bigroupoids. □

We can make the inverse of the isomorphism $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ more explicit: We first have to transfer the action of $\tilde{\mathcal{G}}$ to $\tilde{\mathcal{H}}$ using the functor π, then make that weak action strict by passing to a stabilisation by [11, Theorem 5.3]. We have $U_{(\theta,\xi)} = U^{\tilde{\mathcal{G}}}_{\xi}$ for $\theta \in \Lambda^2\mathbb{R}^n$, $\xi \in \Lambda^3\mathbb{R}^n$ because $\pi(\theta,\xi) = \xi$. For $k_1, k_2 \in \mathbb{Z}^n$, $\eta_1, \eta_2 \in \Lambda^2\mathbb{R}^n$, the natural transformation $\pi(k_1,\eta_1)\cdot\pi(k_2,\eta_2) \Rightarrow \pi((k_1,\eta_1)\cdot(k_2,\eta_2))$ in $\tilde{\mathcal{G}}$ is $k_1 \wedge \eta_2$, so $\omega((k_1,\eta_1),(k_2,\eta_2)) = U^{\tilde{\mathcal{G}}}_{k_1\wedge\eta_2}\cdot\omega^{\tilde{\mathcal{G}}}(k_1,k_2)$.

To turn this action of the crossed module $\tilde{\mathcal{H}}$ by correspondences into a strict action by automorphisms, we take the Hilbert space of L^2-sections of the Fell bundle $(\mathbb{C})_{g\in\tilde{H}^1}$ over $\tilde{H}^1$. In our case, this is simply $\mathcal{K} = L^2(\mathbb{Z}^n \times \Lambda^2\mathbb{R}^n)$. The multiplication in the Fell bundle given by ω forms an action of $\tilde{\mathcal{H}}$ on this Hilbert module over $\mathbb{C}$.

We give $\mathbb{K}(\mathcal{K})$ the induced action, so that $\mathcal{K}$ is an equivariant Morita equivalance between $\mathbb{K}(\mathcal{K})$ and $\mathbb{C}$ with the given action of $\tilde{\mathcal{H}}$. Since the action of $\tilde{\mathcal{H}}$ on $\mathbb{C}$ is non-trivial, $\mathcal{K}$ is not a Hilbert space representation of $\tilde{\mathcal{H}}$, but only a projective Hilbert space representation. On $\tilde{H}^1$, this is given by

$$((k_2,\eta_2)\cdot f)(k_1,\eta_1) = \omega((k_1,\eta_1),(k_2,\eta_2))f(k_1+k_2,\eta_1+\eta_2+k_1\wedge k_2)$$

for $k_1, k_2 \in \mathbb{Z}^n$ and $\eta_1, \eta_2 \in \Lambda^2\mathbb{R}^n$. This induces an action $\alpha\colon \tilde{H}^1 \to \mathrm{Aut}(\mathbb{K}(\mathcal{K}))$. Together with the homomorphism $u\colon \tilde{H}^2 \to U(\mathcal{K})$ given by

$$u_{(\theta,\xi)}(f)(k,\eta) = U^{\tilde{\mathcal{G}}}(\xi + k \wedge \theta)\, f(k, \eta + \theta)$$

this is the strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K})$ that corresponds to the action by correspondences of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ given by $(\omega^{\tilde{\mathcal{G}}}, U^{\tilde{\mathcal{G}}})$.

Finally, we induce the action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K})$ to $\mathcal{H} \ltimes \mathbb{T}^n$. As described above, this produces an action of $\mathcal{H} \ltimes P$ with $P = \mathbb{R}^n/\mathbb{Z}^n$. More precisely, we get an action of $\mathcal{H} \ltimes P$ on a continuous trace C*-algebra over P.

Theorem 5.3. *Let A be the continuous trace* C*-*algebra that corresponds to the element $(1,0) \in \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^3)$. Then the Dixmier–Douady invariant of A is a generator of $H^3(\mathbb{T}^3, \mathbb{Z})$.*

Proof. Let $(0,x) \in \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T})$ and choose a cocycle $\omega\colon \mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{T}$ representing x. The action of $\tilde{\mathcal{H}}$ on $\mathbb{C}$ by correspondences associated to this pair is pulled back from an action of $\mathbb{Z}^3$ on $\mathbb{C}$ by correspondences via the canonical functor $\tilde{\mathcal{H}} \to \mathbb{Z}^3$. Let $\mathcal{K}' = L^2(\mathbb{Z}^3)$. The group $\mathbb{Z}^3$ acts projectively on $\mathcal{K}'$ via ω. This induces an honest representation $\alpha\colon H^1 \to \mathbb{Z}^3 \to \mathrm{Aut}(\mathbb{K}(\mathcal{K}'))$ of H^1 on the compact operators. Let $u\colon H^2 \to U(M(\mathbb{K}(\mathcal{K}')))$ be the trivial homomorphism. The pair (α, u) is a strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K}')$. By the same reasoning as above, we may choose A to be the C*-algebra obtained by inducing this $\tilde{\mathcal{H}}$-action to an $\mathcal{H}$-action. Since $A \cong C(\mathbb{T}^3, \mathbb{K}(\mathcal{K}'))$, its Dixmier–Douady class vanishes. But $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^3) \to \mathrm{Br}(\mathbb{T}^3) \cong H^3(\mathbb{T}^3, \mathbb{Z})$ is surjective, therefore $(1,0)$ has to be mapped to a generator of $H^3(\mathbb{T}^3, \mathbb{Z}) \cong \mathbb{Z}$. □

