

# Uniqueness from discrete data in an inverse spectral problem for a pencil of ordinary differential operators

Malcolm Brown, Marco Marletta and Freddy Symons

## ABSTRACT

We prove a pair of uniqueness theorems for an inverse problem for an ordinary differential operator *pencil* of second order. The uniqueness is achieved from a discrete set of data, namely, the values at the points  $-n^2$  ( $n \in \mathbb{N}$ ) of (a physically appropriate generalization of) the Weyl–Titchmarsh  $m$ -function  $m(\lambda)$  for the problem. As a corollary, we establish a uniqueness result for a physically motivated inverse problem inspired by Berry and Dennis (‘Boundary-condition-varying circle billiards and gratings: the Dirichlet singularity’, *J. Phys. A: Math. Theor.* 41 (2008) 135203).

To achieve these results, we prove a limit-circle analogue to the limit-point  $m$ -function interpolation result of Rybkin and Tuan (‘A new interpolation formula for the Titchmarsh–Weyl  $m$ -function’, *Proc. Amer. Math. Soc.* 137 (2009) 4177–4185); however, our proof, using a Mittag-Leffler series representation of  $m(\lambda)$ , involves a rather different method from theirs, circumventing the  $A$ -amplitude representation of Simon (‘A new approach to inverse spectral theory, I. Fundamental formalism’, *Ann. Math.* (2) 150 (1999) 1029–1057). Uniqueness of the potential then follows by appeal to a Borg–Marčenko argument.

### 1. Introduction: new definitions and problem statements

Let  $H$  denote the Hilbert space  $L^2(0, 1; r \, dr) = \{u : (0, 1) \rightarrow \mathbb{C} \mid \int_0^1 r |u(r)|^2 \, dr < \infty\}$ . Suppose that  $q, w \in L^\infty_{\text{loc}}(0, 1]$ , with  $w > 0$  almost everywhere and  $q$  real-valued. In the space  $H$  we examine the following operator pencil:

$$Lu(r; \lambda) = \lambda Pu(r; \lambda) \quad (r \in (0, 1)). \tag{1.1}$$

Here  $L$  is a realization in  $H$  of the differential expression

$$\ell u(r) = -\frac{1}{r}(ru'(r))' + q(r)u(r)$$

which we shall define precisely below, and  $P$  is the unbounded multiplication operator

$$Pu(r) = w(r)u(r)$$

with domain

$$D(P) = \left\{ u \in L^2(0, 1; r \, dr) \mid \int_0^1 w(r)|u(r)|^2 \, dr < \infty \right\} = L^2(0, 1; w(r) \, dr),$$

in which the weight  $w$  is assumed to have the following singular behaviour:

$$w(r) = \frac{1}{r^\nu}(1 + o(1)) \quad (r \rightarrow 0),$$

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where  $\nu \geq 0$  is fixed. Typically, one might treat equation (1.1) by writing it in the form

$$\frac{1}{rw(r)}(ru'(r))' + \frac{q(r)}{w(r)}u(r) = \lambda u(r)$$

and noting that the expression on the left-hand side is formally symmetric in the space  $L^2(0, 1; rw(r)dr)$ . In this paper, however, we work in the space  $H$ , which is the natural choice in the physical setting from which our problem arises. Briefly, if  $w(r) = r^{-2}$  and  $\lambda = -\lambda_n$ , then (1.1) becomes the  $\mu = 0$  case of

$$-\frac{1}{r}(ru'_n(r; \mu))' + q(r)u_n(r; \mu) + \frac{\lambda_n}{r^2}u_n(r; \mu) = \mu u_n(r; \mu). \tag{1.2}$$

This is a Bessel-type equation with potential, and equations related to it have been studied quite extensively [2, 3, 9, 10, 19]. In addition, if  $\lambda_n$  are the angular eigenvalues of a spherically symmetric time-independent Schrödinger equation in a sub-domain of  $\mathbb{R}^2$ , then separating the same equation into polar coordinates with radial component  $u_n$  yields precisely the system (1.2). These eigenvalues are determined by the domain and boundary conditions. A particular choice of these is discussed later, in relation to a scenario first formulated in [7] and further explored in [22]. We refer the reader to Section 4 and to these two references for further details.

We now describe the domain of the pencil  $L - \lambda P$ . It turns out that, in some cases, the natural choice of domain is  $\lambda$ -dependent, and we require the following definition.

DEFINITION 1.1. We say that the equation

$$-\frac{1}{r}(ru'(r))' + q(r)u(r) = \lambda w(r)u(r) \tag{1.3}$$

is in *pencil-limit-point* or *pencil-limit-circle* at 0, corresponding, respectively, to the  $L^2(0, 1; r dr)$  solution space being one- or two-dimensional. We abbreviate these, respectively, by PLP and PLC.

REMARK 1.1. For our example, with  $w(r) = r^{-\nu}(1 + o(1))$ , it turns out that the problem is always in PLC at 0 if  $\nu \in [0, 2)$ , always in PLP if  $\nu > 2$ , and has  $\lambda$ -dependent classification if  $\nu = 2$ ; see Appendix C. In the case  $\nu = 2$ , (1.3) is in PLP at 0 if  $\text{Im}(\sqrt{\lambda}) \geq 1$  and in PLC at 0 if  $\text{Im}(\sqrt{\lambda}) < 1$ ; we choose the branch of the square root with  $\text{Im}\sqrt{\lambda} > 0$ . The parabola  $\text{Im}\sqrt{\lambda} = 1$  divides  $\mathbb{C}$  into components  $\Omega_p$  and  $\Omega_c$ , and the pencil is in PLP for  $\lambda \in \Omega_p$ , PLC for  $\lambda \in \Omega_c$ ; see Figure C.1 in Appendix C.

The definition of the domain of the pencil  $L - \lambda P$  is slightly simplified by noting that, except when the problem is in PLP for all  $\lambda$ , the point  $\lambda = 0$  always lies in the domain  $\Omega_c$  in which the equation is in PLC: see Figure C.1. Thus, in the PLC case, we can take  $U$  to be any non-trivial real-valued solution of the equation  $\ell U = 0$ , and use it to define a boundary condition in the usual way for the classical limit-circle case. Let  $[\cdot, \cdot]$  denote the Wronskian.

DEFINITION 1.2. In the PLC case let  $U$  be any non-trivial real-valued solution of  $\ell U = 0$ . Then the boundary condition at 0 defined by  $U$  is

$$[u, U](0^+; \lambda) := \lim_{r \searrow 0} \{u(r; \lambda)rU'(r) - ru'(r; \lambda)U(r)\} = 0. \tag{1.4}$$

DEFINITION 1.3 (Domain of  $L - \lambda P$ ). In the PLP case,

$$D(L - \lambda P) = \{u \in H \mid \ell u - \lambda wu \in H, u(1; \lambda) = 0\}.$$

In the PLC case,

$$D(L - \lambda P) = \{u \in H \mid \ell u - \lambda w u \in H, u(1; \lambda) = 0, [u, U](0^+; \lambda) = 0\}.$$

To introduce the *Weyl–Titchmarsh function*  $m(\lambda)$  for our pencil, we first remark that, owing to the asymptotics in Appendix C, equation (1.3) has at least one non-trivial solution in  $H$ . We can therefore make the following definition.

DEFINITION 1.4. In the PLP case, let  $u(\cdot; \lambda)$  denote the unique (up to scalar multiples) solution of (1.3) in  $H$ . In the PLC case, let  $u(\cdot; \lambda)$  denote the unique-up-to-multiples solution with  $[u, U](0^+; \lambda) = 0$ . Then the Dirichlet  $m$ -function is

$$m(\lambda) = u'(1; \lambda)/u(1; \lambda). \quad (1.5)$$

Using the variation-of-parameters formula, one may show that, with the domains as in Definition 1.3, the operator  $L - \lambda P$  is invertible when  $m(\lambda)$  is analytic, and that the eigenvalues of the pencil  $L - \lambda P$ , which are poles of  $(L - \lambda P)^{-1}$ , are the poles of  $m(\lambda)$ . Note that, in the case  $\nu = 2$ , there is generally a discontinuity of  $m$  across the boundary curve between  $\Omega_p$  and  $\Omega_c$ , due to the freedom in choosing the boundary condition function  $U$  for  $\text{Im}(\sqrt{\lambda}) < 1$ .

The main objective of this paper is to obtain a pair of uniqueness theorems for the following.

INVERSE PROBLEM 1.1. Let  $w : (0, 1] \rightarrow (0, +\infty)$  be locally bounded and suppose (1.1) is in PLP or PLC at 0. If in PLC, suppose that we have a boundary condition as in Definition 1.3. Now let  $\mathcal{S} := ((-n^2, m_n))_{n=1}^\infty$  be a sequence of admissible points in the graph of a generalized Titchmarsh–Weyl  $m$ -function for (1.3). Recover the potential  $q$  from the sequence  $\mathcal{S}$  under these conditions.

Our approach is firstly to show that, in both the PLP and PLC cases, the  $m$ -function is uniquely determined by its values at  $-n^2$  ( $n \in \mathbb{N}$ ), then secondly to invoke the Borg–Marčenko-type theorem in Appendix A that uniquely determines a potential from its associated  $m$ -function. In the PLP case  $\nu \geq 2$  (note  $\nu = 2$  turns out to be treatable by a PLP technique), we will transform (1.3) to Liouville normal form on the half-line  $[0, \infty)$ , in PLP at  $\infty$ , regular at 0, before utilizing the Rybkin–Tuan interpolation formula [24] for the classical limit-point  $m$ -function associated with such an equation. This is valid because the PLP and classical limit-point  $m$ -functions are formally the same where their domains overlap, that is, all of  $\mathbb{C}$  when  $\nu > 2$  and  $\Omega_p$  when  $\nu = 2$ ; the Rybkin–Tuan interpolation holds in this region. However, when  $0 \leq \nu < 2$ , the Liouville normal form of (1.3) holds on a finite interval; to our knowledge, there is no interpolation result for such a classical limit-circle problem.

To fill this gap, in Section 2 we will prove an interpolation result similar to that in [24], but which holds in a finite-interval limit-circle case. We will then argue using the same reasoning as in the PLP case that we may use the interpolation to prove our PLC uniqueness theorem. The uniqueness theorems will be stated and proved by the outlined methods in Section 3.

We will conclude the paper with an illustration of the relevance of this result in Section 4, where we explain how it proves a uniqueness theorem for the physically motivated Berry–Dennis PDE inverse problem, which involves boundary singularities and partial Cauchy data at the boundary.

2. *Interpolation of a classical limit-circle  $m$ -function*

We will use Theorem A.1 [24, Theorem 5] to prove our PLP uniqueness result. A drawback of the theorem is that it will not help for the PLC uniqueness, as it only applies to classical limit-point operators on the half-line. The purpose of this section is to establish an analogous result, for a particular finite-interval classical limit-circle Sturm–Liouville problem.

Suppose  $Q \in L^2(0, 1)$  is real-valued. Then considered over  $L^2(0, 1)$  the differential equation

$$-u''(x; \lambda) + \left(Q(x) - \frac{1}{4x^2}\right)u(x; \lambda) = \lambda u(x; \lambda), \quad (x \in (0, 1)) \tag{2.1}$$

is classical limit-circle non-oscillatory at 0 (see Lemma C.3 and note we may formally transform between (2.1) and (1.3) using the Liouville–Green transformation [13, equation (2.5.2)]). Hence, we require a boundary condition at 0: we will use the *Friedrichs* or *principal* one. Let  $U_p$  be a principal solution of (2.1), that is,  $U_p$  is non-trivial, and, for any linearly independent solution  $V$ , we have  $U_p(x) = o(V(x))$  as  $x \rightarrow 0$ . The Friedrichs boundary condition at 0 is the requirement that a solution  $u$  satisfy

$$[u, U_p](0^+; \lambda) = 0. \tag{2.2}$$

Up to a scalar multiple, (2.1) and (2.2) uniquely specify a solution  $u$  (a simple consequence of Lemma C.3). Taking such a non-trivial solution  $u$ , we choose a purely Robin (-to-Robin)  $m$ -function, that is, for  $h \neq H$  both real, the unique  $m_{h,H}(\lambda)$  satisfying

$$u'(1; \lambda) - Hu(1; \lambda) = m_{h,H}(\lambda)(u'(1; \lambda) - hu(1; \lambda)). \tag{2.3}$$

We will interpolate this  $m$ -function, in the style of Theorem A.1.

The proof of Theorem A.1 in [24] relies fundamentally on the observation that the classical limit-point half-line Dirichlet  $m$ -function has a representation using a Laplace transform of the  $A$ -amplitude [4, 14]. This is used by first proving [24, Theorem 4] that a Laplace transform

$$F(z) = \mathcal{L}[f](z) = \int_0^\infty e^{-zx} f(x) dx \quad \left(\operatorname{Re}(z) > \frac{1}{2} + \beta\right)$$

has representation

$$F(z) = \sum_{n=0}^\infty c_n \left(z + \frac{1}{2} - \beta\right) \sum_{k=0}^n a_{nk} F(k + \beta), \tag{2.4}$$

for  $c_n, a_{nk}$  defined as in Theorem A.1 and fixed positive  $\beta$ , provided

$$\int_0^\infty e^{-\delta x} |f(x)| dx < \infty \quad \text{for every } \delta > 0. \tag{2.5}$$

Rybkin and Tuan then show [24, Theorem 5] that interpolation formula (2.4) applies to  $F(\kappa) = m(-\kappa^2) - \kappa$ .

We shall follow a similar line of attack, and eventually implement (2.4). Unfortunately, the  $A$ -amplitude Laplace transform representation in [14] is not valid in the classical limit-circle case at one endpoint of a finite interval, since one cannot transform such a problem to the half-line whilst retaining the Liouville normal form. Another approach must be used.

We will find a Laplace transform representation of  $m_{h,H}(\lambda)$  by showing that its so-called *Mittag-Leffler series expansion* (see, for example, [12, Chapter 8]) is simply related to a Laplace transform. We then prove that condition (2.5) holds, implying the validity of interpolation formula (2.4).

Self-adjoint operators associated with a classical limit-circle non-oscillatory Sturm–Liouville problem on a finite interval, with separated boundary conditions, have purely discrete spectrum comprising simple eigenvalues. One way to observe this is to use the Niessen–Zettl transformation [23] of such a problem to a regular problem on the same interval, then recall

that spectra of regular Sturm–Liouville problems comprise simple eigenvalues (see, for example, [11]). This holds under any choice of separated boundary conditions, whence we see that  $m_{h,H}$  has, as its only singular behaviour, simple poles at the eigenvalues  $\lambda_n$  of (2.1) and (2.2) with the further boundary condition

$$u'(1; \lambda) = hu(1; \lambda), \quad (2.6)$$

since these are where the denominator of  $m_{h,H}(\lambda)$  vanishes.

In Lemma B.1, we show that in the classical limit-circle non-oscillatory case, enumerating the eigenvalues as  $\lambda_n$ ,  $n = 1, 2, 3, \dots$ , we have

$$\begin{aligned} \lambda_n &= (n + 1/4)^2 \pi^2 + O(1), \\ \sqrt{\lambda_n} &= (n + 1/4)\pi + O(1/n) \quad (n \rightarrow \infty). \end{aligned} \quad (2.7)$$

For each  $n$ , the eigenfunction  $\varphi_n$  corresponding to  $\lambda_n$  is defined by

$$\varphi_n = \varphi(\cdot; \lambda_n),$$

where  $\varphi$  solves (2.1) with initial conditions  $\varphi(1; \lambda) = 1$ ,  $\varphi'(1; \lambda) = h$ . Suppose that  $\psi$  is the linearly independent solution with  $\psi(1; \lambda) = 1$ ,  $\psi'(1; \lambda) = H$ , and

$$\begin{aligned} \Phi(\lambda) &:= \varphi(0^+; \lambda), \\ \Psi(\lambda) &:= \psi(0^+; \lambda). \end{aligned}$$

Then, by checking that  $f(x; \lambda) := \psi(x; \lambda) + m_{h,H}(\lambda)\varphi(x; \lambda)$  satisfies the ‘boundary condition’ in (2.3), it follows that

$$m_{h,H}(\lambda) = -\frac{\Psi(\lambda)}{\Phi(\lambda)}. \quad (2.8)$$

Therefore,  $\Phi(\lambda_n)$  being 0 implies via integration by parts that

$$\begin{aligned} H - h &= f'(1; \lambda)\varphi_n(1) - f(1; \lambda)\varphi_n'(1) \\ &= (\lambda - \lambda_n) \int_0^1 f(\cdot; \lambda)\varphi_n \\ &= (\lambda - \lambda_n) \int_0^1 \psi(\cdot; \lambda)\varphi_n - (\lambda - \lambda_n) \frac{\Psi(\lambda)}{\Phi(\lambda) - \Phi(\lambda_n)} \int_0^1 \varphi(\cdot; \lambda)\varphi_n \\ &\rightarrow -\frac{\Psi(\lambda_n)}{\Phi'(\lambda_n)} \int_0^1 \varphi_n^2 \quad \text{as } \lambda \rightarrow \lambda_n. \end{aligned} \quad (2.9)$$

If we denote the norming constants associated with  $\lambda_n$  by  $\alpha_n := \int_0^1 \varphi_n^2$ , then we see from (2.8) and (2.9) that the residue of the  $m$ -function at its poles is given by

$$\text{Res}(m_{h,H}; \lambda_n) = \frac{H - h}{\alpha_n}. \quad (2.10)$$

Furthermore, in Lemma B.2 we prove that

$$\alpha_n = 1/2 + O(1/n). \quad (2.11)$$

The asymptotics (2.11) and (2.7) immediately imply that  $\sum_{n=1}^{\infty} 1/\alpha_n(\lambda - \lambda_n)$  is convergent, uniformly for  $\lambda$  in any compact set bounded away from  $\{\lambda_n\}_{n=1}^{\infty}$ . Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)} \rightarrow 0 \quad (\text{Im}(\lambda) \rightarrow \infty). \quad (2.12)$$

This will ultimately turn out to be the Mittag-Leffler series we seek, but we need to link this result to the  $m$ -function. We can achieve this via Nevanlinna-type properties of  $m_{h,H}$ . For completeness, we briefly repeat here the following well-known calculation, showing that  $m_{h,H}$

is (anti-)Nevanlinna. Observe that, for any solution  $u$  of (2.1) and (2.2), we have

$$\begin{aligned} (\lambda - \bar{\lambda}) \int_0^1 u(\cdot; \lambda) u(\cdot; \bar{\lambda}) &= u'(0; \lambda) u(0; \bar{\lambda}) - u(0; \lambda) u'(0; \bar{\lambda}) \\ &= (h - H)(m_{h,H}(\lambda) - m_{h,H}(\bar{\lambda})); \end{aligned}$$

subtracting the complex conjugate of this whilst noting  $u(\cdot; \bar{\lambda}) = \overline{u(\cdot; \lambda)}$  shows that

$$\int_0^1 |u(\cdot; \lambda)|^2 = (h - H) \frac{\text{Im}(m_{h,H}(\lambda))}{\text{Im}(\lambda)}.$$

Hence, if  $h > H$ , then  $m_{h,H}$  is in the Nevanlinna class of functions that map the upper and lower half-planes to themselves, whilst if  $h < H$ , then  $m_{h,H}$  is the negative of such a function, known as anti-Nevanlinna.

It is known that all (anti-)Nevanlinna functions have a Stieltjes integral representation; here

$$m_{h,H}(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\rho(t),$$

where  $\rho$  is the spectral measure associated with the problem (2.1), and

$$A = \text{Re}(m_{h,H}(i)), \quad B = \lim_{\tau \rightarrow +\infty} \frac{m_{h,H}(i\tau)}{i\tau}.$$

Note that  $\rho$  is increasing if and only if  $h > H$ . Furthermore, as a measure it assigns ‘mass’ only at points in the spectrum of the Sturm–Liouville operator associated with (2.1), that is, for any  $d\rho$ -integrable  $g$ ,

$$\int_{\mathbb{R}} g(t) d\rho(t) = \sum_{n=1}^{\infty} \gamma_n g(\lambda_n),$$

$\gamma_n$  being the mass at  $\lambda_n$ . Thus

$$m_{h,H}(\lambda) = A + B\lambda + \sum_{n=1}^{\infty} \gamma_n \left( \frac{1}{\lambda_n - \lambda} - \frac{\lambda_n}{1 + \lambda_n^2} \right).$$

Integrating anti-clockwise along a sufficiently small, simple, closed contour around  $\lambda_n$  and comparing with (2.10) shows that  $\gamma_n = -\text{Res}(m_{h,H}; \lambda_n) = (h - H)/\alpha_n$ . Hence, we may split up the sum and write

$$m_{h,H}(\lambda) = \tilde{A} + B\lambda + \sum_{n=1}^{\infty} \frac{h - H}{\alpha_n(\lambda_n - \lambda)}.$$

To proceed, we need large- $\text{Im}(\lambda)$  asymptotics of  $m_{h,H}(\lambda)$ . Expressing  $m_{h,H}$  in terms of the Neumann  $m$ -function  $m_N(\lambda) := u(1; \lambda)/u'(1; \lambda)$  and using Lemma B.3, we see

$$m_{h,H}(\lambda) = \frac{1 - Hm_N(\lambda)}{1 - hm_N(\lambda)} \sim \frac{1 - H/C\sqrt{\lambda}}{1 - h/C\sqrt{\lambda}} \longrightarrow 1 \quad (\text{Im}(\lambda) \longrightarrow +\infty).$$

From this and (2.12), we see that  $\tilde{A} = 1$  and  $B = 0$ . Thus we have proved the following lemma.

LEMMA 2.1. *Uniformly for  $\lambda$  in any compact set that is non-intersecting with  $\{\lambda_n\}_{n=0}^{\infty}$ , we have a Mittag-Leffler series representation for the Robin  $m$ -function given by*

$$m_{h,H}(\lambda) - 1 = \sum_{n=1}^{\infty} \frac{h - H}{\alpha_n(\lambda_n - \lambda)}. \tag{2.13}$$

REMARK 2.1. Our calculations proving this result are adapted from parts of a calculation in [20, Chapter 3] for a regular Sturm–Liouville problem in normal form.

Lemma 2.1 gives us enough to deduce a Laplace transform representation of  $m_{h,H}$ , and hence our interpolation result. For the reader's convenience we state the theorem in full.

**THEOREM 2.1** (Classical limit-circle  $m$ -function interpolation). *Under the hypothesis that  $Q \in L^2(0, b)$  is real-valued, the Robin  $m$ -function*

$$m_{h,H}(\lambda) = \frac{u'(1; \lambda) - Hu(1; \lambda)}{u'(1; \lambda) - hu(1; \lambda)},$$

for any square-integrable solution  $u$  of the limit-circle non-oscillatory problem

$$\begin{cases} -u''(x; \lambda) + \left(Q(x) - \frac{1}{4x^2}\right)u(x; \lambda) = \lambda u(x; \lambda) & (x \in (0, 1)), \\ [U_p, u](0^+; \lambda) = 0, \end{cases} \tag{2.14}$$

satisfies the interpolation formula

$$m_{h,H}(\lambda) - 1 = \sum_{n=0}^{\infty} c_n(1/2 - \beta - i\sqrt{\lambda}) \sum_{k=0}^n a_{nk} \{m(-(k + \beta)^2) - 1\}.$$

Here  $\beta > 0$  is fixed,

$$\begin{aligned} c_n(z) &:= (2n + 1) \frac{(1/2 - z)_n}{(1/2 + z)_{n+1}} \quad (\text{for a.e. } z \in \mathbb{C}), \\ (z)_n &:= z(z + 1) \cdots (z + n - 1) \quad (z \in \mathbb{C}), \\ a_{nk} &:= \frac{(-n)_k (n + 1)_k}{(k!)^2} \quad (n, k \geq 0), \end{aligned}$$

and the convergence of the series is uniform in any compact subset of  $\text{Im}(\sqrt{\lambda}) > 1/2 + \beta$ .

The proof uses Lebesgue's dominated convergence theorem. We need the following lemma.

**LEMMA 2.2.** *Let  $\varepsilon > 0$  and define  $\rho_n = \sqrt{\lambda_n}$ . Then  $g_N(t) := e^{-\varepsilon t} \sum_{n=1}^N \sin(\rho_n t) / \alpha_n \rho_n$  is uniformly bounded, in  $t \in (0, \infty)$  and  $N \in \mathbb{N}$ , by a fixed integrable function.*

*Proof.* First note that the asymptotic expansion (2.7) may be written as  $\rho_n = (n + 1/4)\pi + \varepsilon_n$ , where  $\varepsilon_n = O(1/n)$ . Then, for each fixed  $t \geq 0$ ,

$$\sin(\rho_n t) = \sin((n + 1/4)\pi t) \cos(\varepsilon_n t) + \cos((n + 1/4)\pi t) \sin(\varepsilon_n t). \tag{2.15}$$

Write  $\varepsilon = 2\sigma$ . It would be enough to find an  $L^1(0, 2\pi)$  function that bounds, uniformly in  $N$ , the expression

$$s_N(t) := e^{-\sigma t} \sum_{n=1}^N \frac{\sin(\rho_n t)}{\alpha_n \rho_n} \quad (t \in (0, 2\pi)),$$

so that  $g_N(t) = e^{-\sigma t} s_N(t)$  ( $t \in (0, \infty)$ ) is dominated by an  $L^1(0, \infty)$  function, owing to the exponential decay of  $e^{-\sigma t}$ . So, note that

$$\text{if } t \in [0, \sigma^{-1} \log(n)], \text{ then } |\varepsilon_n t| = O(\log(n)/n),$$

and

$$\text{if } t \geq \sigma^{-1} \log(n), \text{ then } e^{-\sigma t} \leq e^{-\sigma \sigma^{-1} \log(n)} = 1/n,$$

so that  $e^{-\sigma t} \sin(\varepsilon_n t) = O(1/\sqrt{n})$ ; by a similar argument  $e^{-\sigma t} (\cos(\varepsilon_n t) - 1) = O(1/n)$ . Both estimates are uniform in  $t \geq 0$ . With (2.15), these are enough to ensure a constant bound for

$$s_N(t) - e^{-\sigma t} \sum_{n=1}^N \frac{\sin((n + 1/4)\pi t)}{\alpha_n \rho_n} \quad (t \in (0, 2\pi)).$$

Hence, substituting the asymptotic expansions (2.7) and (2.11) into the second sum in the above expression means the following: if we can show that both  $\sum_{n=1}^N \cos(nx)/n$  and  $\sum_{n=1}^N \sin(nx)/n$  ( $x \in (0, 2\pi)$ ) are bounded, uniformly in  $N$ , by some fixed element of  $L^1(0, 2\pi)$ , then it will follow that so is  $s_N(t)$  ( $t \in (0, 2\pi)$ ), proving the lemma.

We will prove the uniform  $L^1(0, 2\pi)$  bound for the cos-series; the same approach produces a similar bound for the sin-series. Denote by  $c_N(x)$  the partial sum  $\sum_{n=1}^N \cos(nx)/n$  and note

$$c'_N(x) = - \sum_{n=1}^N \sin(nx) = \frac{\cos((N + 1/2)x) - \cos(x/2)}{2 \sin(x/2)}.$$

Thus  $c'_N(x)$  is bounded by  $1/\sin(x/2)$ . Noting  $|c_N(\pi)| \leq 1$ , we see  $|c_N(x)| \leq 1 + \int_{\pi}^x |c'_N| \leq 1 + 2 \log |\cot(x/4)|$ , which is certainly integrable over  $(0, 2\pi)$  since to leading-order it is  $-\log(x)$  for  $x$  near 0 and  $-\log(2\pi - x)$  near  $2\pi$ . □

*Proof of Theorem 2.1.* We first observe that the Mittag-Leffler series (2.13) may be written as

$$\begin{aligned} m_{h,H}(-\kappa^2) - 1 &= \sum_{n=1}^{\infty} \frac{h - H}{\alpha_n(\rho_n^2 + \kappa^2)} \\ &= (h - H) \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\kappa t} \frac{\sin(\rho_n t)}{\alpha_n \rho_n} dt \quad (\operatorname{Re}(\kappa) > 0). \end{aligned} \tag{2.16}$$

Assuming that integration and summation may be interchanged (we show this below), we see that  $m_{h,H}(-\kappa^2) - 1/(h - H)$  is the Laplace transform  $\mathcal{L}[f](\kappa)$  of the series

$$f(t) := \sum_{n=1}^{\infty} \frac{\sin(\rho_n t)}{\alpha_n \rho_n} \quad (t \geq 0). \tag{2.17}$$

We now prove the convergence of (2.17) and justify the interchange of summation and integration in (2.16).

From (2.15), we have  $\sin(\rho_n t) = \cos(\pi t/4) \sin(n\pi t) + \sin(\pi t/4) \cos(n\pi t) + O(1/n)$ . Hence, by (2.7) and (2.11), the pointwise convergence of (2.17) is determined by that of  $\sum_{j=1}^{\infty} e^{ijx}/j$ . But this is simply the Fourier series for the  $2\pi$ -periodic extension of the expression  $-\log |2 \sin(x/2)| + i(\pi - x)/2$  ( $x \in (-\pi, \pi)$ ) so the pointwise convergence of (2.17) is immediate.

We may now simply apply Lemma 2.2 to see that  $g_N(t) := e^{-\operatorname{Re}(\kappa)t} \sum_{n=1}^N \sin(\rho_n t)/\alpha_n \rho_n$  is dominated by an integrable function. Dominated convergence follows, and hence we may write

$$m_{h,H}(-\kappa^2) - 1 = (h - H) \int_0^{\infty} e^{-\kappa t} f(t) dt \quad (\operatorname{Re}(\kappa) < 0).$$

All that remains is to check condition (2.5). But this is obvious, since, by dominated convergence,  $e^{-\delta t} |f(t)|$  is integrable for every  $\delta > 0$ . Therefore, by application of the interpolation result (2.4) to  $F(\kappa) = m_{h,H}(-\kappa^2) - 1$ , the theorem follows, with uniform convergence in any compact subset of the parabolic  $\lambda$ -region  $\operatorname{Im}(\sqrt{\lambda}) > 1/2 + \beta$ . □

### 3. Uniqueness theorems for the inverse problem

The main result of this paper is a pair of uniqueness theorems for Inverse Problem 1.1. We will state and prove these here, by means of Theorem A.1 and our interpolation result in Theorem 2.1. The uniqueness theorems are kept separate due to certain technical conditions in both being similar in representation, but fundamentally different in structure.

**THEOREM 3.1** (Uniqueness in the PLP case). *Fix  $\nu \geq 2$ ,  $c > 0$  and  $\alpha > \nu/2 - 1 \geq 0$ , and let  $w, q \in L_{\text{loc}}^\infty(0, 1]$ , with  $q$  real-valued and  $w \geq c$  almost everywhere (a.e.). Suppose that  $w, w' \in AC(0, 1]$  with  $w', w'' \in L_{\text{loc}}^\infty(0, 1]$ . Suppose also that, as  $r \rightarrow 0$ ,*

- (i)  $w(r) = (1/r^\nu)(1 + O(r^\alpha))$ ;
- (ii)  $q(r) = w(r)O(r^\alpha)$ ;
- (iii)  $(w(r)r^\nu)' = O(r^{-\nu/2})$ , and  $(w(r)r^\nu)'' = O(r^{-\nu})$ .

*If  $w$  is known, then the interpolation sequence  $((-n^2, m_n))_{n=1}^\infty$ , of values (in the graph) of the PLP Dirichlet  $m$ -function (1.5) for (1.3), uniquely determines the potential  $q$ .*

*Proof.* We perform a Liouville–Green transformation:

$$t(r) = \int_r^1 \sqrt{w},$$

$$z(t(r)) = r^{1/2}w(r)^{1/4}u(r) \quad (r \in (0, 1)).$$

This leads to the corresponding solution space  $L^2(0, \infty; r(t)^\nu dt)$  in which we seek  $z(\cdot; \lambda)$ ; further, over this space, the transformed equation is in PLP at  $\infty$  (or, in the case  $\nu = 2$ , has the PLP/PLC behaviour outlined in Appendix C, to which the reader is directed for details). That the domain in which  $t$  lies is  $(0, \infty)$  follows from the fact that, as  $r \rightarrow 0$ ,  $t(r) \sim \int_r^1 s^{-\nu/2} ds \rightarrow \infty$ .) The equation satisfied by  $z$  is

$$-z''(t; \lambda) + Q(t)z(t; \lambda) = \lambda z(t; \lambda) \quad (t \in (0, \infty)), \quad (3.1)$$

where

$$Q(t(r)) := \frac{q(r)}{w(r)} - r^{-1/2}w(r)^{-3/4} \frac{d}{dr} \left\{ r \frac{d}{dr} (r^{-1/2}w(r)^{-1/4}) \right\}$$

$$= - \left( \frac{\nu - 2}{4} \right)^2 r^{\nu-2} (1 + \zeta(r)) + \varepsilon_2(r) \quad (r \in (0, 1)),$$

and

$$\varepsilon_1(r) = w(r)r^\nu - 1, \quad (3.2)$$

$$\varepsilon_2(r) = \frac{q(r)}{w(r)}, \quad (3.3)$$

$$\zeta(r) = - \frac{\varepsilon_1(r)}{1 + \varepsilon_1(r)} - \left( \frac{2}{\nu - 2} \right)^2 \frac{r^2 \varepsilon_1''(r)}{(1 + \varepsilon_1(r))^2}$$

$$+ \frac{5}{(\nu - 2)^2} \frac{r^2 \varepsilon_1'(r)^2}{(1 + \varepsilon_1(r))^3} - \frac{2\nu}{(\nu - 2)^2} \frac{r \varepsilon_1'(r)}{(1 + \varepsilon_1(r))^2}. \quad (3.4)$$

We now want to apply Theorem A.1 to the  $m$ -function of equation (3.1); for this we need  $\int_x^{x+1} |Q|$  to be a bounded expression in  $x \in (0, \infty)$ , that is,  $Q \in l^\infty(L^1)(0, \infty)$ . It would suffice that  $Q \in L^\infty(0, \infty)$ . Note that

$$\int_x^{x+1} |Q(t)| dt = \int_x^{x+1} \left| \left( \frac{\nu - 2}{4} \right)^2 r(t)^{\nu-2} (1 + \zeta(r(t))) - \varepsilon_2(r(t)) \right| dt. \quad (3.5)$$

By applying the hypotheses (i) and (iii) to (3.2), we easily observe that

$$\varepsilon_1'(r) = O(r^{-\nu/2}) \subset O(r^{1-\nu}) \quad \text{and} \quad \varepsilon_1''(r) = O(r^{-\nu}) \quad (r \rightarrow 0).$$

Thus  $\zeta(r) \in L_{\text{loc}}^\infty(0, 1]$  and is  $O(r^{2-\nu})$  as  $r \rightarrow 0$ . Further,  $w, q \in L_{\text{loc}}^\infty(0, 1]$  implies that  $\varepsilon_2(r)$  is bounded. Therefore  $Q \in L^\infty(0, \infty) \subset l^\infty(L^1)(0, \infty)$ , so (3.1) is in classical limit-point at  $\infty$ . Formally the classical limit-point and PLP  $m$ -functions of (3.1), respectively, over the spaces

$L^2(0, \infty)$  and  $L^2(0, \infty; r(t)^\nu dt)$ , have the same expression. Since the integral hypothesis of Theorem A.1 is satisfied,  $m(\cdot)$  can be interpolated from its values at the points  $(-n^2)_{n=1}^\infty$ .

In particular, given any non-real ray through the origin and the sequence of interpolation pairs

$$((-n^2, m_n))_{n=1}^\infty, \tag{3.6}$$

for any  $\lambda$  on this ray we can calculate the value of  $m(\lambda)$ . Choosing any such ray in the first quadrant and applying Corollary A.1, we have immediately that  $Q$  is uniquely determined by the sequence (3.6), and by the reverse transformation it follows that  $q$  is as well.  $\square$

**THEOREM 3.2 (Uniqueness in the PLC case).** *Let  $0 \leq \nu < 2$ ,  $c > 0$  and  $\alpha > 3/2 - 3\nu/4 > 0$ , and fix  $w, q \in L^\infty_{\text{loc}}(0, 1]$ , with  $q$  real-valued and  $w \geq c$  almost everywhere (a.e.). Suppose that  $w, w' \in AC(0, 1]$  with  $w', w'' \in L^\infty_{\text{loc}}(0, 1]$ , and that, as  $r \rightarrow 0$ ,*

- (i)  $w(r) = (1/r^\nu)(1 + O(r^\alpha))$ ;
- (ii)  $q(r) = w(r)O(r^{\alpha-2})$ ;
- (iii)  $(w(r)r^\nu)' = O(r^{\alpha-1})$ , and  $(w(r)r^\nu)'' = O(r^{\alpha-2})$ .

*If  $w$  is known, then the interpolation sequence  $((-n^2, m_n))_{n=1}^\infty$ , of values (in the graph) of the PLC Dirichlet  $m$ -function (1.5) for (1.3) with boundary condition (1.4), uniquely determines the potential  $q$ .*

*Proof.* First note that, under these assumptions,  $\sqrt{w}$  is integrable. All asymptotic estimates are as  $r$  or  $t \rightarrow 0$ . We use a different transformation from that in the proof of Theorem 3.1, namely

$$\begin{aligned} t(r) &= \int_0^r \frac{\sqrt{w}}{\|\sqrt{w}\|_{L^1(0,1)}}, \\ z(t(r)) &= r^{1/2}w(r)^{1/4}u(r) \quad (r \in (0, 1)). \end{aligned}$$

This gives rise to

$$-z''(t; \lambda) + \tilde{Q}(t)z(t; \lambda) = \lambda z(t; \lambda) \quad (t \in (0, 1)), \tag{3.7}$$

where this time

$$\tilde{Q}(t(r)) = -\left(\frac{2-\nu}{4}\right)^2 r^{-(2-\nu)}(1 + \zeta(r)) + \varepsilon_2(r),$$

with  $\varepsilon_2$  and  $\zeta$  defined as in (3.3) and (3.4). Note  $\varepsilon_2(r) = O(r^\alpha)$ , and that

$$-\left(\frac{2-\nu}{4}\right)^2 r(t)^{-(2-\nu)} = -\frac{1}{4t^2},$$

whilst

$$t(r) = \frac{2}{(2-\nu)\|\sqrt{w}\|_{L^1(0,1)}} r^{1-\nu/2}(1 + O(r^\alpha)).$$

Our aim is to apply Theorem 2.1, for which we need  $Q(t) := \tilde{Q}(t) + 1/4t^2 \in L^2(0, 1)$ . Recalling (3.2), we use condition (iii) to observe  $\varepsilon_1'(r) = O(r^{\alpha-1})$  and  $\varepsilon_1''(r) = O(r^{\alpha-2})$ . Thus, by (3.4),  $\zeta(r) = O(r^\alpha)$ . Since  $f(t) \in L^2(0, 1)$  if and only if  $f(t(r)) \in L^2(0, 1; \sqrt{w(r)} dr)$  (easily checked), and  $2(\alpha - 2 + \nu) - \nu/2 > -1$ , we see  $Q \in L^2(0, 1)$ , as required.

Hence, by Theorem 2.1 the Robin  $m$ -function (and by a fractional linear transformation, any  $m$ -function) is uniquely determined by the sequence (3.6). Corollary A.1 concludes the proof.  $\square$

COROLLARY 3.1. Any finite number of values  $m(-n^2)$  in the interpolation sequence may be discarded, yet the  $m$ -function, and hence the potential, will still be uniquely determined.

*Proof.* Since, in (2.4), the parameter  $\beta > 0$  may be chosen freely, one may choose  $\beta$  to be any positive integer. The resulting interpolation formula does not require the values  $F(1), \dots, F(\beta - 1)$ , and so the values  $m(-1), \dots, m(-(\beta - 1)^2)$  are not needed.  $\square$

#### 4. The Berry–Dennis problem

We now explain the claim made in the introduction, namely, that uniqueness for a physically inspired inverse problem is achieved as a corollary of the above result on pencils. The setup is that of [7, 22]. Consider the two-dimensional Schrödinger equation with spherically symmetric potential  $q \in L^1_{\text{loc}}(0, 1]$ ,

$$-\Delta U(\mathbf{x}) + q(|\mathbf{x}|)U(\mathbf{x}) = 0 \quad (\mathbf{x} \in \Omega), \quad (4.1)$$

where  $\Omega$  is the semi-circular region  $\{\mathbf{x} = (\chi, \eta) \in \mathbb{R}^2 \mid \chi^2 + \eta^2 \leq 1, \chi \geq 0\}$ . Let

$$\Gamma := \{(\chi, \eta) \in \mathbb{R}^2 \mid \chi^2 + \eta^2 = 1, \chi > 0\} \subset \partial\Omega,$$

and take  $0 < \varepsilon < 1$ ,  $g \in H^{1/2}(\Gamma)$ . Write  $\mathbf{x} = (x, y)$  and assign to the differential equation (4.1) the boundary conditions

$$U(\mathbf{x}) = g \quad (\mathbf{x} \in \Gamma), \quad (4.2)$$

$$U(\mathbf{x}) + \varepsilon y \frac{\partial U}{\partial \nu}(\mathbf{x}) = 0 \quad (\mathbf{x} \in \partial\Omega \setminus \Gamma), \quad (4.3)$$

where  $\partial/\partial\nu$  is the outward-pointing normal derivative on  $\partial\Omega$ . These considerations define an operator  $\mathcal{L}$  over  $L^2(\Omega)$ , taking values  $\mathcal{L}U = (-\Delta + q)U$  and having domain  $D(\mathcal{L}) := \{U \in L^2(\Omega) \mid \Delta U \in L^2(\Omega); (4.2), (4.3) \text{ hold}\}$ .

In polar coordinates  $\mathbf{x} = (r, \theta)$ , the action of  $\mathcal{L}$  is that of

$$-\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + q(r).$$

Since on  $\partial\Omega \setminus \Gamma$  the normal derivative is given by  $-\partial/\partial x = \pm r^{-1} \partial/\partial \theta$  ( $\theta = \pm\pi/2$ ), we find that (4.3) becomes

$$U(\mathbf{x}) + \varepsilon \frac{\partial U}{\partial \theta}(\mathbf{x}) = 0 \quad \left(r \in (0, 1), \theta = \pm \frac{\pi}{2}\right).$$

Hence, after performing the separation of variables  $U(r, \theta) = u(r)\Theta(\theta)$ , we arrive at the angular eigenvalue problem

$$\begin{aligned} -\Theta'' &= \lambda\Theta \quad \text{on } (-\pi/2, \pi/2) \\ \Theta(-\pi/2) + \varepsilon\Theta'(-\pi/2) &= 0 = \Theta(\pi/2) + \varepsilon\Theta'(\pi/2), \end{aligned}$$

which, it is easily calculated, has eigenvalues and eigenfunctions

$$\begin{aligned} \lambda_0 &= -\frac{1}{\varepsilon^2}, \quad \lambda_n = n^2 \quad (n \in \mathbb{N}); \\ \Theta_n(\theta) &= \begin{cases} e^{-\theta/\varepsilon} & (n = 0), \\ \cos(n\theta) - (n\varepsilon)^{-1} \sin(n\theta) & (n \text{ even}), \\ \cos(n\theta) + n\varepsilon \sin(n\theta) & (n \text{ odd}). \end{cases} \end{aligned}$$

Feeding this information back into the problem, one can find (as remarked in [22]) that  $\mathcal{L}$  is isometrically equal to the orthogonal direct sum of the ordinary differential operators

$\mathcal{L}_n$  ( $n = 0, 1, 2, \dots$ ) given by

$$\mathcal{L}_n u(r) := -\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) + q(r)u(r) + \frac{\lambda_n}{r^2} u(r),$$

and equipped with the domains

$$D(\mathcal{L}_n) := \{u \in L_r^2(0, 1) \mid \mathcal{L}_n u \in L_r^2(0, 1), u(1) = 0\}.$$

We are concerned with an associated inverse problem. Consider the generalization  $L_\lambda$  ( $\lambda \in \mathbb{C}$ ) of  $\mathcal{L}_n$ :

$$L_\lambda u(r) := -\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) + q(r)u(r) - \frac{\lambda}{r^2} u(r),$$

with domain

$$D(L_\lambda) := \{u \in L_r^2(0, 1) \mid L_\lambda u \in L_r^2(0, 1), u(1) = 0\}.$$

The differential equation  $L_\lambda u = 0$  is precisely (1.3) with  $w(r) = 1/r^2$ , and hence displays the PLP/PLC behaviour outlined in Lemma C.1 and Figure C.1. We define the Dirichlet  $m$ -function  $m(\lambda)$  as in (1.5).

Now recall, from the theory of inverse problems in PDEs, the Dirichlet-to-Neumann operator  $\Lambda_\Gamma : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ . This maps Dirichlet data  $U|_\Gamma$  to Neumann data  $\partial U/\partial \nu|_\Gamma$  for any solution  $U \in H^1(\Omega)$  of (4.1) and (4.3). We may write any such solution using the generalized Fourier basis  $(u_n(r)\Theta_n(\theta))_{n=0}^\infty$ :

$$U(r, \theta) = \sum_{n=0}^\infty u_n(r)\Theta(\theta). \tag{4.4}$$

By differentiating, it follows that, in this basis,  $\Lambda_\Gamma$  takes the form of the diagonal matrix  $\text{diag}(m(-\lambda_0), m(-\lambda_1), m(-\lambda_2), \dots)$ .

**INVERSE PROBLEM 4.1.** Given an admissible Dirichlet-to-Neumann map  $\Lambda_\Gamma$  for

$$\begin{aligned} (-\Delta + q(|\mathbf{x}|))U(\mathbf{x}) &= 0 \quad (\mathbf{x} \in \Omega), \\ U(\mathbf{x}) + \varepsilon y \frac{\partial U}{\partial \nu} &= 0 \quad (\mathbf{x} \in \partial\Omega \setminus \Gamma), \end{aligned}$$

recover the radially symmetric potential  $q$ .

Uniqueness for this inverse problem is immediate from Theorem 3.1, under the conditions  $q \in L_{\text{loc}}^\infty(0, 1]$  and  $q(r) = O(r^{\alpha-2})$  ( $r \rightarrow 0$ ) for some fixed  $\alpha > 0$ . The uniqueness follows since, for positive  $n$ , the restrictions on  $q$  make the type (1.3) pencil, associated with each operator  $\mathcal{L}_n$ , be in PLP at 0 (see [22]), whilst the sequences  $-\lambda_n = -n^2$  and  $m(-n^2)$  form the interpolation sequence required in Theorem 3.1. Thus we have proved the following theorem.

**THEOREM 4.1** (Uniqueness for the Berry–Dennis inverse problem). *Any given Dirichlet-to-Neumann map  $\Lambda_\Gamma$  for the system (4.1) and (4.3) may have arisen from at most one radially symmetric potential  $q \in L_{\text{loc}}^\infty(0, 1] \cap O(r^{\alpha-1}; r \rightarrow 0)$ .*

The 0th term  $(1/\varepsilon^2, m(1/\varepsilon^2))$  is superfluous for our needs. However, we can go farther. Following Corollary 3.1, we may discard arbitrarily many of the diagonal terms of  $\Lambda$  and still retain uniqueness of  $q$ .

**REMARK 4.1.** Theorem 4.1 is markedly different from existing results for inverse problems involving partial-boundary Dirichlet-to-Neumann measurements in two-dimensional domains. Such existing results, for example, [16–18] all deal with problems in which the portion of

the boundary where the measurements are not made,  $\partial\Omega \setminus \Gamma$ , has a homogeneous Dirichlet or Neumann condition assigned; the Berry–Dennis setup has a singular boundary condition here.

#### Appendix A. Limit-point interpolation and a Borg–Marčenko theorem

We collect here some useful theorems. The first is the interpolation result from [24] mentioned in Section 2 and applied in the proof of Theorem 3.1, whilst the second is a general Borg–Marčenko uniqueness result and a simple corollary, the latter being what we need in Section 3. We state the first in full to highlight its similarities with Theorem 2.1.

**THEOREM A.1** (Rybkin–Tuan; classical limit-point  $m$ -function interpolation). *Let  $Q$  be a real-valued function in  $l^\infty(L^1)(0, \infty)$ , that is,*

$$\|Q\| := \sup_{x \geq 0} \int_x^{x+1} |Q| < \infty.$$

*Suppose that  $m$  is the Weyl–Titchmarsh  $m$ -function associated with the limit-point Schrödinger operator*

$$S := -d^2/dx^2 + Q(x) \quad (x \in (0, \infty))$$

*on  $L^2(0, \infty)$ , that is,  $m(\lambda) := u'(0, \lambda)/u(0, \lambda)$  ( $\text{Im}\lambda > 0$ ) for any square-integrable solution  $u$  of  $Su(\cdot; \lambda) = \lambda u(\cdot; \lambda)$ . If  $\lambda$  is from the parabolic domain with  $(\text{Im}\lambda)^2 > 4\beta_0^2 \text{Re}\lambda + 4\beta_0^4$ , then*

$$m(\lambda) - i\sqrt{\lambda} = \sum_{n \geq 0} c_n(-i\sqrt{\lambda} - \beta_0 + 1) \sum_{k=0}^n a_{nk}(m(-\omega_k^2) + \omega_k),$$

*where  $\varepsilon > 0$  is a fixed parameter,  $\beta_0 := \max\{\sqrt{2\|Q\|}, e\|Q\|\} + \frac{1}{2} + \varepsilon$ ,*

$$c_n(z) := (2n+1) \frac{(1/2-z)_n}{(1/2+z)_{n+1}} \quad (\text{for a.e. } z \in \mathbb{C}),$$

$$(z)_n := z(z+1) \cdots (z+n-1) \quad (z \in \mathbb{C}),$$

$$a_{nk} := \frac{(-n)_k (n+1)_k}{(k!)^2} \quad (n, k \geq 0),$$

*and*

$$\omega_k := k + \beta_0 - \frac{1}{2}.$$

**REMARK A.1.** The parabolic domain  $(\text{Im}\lambda)^2 > 4\beta_0^2 \text{Re}\lambda + 4\beta_0^4$  may be written more succinctly as  $\text{Im}\sqrt{\lambda} > \beta_0$ , where  $\arg(\sqrt{\lambda}) \in [0, \pi)$ . Furthermore, it is concave, and its intersection with any non-real ray through the origin is an infinite complex interval.

**THEOREM A.2** (Simon, Gesztesy–Simon, Bennewitz; Borg–Marčenko-type uniqueness). *Let  $Q_j \in L^1_{\text{loc}}[0, b)$  ( $j = 1, 2$ ) be real-valued,  $b \in (0, \infty]$ , and  $m_j(\lambda)$  ( $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ,  $j = 1, 2$ ) be the Titchmarsh–Weyl  $m$ -functions associated, respectively, with the differential expressions*

$$-d^2/dx^2 + Q_j(x) \quad (x \in (0, \infty), j = 1, 2)$$

*(with self-adjoint boundary conditions at  $b$  if needed). In addition, let  $a \in (0, b)$ ,  $0 < \varepsilon < \pi/2$  and suppose that as  $\lambda \rightarrow \infty$  along the ray  $\arg(\lambda) = \pi - \varepsilon$ , we have*

$$|m_1(\lambda) - m_2(\lambda)| = O(\exp(-2\text{Im}(\sqrt{\lambda})a)).$$

*Then  $Q_1 = Q_2$  a.e. in  $[0, a]$ .*

Theorem A.2 was originally stated in a slightly weaker form (without the ray condition) by Simon in 1999 [25]; the above improvement was first published, with a shorter proof, by Gesztesy and Simon in 2000 [15]. An alternative, even shorter, limit-point proof was found by Bennewitz in 2001 [6]. All are generalizations of the original, much-celebrated uniqueness theorem proved separately in 1952 by Borg [8] and Marchenko [21]. As an immediate consequence we have the result we need in this paper.

COROLLARY A.1. *If  $m_1 = m_2$  in an infinite sub-interval of the ray  $\{r e^{i(\pi-\varepsilon)} \mid r \in (0, \infty)\}$  with fixed  $0 < \varepsilon < \pi/2$ , then  $Q_1 = Q_2$  a.e. in  $[0, b)$ .*

Appendix B. *Various asymptotics for a Bessel-type equation*

In this appendix, we collect some necessary results on the large- $n$  asymptotics of the eigenvalues and norming constants defined in Section 2, as well as a result on asymptotics of the  $m$ -function, needed in the same section.

The eigenvalues of the Bessel equation of zeroth order, with Dirichlet and Neumann boundary conditions at the left and right endpoints, respectively, of  $(0, 1)$ , are well-studied, and are algebraically equivalent to the positive zeros of the Bessel function  $J_1$ . This information is enough to determine the eigenvalues  $\lambda_n$  for the boundary value problem (2.1), (2.2) and (2.6), asymptotically to order  $1/n$ . We calculate these first for the unperturbed equation, and then use a result from [10] to move to the perturbed version.

LEMMA B.1. *Let  $Q \in L^2(0, 1)$ ,  $h \in \mathbb{R}$  and denote by  $U_p(\cdot; \lambda)$  the principal solution at 0 of*

$$-u''(x; \lambda) + \left\{ Q(x) - \frac{1}{4x^2} \right\} u(x; \lambda) = \lambda u(x; \lambda) \quad (x \in (0, 1)), \tag{B.1}$$

that is,  $U_p$  is non-trivial, and for all linearly independent solutions  $V$  we have  $U_p(0^+) = o(V(0^+))$ . When ordered by size and enumerated by  $n = 1, 2, 3, \dots$  the eigenvalues  $\lambda_n$  of the above differential equation with the boundary conditions

$$\begin{cases} [u, U_p](0^+; \lambda) = 0, \\ u'(1; \lambda) = hu(1; \lambda), \end{cases}$$

satisfy the asymptotics

$$\sqrt{\lambda_n} = (n + 1/4)\pi + O(1/n).$$

*Proof.* Suppose firstly that  $Q \equiv 0$ , and denote the corresponding eigenvalues by  $\lambda_n^0$ . The boundary condition at 0 allows us to choose any constant multiple of  $x^{1/2}J_0(\sqrt{\lambda}x)$  as our solution. The condition at 1 then forces the eigenvalues to be the positive zeros of

$$\sqrt{\lambda}J_1(\sqrt{\lambda}) + (h - 1/2)J_0(\sqrt{\lambda}).$$

Thus, for each fixed  $c$ , we seek asymptotics for the zeros of

$$f(z) := zJ_1(z) - cJ_0(z).$$

Recall that  $J_0$  and  $J_1$  have only simple positive zeros [1, Subsection 9.5], and note  $f(j_{0,n}) = j_{0,n}J_1(j_{0,n})$ , which alternates in sign as  $n$  is incremented because  $j_{0,n}$  interlace with  $j_{1,n}$ . The intermediate value theorem then gives a zero  $z_n \in (j_{0,n}, j_{0,n+1})$  for  $f$ , whilst the fact that  $J_0$  and  $J_1$  oscillate with asymptotically the same ‘period’ [1, Subsection 9.2] means  $z_n$  is unique. Since  $j_{0,n} = (n - 1/4)\pi + O(1/n)$  and  $j_{1,n} = (n + 1/4)\pi + O(1/n)$  [1, equation 9.5.12], the positive

zeros of  $f$  are

$$\begin{aligned} z_n &= (n + 1/4)\pi + \varepsilon_n \\ &\sim n, \end{aligned}$$

with the leading-order behaviour following from  $|\varepsilon_n| \leq \pi/2 + O(1/n)$ . We now use the asymptotic expansion [1, equation 9.2.1] of the first-order Bessel function  $J_\mu(x) = \sqrt{(2/\pi x)}(\cos(x - \mu\pi/2 - \pi/4) + O(1/x))$  ( $x \rightarrow +\infty$ ) to observe that

$$\begin{aligned} O(1/n) &\ni \frac{c}{\sqrt{z_n}} J_0(z_n) = \sqrt{z_n} J_1(z_n) \\ &= -\sqrt{\frac{2}{\pi}} (\cos(z_n + \pi/4) + O(1/n)). \end{aligned}$$

Taylor-expanding around the zeros of cosine implies  $\sqrt{\lambda_n^0} = z_n = (n + 1/4)\pi + O(1/n)$ . Finally, the second equation of [10, p. 17] is precisely  $\sqrt{\lambda_n} = \sqrt{\lambda_n^0} + O(1/n)$ .  $\square$

The next lemma provides a powerful asymptotic representation of the norming constants in Section 2. For its proof we will relate our notation to that of [10] and then utilize some results from the same paper.

LEMMA B.2. *Let  $Q \in L^2(0, 1)$ ,  $h \in \mathbb{R}$  and suppose that  $\varphi(\cdot; \lambda)$  solves (2.1) with initial conditions  $\varphi(1; \lambda) = 1, \varphi'(1; \lambda) = h$ . Then the norming constants  $\alpha_n := \int_0^1 \varphi(\cdot; \lambda_n)$  satisfy*

$$\alpha_n = 1/2 + O(1/n).$$

*Proof.* By checking the boundary conditions, one may easily see that  $\varphi_n = y_2(\cdot; \lambda_n)/y_2(1; \lambda_n)$ , where  $y_2(\cdot; \lambda)$  is the solution of the differential equation (B.1) satisfying the boundary condition  $t^{-1/2}y_2(t; \lambda) \rightarrow 1$  ( $t \rightarrow 0$ ). In the second-to-last equation of [10, p. 16] it is observed that, as  $\rho \rightarrow +\infty$ , we have

$$\int_0^1 y_2(\cdot; \rho^2)^2 = \frac{1}{\rho} \left[ \frac{1}{2} + O\left(\frac{\log(\rho)}{\rho}\right) \right]. \quad (\text{B.2})$$

Defining  $\rho_n = \sqrt{\lambda_n}$ , we see that the lemma would follow if  $y_2(1; \lambda_n)^{-2} = \rho_n(1 + O(1/n))$ . To justify this we appeal to [10, Lemma 3.2], which implies that

$$\left| y_2(1; \lambda_n) - \sqrt{\frac{\pi}{2}} J_0(\rho_n) \right| \leq \frac{C}{\sqrt{n}} (e^{I(n)} - 1), \quad (\text{B.3})$$

where (using Cauchy–Schwarz for the third line)

$$\begin{aligned} 0 \leq I(n) &:= \int_0^1 \frac{t}{1 + \rho_n t} (1 - \log(t)) |Q(t)| dt \\ &\leq \frac{1}{\rho_n} \int_0^1 (1 - \log(t)) |Q(t)| dt \\ &\leq \frac{1}{\rho_n} \left( \int_0^1 (1 - \log(t))^2 dt \right)^{1/2} \|Q\|_{L^2(0,1)} \\ &< \frac{3\|Q\|_{L^2(0,1)}}{\rho_n}. \end{aligned} \quad (\text{B.4})$$

Owing to (B.3), (B.4), and Lemma B.1, we find

$$y_2(1; \lambda_n) = \sqrt{\frac{\pi}{2}} J_0(\rho_n) + O(n^{-3/2}). \tag{B.5}$$

Lemma B.1 shows, furthermore, that  $\rho_n = j_{1,n} + O(1/n) = (n + 1/4)\pi + O(1/n)$  which, owing to  $J'_0 = -J_1$ , are asymptotically the local extrema of  $J_0$ . Hence, by expanding the cosine part of [1, equation (9.2.1)] in a first-order Taylor approximation around  $n\pi$ , it follows that

$$J_0(\rho_n) = \sqrt{\frac{2}{\pi\rho_n}} \left[ (-1)^n + O\left(\frac{1}{n}\right) \right].$$

Upon substitution into (B.5), this yields the desired result. □

The large-imaginary-part asymptotics of  $m$ -functions is also a well-studied topic. The result we use in Section 2 is an application of the very general Theorem 4.1 of [5] to (2.1); we state it as a lemma.

LEMMA B.3. *Let  $m_N$  be the Neumann  $m$ -function for (2.1), (2.2) with  $Q \in L^2(0, b)$ , that is,  $m_N(\lambda) := u(1; \lambda)/u'(1; \lambda)$  for a non-trivial solution  $u(\cdot; \lambda)$ . Then, as  $\lambda \rightarrow \infty$  along any non-real ray through 0,*

$$m_N(\lambda) = \frac{1}{i\sqrt{\lambda}}(1 + o(1)).$$

### Appendix C. PLP and PLC behaviour; dimension of solution space

We will analyse here the dimension of the solution space of (1.3) with  $w(r) \sim r^{-\nu}$  and  $\nu \geq 0$ . It will be helpful to treat the two cases  $\nu \geq 2$  and  $0 \leq \nu < 2$  separately, respectively, in Lemmas C.1 and C.3. The first analysis is by transforming the problem to Liouville normal form on the half-line and using known large- $x$  asymptotics of solutions. The second follows a different approach, using asymptotic analysis and variation of parameters to build recursion formulae that can be used to construct a pair of linearly independent solutions.

LEMMA C.1. *Suppose  $\nu \geq 2$  and  $\alpha > (\nu - 2)/2$ , and, furthermore, let  $q, w \in L^\infty_{\text{loc}}(0, 1]$  be real-valued with  $w > 0$  a.e., satisfying, as  $r \rightarrow 0$ ,*

- (i)  $w(r) = (1/r^\nu)(1 + O(r^\alpha))$ ;
- (ii)  $w(r)$  is a.e. bounded away from 0; and
- (iii)  $q(r) = w(r)O(r^\alpha)$ .

Then equation (1.3) is in

- (i) PLP at 0 when  $\nu > 2$  or  $\text{Im}\sqrt{\lambda} \geq 1$ , and in
- (ii) PLC at 0 when  $\nu = 2$  and  $1 > \text{Im}\sqrt{\lambda} > 0$ .

To prove this, we will use a result given by Eastham [13, Ex. 1.9.1], which, by providing asymptotic expressions for the solutions of equation (1.3), will give us the means to determine when any solution is in  $L^2(0, 1; r dr)$ . For convenience and completeness we state the form of this result, which provides the most generality when applied here.

LEMMA C.2 (Eastham; one-dim. Schrödinger equation solution asymptotics). *Let  $c$  be non-real and  $R \in L^2(a, \infty)$ . Then the differential equation*

$$-y'' + Ry = c^2y \quad \text{on } (a, \infty)$$

has solutions  $y_{\pm}$  asymptotically given, as  $x \rightarrow \infty$ , by

$$y_{\pm}(x) = \exp\left(\pm i \left\{ cx - \frac{1}{2c} \int_a^x R \right\}\right) (1 + o(1)).$$

With this in mind, we proceed with the proof.

*Proof of Lemma C.1.* We write

$$\begin{aligned} w(r) &= \frac{1}{r^{\nu}}(1 + \varepsilon_1(r)), \\ \frac{q(r)}{w(r)} &= \varepsilon_2(r) \quad (r \in (0, 1)), \end{aligned}$$

where

$$\varepsilon_j(r) = O(r^{\alpha}) \quad (r \rightarrow 0, \quad j = 1, 2). \quad (\text{C.1})$$

By performing a Liouville–Green-type transformation, with

$$\begin{aligned} t(r) &= \int_r^1 \rho^{-\nu/2} d\rho \quad (r \in (0, 1)), \\ z(t) &= r(t)^{(2-\nu)/4} u(r(t)) \quad (t \in (0, \infty)), \end{aligned} \quad (\text{C.2})$$

we arrive at the following equation, for  $t \in (0, \infty)$ ,

$$-z''(t; \lambda) + \underbrace{\left[ (\varepsilon_1 \varepsilon_2 - \lambda \varepsilon_1 + \varepsilon_2)(r) - \left( \frac{\nu-2}{4} \right)^2 r^{\nu-2} \right]}_{=: Q(t; \lambda)} z(t; \lambda) = \lambda z(t; \lambda). \quad (\text{C.3})$$

Note that if  $\nu > 2$ , then we have  $t(r) = ((\nu-2)/2)(r^{1-\nu/2} - 1)$ , whereas if  $\nu = 2$ , then  $t(r) = -\log(r)$ . For conciseness, we will treat both cases  $\nu = 2$  and  $\nu > 2$  at the same time, as the only difference between them arises near the end of the reasoning, and will be highlighted clearly.

We want to apply Lemma C.2 to equation (C.3), for which we need  $Q(\cdot; \lambda) \in L^2(0, \infty)$ . Since  $Q(\cdot; \lambda) \in L_{\text{loc}}^{\infty}[0, \infty)$ , the large- $t$  behaviour of  $Q(t; \lambda)$  determines its square-integrability. Hence,  $Q(\cdot; \lambda) \in L^2(0, \infty)$  if and only if

$$\infty > \int_0^{\infty} |(\varepsilon_1 \varepsilon_2 - \lambda \varepsilon_1 + \varepsilon_2)(r(t))|^2 dt = \int_0^1 r^{-\nu/2} |(\varepsilon_1 \varepsilon_2 - \lambda \varepsilon_1 + \varepsilon_2)(r)|^2 dr. \quad (\text{C.4})$$

But this holds automatically, due to (C.1). Thus we have a pair of solutions  $z_{\pm}(\cdot; \lambda)$  for equation (C.3) given, as  $t \rightarrow \infty$ , by

$$z_{\pm}(t; \lambda) = \exp\left(\pm i \left\{ \sqrt{\lambda} t - \frac{1}{2\sqrt{\lambda}} \int_0^t Q(\cdot; \lambda) \right\}\right) (1 + o(1)).$$

Now, the integral in the argument of this exponential is easily calculated to be

$$\int_{r(t)}^1 \left\{ (\varepsilon_1 \varepsilon_2 - \lambda \varepsilon_1 + \varepsilon_2)(\rho) - \left( \frac{\nu-2}{4} \right)^2 \rho^{\nu-2} \right\} \rho^{-\nu/2} d\rho.$$

By (C.1), the first part of this integral is convergent to a finite limit as  $t \rightarrow \infty$ . The second part is 0 if  $\nu = 2$  and convergent if  $\nu > 2$ , since  $\nu - 2 - \nu/2 > -1$ . Thus, in fact, we have the leading-order asymptotics

$$z_{\pm}(t; \lambda) \sim e^{\pm i \sqrt{\lambda} t}. \quad (\text{C.5})$$

For any solution  $u(\cdot; \lambda)$  of equation (1.3) and its corresponding transformed solution  $z(\cdot; \lambda)$  of (C.3), we have  $\int_0^1 r |u(r; \lambda)|^2 dr = \int_0^{\infty} r(t)^2 |z(t; \lambda)|^2 dt$ . But the leading-order asymptotics (C.5)

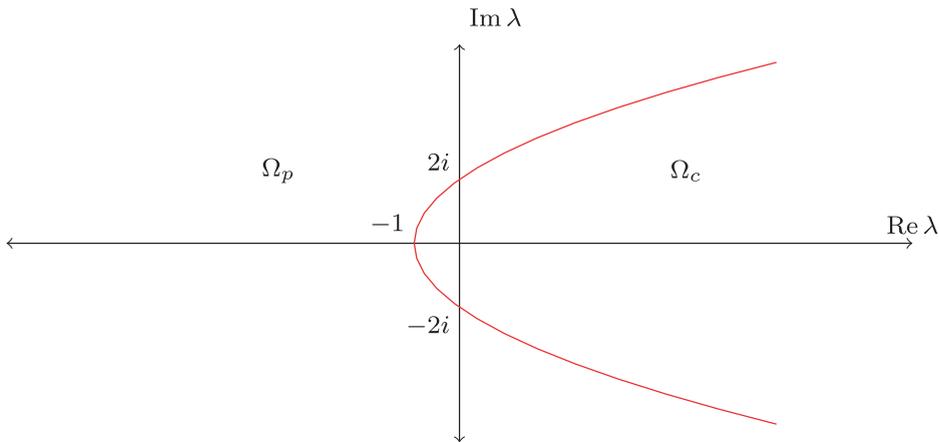


FIGURE C.1. A partition of the  $\lambda$ -plane for equation (1.3) with  $\nu = 2$ .

show that  $\int_0^\infty r(t)^2 |z_\pm(t; \lambda)|^2 dt < \infty$  if and only if

$$\infty > \int_0^\infty r(t)^2 |e^{\pm i\sqrt{\lambda}t}|^2 dt = \int_0^\infty r(t)^2 e^{\pm 2\text{Im}\sqrt{\lambda}t} dt. \tag{C.6}$$

When  $\nu > 2$ , the transformation (C.2) simplifies to  $r(t) = (1 - ((2 - \nu)/2)t)^{2/(2-\nu)}$ , which will not affect the exponential large- $t$  asymptotics of the integrand in (C.6). This implies that precisely one solution of equation (1.3) (up to scaling by a constant), namely,  $u_-(\cdot; \lambda)$ , is in  $L^2(0, 1; r dr)$ . In other words, for  $\nu > 2$ , (1.3) is in PLP at 0.

On the other hand, when  $\nu = 2$ , we find  $r(t)^2 = e^{-2t}$ , which when multiplied with the other exponential factor  $e^{\pm 2\text{Im}\sqrt{\lambda}t}$  in (C.6) means that  $\text{Im}\sqrt{\lambda} \geq 1$  makes (1.3) in PLP at 0, whilst if  $\text{Im}\sqrt{\lambda} < 1$ , the latter must be in PLC at 0.  $\square$

REMARK C.1. When  $\nu = 2$ , we may represent graphically the  $L^2(0, 1; r dr)$  nature of the solutions of (1.3); see Figure C.1. Here,  $\Omega_p := \{\lambda \in \mathbb{C} \mid \text{Im}\sqrt{\lambda} \geq 1\}$  and  $\Omega_c := \{\lambda \in \mathbb{C} \mid \text{Im}\sqrt{\lambda} < 1\}$ , so that if  $\lambda \in \Omega_p$  or  $\Omega_c$ , then equation (1.3) is, respectively, in PLP or PLC.

LEMMA C.3. Consider equation (1.3) with real-valued  $w, q \in L^1_{\text{loc}}(0, 1]$ . Let  $0 \leq \nu < 2$ . Define  $\varepsilon_1(r) = r^\nu w(r) - 1$  and  $\varepsilon_2(r) = q(r)/w(r)$ , and suppose that

$$\varepsilon_j(r) = o(1) \quad (r \rightarrow 0, j = 1, 2). \tag{C.7}$$

Then there is a fundamental system  $\{u_1(\cdot; \lambda), u_2(\cdot; \lambda)\}$  satisfying  $u_1(r; \lambda) \rightarrow 1, u_2(r; \lambda) \sim \log(r)$  as  $r \rightarrow 0$ , and both  $u_1$  and  $u_2$  are in  $L^2(0, 1; r dr)$ .

*Proof.* Transform by  $v(r) = r^{1/2}u(r)$ , so that (1.3) becomes

$$\begin{aligned} -v''(r; \lambda) - \frac{1}{4r^2}v(r; \lambda) &= (\lambda w - q)(r)v(r; \lambda) \\ &= r^{-\nu}(1 + \varepsilon_1(r))(\lambda - \varepsilon_2(r))v(r; \lambda). \end{aligned} \tag{C.8}$$

Consider the sequences  $(v_k(\cdot; \lambda))_{k=0}^\infty$  and  $(y_k(\cdot; \lambda))_{k=0}^\infty$  defined by

$$-v''_{k+1}(r; \lambda) - \frac{1}{4r^2}v_{k+1}(r; \lambda) = (\lambda w - q)(r)v_k(r; \lambda),$$

and an equation of the same form for  $y_k$ , satisfying  $v_0(r; \lambda) = r^{1/2}$  and  $y_0(r; \lambda) = r^{1/2} \log(r)$ .

We now suppress the  $\lambda$ -dependence to simplify notation. If we can show that the series (note, starting from  $k = 1$ )  $V := \sum_{k=1}^{\infty} v_k$  and  $Y := \sum_{k=1}^{\infty} y_k$  converge uniformly near 0, and satisfy the asymptotics

$$V(r) = o(r^{1/2}), \quad Y(r) = o(r^{1/2} \log(r)) \quad (r \rightarrow 0), \quad (\text{C.9})$$

then  $r^{-1/2}v = r^{-1/2}v_0 + r^{-1/2}V$  and  $r^{-1/2}y = r^{-1/2}y_0 + r^{-1/2}Y$  is the required solution pair.

Note that  $v_0, y_0$  form a fundamental system in the kernel of the left-hand side of (C.8), and their Wronskian is 1. Therefore, by variation of parameters,  $v_k$  (and  $y_k$  in place of  $v_k$ ) must satisfy

$$\begin{aligned} v_{k+1}(r) &= \int_0^r (v_0(r)y_0(s) - y_0(r)v_0(s))(\lambda w - q)(s)v_k(s) ds \\ &= r^{1/2} \int_0^r s^{1/2-\nu} (\log(s) - \log(r))(1 + \varepsilon_1(s))(\lambda - \varepsilon_2(s))v_k(s) ds. \end{aligned}$$

We want to estimate this integrand. By (C.7), this is straightforward. For each fixed  $\lambda$  there is  $\delta_1 > 0$  such that

$$|(1 + \varepsilon_1(r))(\lambda - \varepsilon_2(r))| < 2|\lambda| \quad (0 < r < \delta_1). \quad (\text{C.10})$$

Furthermore, there is  $\delta_2 > 0$  with

$$|\log(r)| < r^{-1+\nu/2} \quad (0 < r < \delta_2). \quad (\text{C.11})$$

Take  $\delta(\varepsilon) = \min\{\delta_1, \delta_2\}$ , where  $\varepsilon = 1 - \nu/2 > 0$ .

We first consider  $v_k$ . By the triangle inequality, (C.10) and (C.11), we have the estimate

$$|v_{k+1}(r)| \leq 2|\lambda|r^{1/2} \left\{ \int_0^r s^{\varepsilon-3/2} |v_k(s)| ds + r^{-\varepsilon} \int_0^r s^{2\varepsilon-3/2} |v_k(r)| ds \right\} \quad (0 < r < \delta, k = 0, 1, 2, \dots).$$

From this and  $|v_0(r)| \leq r^{1/2}$ , we derive inductively that

$$|v_k(r)| \leq \frac{(3/2)_k}{(k+1)!k!} r^{1/2} \left( \frac{4|\lambda|r^\varepsilon}{\varepsilon} \right)^k \quad (0 < r < \delta, k = 1, 2, 3, \dots), \quad (\text{C.12})$$

where  $(z)_k = z(z+1)\cdots(z+k-1)$  is the Pochhammer symbol. But, for all  $j \geq 0$ , we have

$$\frac{3/2+j}{2+j} < 1 \implies \frac{(3/2)_k}{(k+1)!} < 1 \quad (k \in \mathbb{N}).$$

Thus (C.12) simplifies to

$$|v_k(r)| < r^{1/2} \frac{1}{k!} \left( \frac{4|\lambda|r^\varepsilon}{\varepsilon} \right)^k \quad (0 < r < \delta, k \in \mathbb{N}), \quad (\text{C.13})$$

implying, by Weierstrass' M-test for convergence of functional series, that  $V$  is uniformly convergent on the interval  $(0, \delta)$ . Furthermore, by (C.13), all terms in  $V$  are  $O(r^{1/2+\varepsilon}) = o(r^{1/2})$ , so one-half of (C.9) is satisfied; it follows that  $v(r) = r^{1/2}(1 + O(r^\varepsilon))$  ( $r \rightarrow 0$ ), as required.

We appeal to a similar argument in the case of  $y$ , using (C.10) alongside the slightly different estimates

$$\begin{aligned} |\log(r)| &< r^{-\varepsilon/2} \quad (0 < r < \delta(\varepsilon/2)), \\ |y_1(r)| &\leq \frac{10|\lambda|r^{1/2+\varepsilon}}{3\varepsilon} \quad (0 < r < \delta(\varepsilon/2)), \\ |y_{k+1}(r)| &\leq 2|\lambda|r^{1/2} \left\{ \int_0^r s^{\varepsilon-3/2}|y_k(s)| ds + r^{-\varepsilon} \int_0^r s^{2\varepsilon-3/2}|y_k(s)| ds \right\} \quad (0 < r < \delta(\varepsilon), k \in \mathbb{N}). \end{aligned}$$

These can be used inductively to show that

$$|y_k(r)| \leq \frac{2(3/2)_k}{(k+1)!k!} r^{1/2} \left( \frac{4|\lambda|r^\varepsilon}{\varepsilon} \right)^k \quad (0 < r < \delta(\varepsilon/2), k \in \mathbb{N}).$$

Thus, as with  $v_k$ , the series  $Y$  is uniformly convergent on  $(0, \delta(\varepsilon/2))$ , and the estimates show that the remaining half of (C.9) is satisfied:  $Y(r) = O(r^{1/2+\varepsilon}) = o(r^{1/2} \log(r))$ .

The last claim is that both  $u_1(r) = r^{-1/2}v(r)$  and  $u_2(r) = r^{-1/2}y(r)$  are in  $L^2(0, 1; r dr)$ . Clearly, for any  $\delta > 0$ , on the interval  $(\delta, 1)$  the equation (1.3) is regular, so its solutions are all continuous. We now see that  $u_1(r) \rightarrow 1, u_2(r) \sim \log(r)$  as  $r \rightarrow 0$ , so the claim follows immediately.  $\square$

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Malcolm Brown  
Cardiff School of Computer Science and  
Informatics  
Queens Buildings  
5, The Parade  
Cardiff  
CF24 3AA  
United Kingdom  
brownbm@cardiff.ac.uk

Marco Marletta and Freddy Symons  
Cardiff School of Mathematics  
21-23, Senghennydd Road  
Cardiff  
CF24 4AG  
United Kingdom  
marlettam@cardiff.ac.uk  
symonsfj@cardiff.ac.uk