Corollary 5.4. *Let $k = \binom{n}{3}$. The group homomorphism*

$$\mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \to H^3(\mathbb{T}^n, \mathbb{Z}),$$

which maps an element $([U], [\omega])$ to the Dixmier–Douady class of the associated continuous trace C*-*algebra restricts to an isomorphism $\mathbb{Z}^k \to H^3(\mathbb{T}^n, \mathbb{Z})$ and maps $H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ to zero.*

Proof. Each projection $p_{ijk}\colon \mathbb{T}^n \to \mathbb{T}^3$ for $1 \le i < j < k \le n$ induces a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \arrow[r] & H^3(\mathbb{T}^n, \mathbb{Z}) \\ \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T}) \arrow[r] \arrow[u, "p^*_{ijk}"] & H^3(\mathbb{T}^3, \mathbb{Z}). \arrow[u, "p^*_{ijk}"'] \end{tikzcd}$$

Since the vertical arrows are split by corresponding inclusions $\mathbb{T}^3 \to \mathbb{T}^n$, they are injective. In particular, $p^*_{ijk}(1) \in \{\pm e_i \wedge e_j \wedge e_k\} \subset \Lambda^3\mathbb{Z}^n \cong \mathbb{Z}^k$. The statement now follows from Theorem 5.3 because $\mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ is generated by the images of all p^*_{ijk}. □

6. Crossed module actions and T-duality

Let A be a continuous trace C*-algebra whose spectrum is a $\mathbb{T}^n$-space P with orbit space X. A second such pair A' with spectrum P' and the same orbit space is said to be (topologically) *T-dual* to A if the $\mathbb{R}^n$-action on P lifts to one on A with $A' \cong A \rtimes \mathbb{R}^n$. In particular, this implies an isomorphism of twisted K-groups $K_*(A) \cong K_{*-n}(A')$ by the Connes–Thom isomorphism. In the case of principal $\mathbb{T}^1$-bundles, any pair (A, P) has a unique T-dual.

A T-dual need no longer exist for higher-dimensional torus bundles. The first obstruction against it is the lifting obstruction discussed above. Even if it vanishes, the crossed product need not be a continuous trace C*-algebra. Whether this is the case is determined by a class in $H^1(X, \mathbb{Z}^\ell)$ for $\ell = \binom{n}{2}$, which is derived from the Mackey obstruction as in [17, Theorem 3.1]. If it does not vanish, the crossed product turns out to be a bundle of non-commutative tori.

A (non-associative) Fell bundle (E, ω, U) given by an action of $\mathcal{H} \ltimes P$ on A by correspondences combines all of the obstructions into one structure: We have already identified the lifting obstruction. By Theorem 5.2, the Brauer group of the fibre $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \times H^3(\mathbb{T}^n, \mathbb{Z})$ may be interpreted as the group of all possible Dixmier–Douady classes and Mackey obstructions of $\mathbb{T}^n$. As described in [19], the group $H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ can be equipped with a natural topology, and [19, Lemma 3.3] gives a homomorphism

$$M\colon \mathrm{Br}_{\mathcal{H}}(P) \to C(X, H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})),$$

which sends $[A] \in \mathrm{Br}_{\mathcal{H}}(P)$ to the function that maps x to the Mackey obstruction of $[A(x)] \in \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$. The homotopy class $[M(A)] \in \pi_0(C(X, H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}))) \cong H^1(X, \mathbb{Z}^\ell)$ for $\ell = \binom{n}{2}$ vanishes if and only if there is a classical T-dual, see [17, Theorem 3.1].

To summarise: The lifting obstruction of the Fell bundle (E, ω, U) vanishes if and only if the multiplication in the Fell bundle is associative. If it is, then we can form the section C*-algebra associated to the Fell bundle. This will again be of continuous trace if and only if $[M(A)]$ vanishes. If it does, then the section algebra is the classical T-dual of A, else it is a non-commutative T-dual. As can be seen from this, the non-associative Fell bundle obtained from the crossed module action contains all information of the T-dual in case it exists and all residual information in case it does not.

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E-mail address: `rmeyer2@uni-goettingen.de`

Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany

E-mail address: `u.pennig@uni-muenster.de`

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany