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RESEARCH PAPER

FRACTIONAL-IN-TIME AND MULTIFRACTIONAL-IN-SPACE STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS *

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Abstract

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This paper derives the weak-sense Gaussian solution to a family of fractional-in-time and multifractional-in-space stochastic partial differential equations, driven by fractional-integrated-in-time spatiotemporal white noise. Some fundamental results on the theory of pseudodifferential operators of variable order, and on the Mittag-Leffler function are applied to obtain the temporal, spatial and spatiotemporal Hölder continuity, in the mean-square sense, of the derived solution.

MSC 2010: Primary 60G60, 60G15, 60G22; Secondary 60G20, 60G17, 60G12

Key Words and Phrases: Caputo-Djrbashian fractional-in-time derivative, elliptic pseudodifferential operator of variable order, Gaussian spatiotemporal white noise measure, Mittag-Leffler function, spatiotemporal Hölder continuity

1. Introduction

Anomalous diffusion arises in many applied fields, for example, in fluid mechanics, statistical mechanics, surface growth, particle transport in heterogeneous media, turbulence phenomena, seismic wave propagation, financial markets (see Barndorff–Nielsen, Mikosch and Resnick [6]; Bouchaud

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and Georges [10]; Carpinteri and Mainardi [11]; Hambly and Jones [14]; Hilfer [17]; Mainardi *et al.* [33]; Metzler and Klafter [36]; [37]; Ruzicka [45], among others). Classical differential calculus does not offer suitable models to represent the non-standard local behavior of such processes. Fractional pseudodifferential calculus provides the required framework for the derivation of probabilistic models to represent anomalous diffusions, e.g., stable Lévy processes (see, for instance, Baeumer, Meerschaert and Mortensen [5]; Becker-Kern, Meerschaert and Scheffler [8]; Kiryakova [23]; Machado, Kiryakova and Mainardi [32]; Mainardi *et al.* [33]; Samorodnitsky and Taqqu [49]; Scalas, Gorenflo and Mainardi [50]).

Multifractional pseudodifferential models allow the representation of heterogeneous local behaviors (see Almeida and Samko [1]; Lorenzo and Hartley [31]; Odzijewicz, Malinowska and Torres [39]; Samko [46]; Samko and Ross [48], among others). The solutions to such models are defined in fractional Besov spaces of variable order (see Leopold [27]; [28] and [29]). Several special classes of multifractional Markov processes have been studied, for example, in Jacob and Leopold [19]; Jacob and Schilling [20]; Kikuchi and Negoro [22]; Kolokoltsov [24] and Komatsu [26]. The order of differentiation of functions in multifractional Besov spaces varies regularly with space and/or time. The smoothness assumption on the exponent function defining the local Hölder exponent does not allow a straightforward introduction of multifractal processes in this framework (see, for instance, Harte [15]; Mandelbrot [34]). In the context of Gaussian processes, the most famous example of multifractional process is multifractional Brownian motion (see, for example, Lévy-Véhel and Peltier [30]), and generalized multifractional Brownian motion (see Ayache [3]; Ayache and Lévy-Véhel [4]). An extended formulation of the class of elliptic Gaussian random fields, introduced in Benassi, Jaffard and Roux [9], and Ruiz-Medina, Angulo and Anh [41], to the context of fractional elliptic pseudodifferential operators of variable order is provided in Ruiz-Medina, Anh and Angulo [42]. The weak-sense definition and properties of this class of variable order fractional Gaussian random fields restricted to compact $d(\cdot)$-sets are derived in Ruiz-Medina, Anh and Angulo [44].

In another direction, Markov processes associated with pseudodifferential operators with smooth symbols were studied by Bass [7]; Jacob [18]; Jacob and Leopold [19]; Kikuchi and Negoro [21]; Kolokoltsov [25]; Komatsu [26], among others. In particular, Bass [7] considered the generator $-(-\Delta)^{\sigma(x)/2}$, where Δ is the Laplacian, for a continuous function $\sigma(x)$, with $0 < \sigma(x) < 2$. The generated process is called an isotropic *stable-like process*. This class is extended in Komatsu [26], where alternative generalized stable-like processes are introduced. The existence of a

Feller semigroup generated by the pseudodifferential operator, whose symbol is the function $-(1+|\xi|^2)^{\sigma(x)}$, is proved in Jacob and Leopold [19], for $0 < \inf \sigma \leq \sup \sigma \leq 2$. This result is extended in Kikuchi and Negoro [21], where strongly elliptic pseudodifferential operators with suitable variable order are considered. The definition and properties of Markov random fields, generated by Feller semigroups, defined from elliptic pseudofferential operators of variable order, restricted to compact $d(\cdot)$-sets, are established in Ruiz-Medina, Anh and Angulo [43].

This paper derives new results in the context of stochastic fractional-in-time and multifractional-in-space pseudodifferential equations, driven by fractional-integrated-in-time spatiotemporal white noise. In particular, the cases where the spatial pseudodifferential operator of variable order is given by $-(-\Delta)^{\sigma(x)/2}$, studied in Bass [7], and by a pseudodifferential operator with symbol $-(1+|\boldsymbol{\xi}|^2)^{\sigma(x)}$, studied in Jacob and Leopold [19], are considered. In Proposition **3.3** below, a weak-sense Gaussian solution, in the mean-square sense, is obtained for a wide class of spatial multifractional pseudodifferential operators, including the two-above-referred cases. For the case $-(-\Delta)^{\sigma(x)/2}$, we prove that the weak-sense Gaussian solution is Hölder continuous, in the mean-square sense, in time and in space. For the family of spatial pseudodifferential operators with symbols $p(\mathbf{x},\boldsymbol{\xi}) = [f(|\boldsymbol{\xi}|)]^{\sigma(\mathbf{x})}$, for $\boldsymbol{\xi}, \mathbf{x} \in \mathbb{R}^n$, with f satisfying certain regularity conditions (see Corollary **4.1** below), the Hölder continuity in space and time of the weak-sense Gaussian solution is proved as well. An example of spatial pseudodifferential operator of variable order in this family is given by the symbol $p(\mathbf{x},\boldsymbol{\xi}) = (|\boldsymbol{\xi}|^2 - 1)^{\sigma(\mathbf{x})}$ (see Example 2 below). Mean-square Hölder continuity in time is also derived for the class of spatial multifractional pseudodifferential operators introduced in Corollary **4.2** below, which includes as particular example the symbol $p(\mathbf{x},\boldsymbol{\xi}) = L(\mathbf{x}, |\boldsymbol{\xi}|)\left(1+|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}\right)^{1/\gamma}$, for $\gamma > 0$, and $\boldsymbol{\xi}, \mathbf{x} \in \mathbb{R}^n$, where, for each $\mathbf{x} \in \mathbb{R}^n$, $L(\mathbf{x}, \cdot)$ is a positive slowly varying function at infinity (see Example 3 below). Finally, Theorem **6.1** shows that the spatiotemporal Hölder continuity, in the mean-square sense, of the formulated weak-sense Gaussian solution follows from its mean-square Hölder continuity in space and in time.

The main contents of this paper are now outlined. The needed preliminary elements on multifractional pseudodifferential operators, and the properties of the Mittag-Leffler function are given in Section **2**. A weak-sense space-time Gaussian solution to a fractional-in-time and multifractional-in-space stochastic partial differential equation is derived in Section **3**, under certain conditions on the involved pseudodifferential operator of variable order. Mean-square continuity of the temporal increments is proved in Section **4**. Mean-square continuity of the spatial increments is analyzed in

Section **5**. The mean-square continuity of the spatiotemporal increments is studied in Section **6**. Final comments are given in Section **7**.

In the derivation of the main results of this paper, we have used different notations for different positive constants that appear in the proofs of such results, with different meanings or values, that are explicitly defined, or they can be directly obtained, from the steps given in those proofs. Note that, in the next section, the notations used for certain positive constants, in the definitions listed below, are kept through the rest of the paper.

2. Preliminaries

Preliminary definitions, properties and results, about pseudodifferential operators of variable order on multifractional Sobolev spaces, are introduced in Section **2.1**. The Mittag-Leffler function, and some of its properties are given in Section **2.2**.

2.1. Pseudodifferential operators of variable order

Let δ and ϖ be real numbers with $0 \leq \delta < \varpi \leq 1$, and let σ be a real-valued function in $\mathcal{B}^{\infty}(\mathbb{R}^n)$, the space of all C^{∞} functions on $\mathbb{R}^n$ whose derivatives of each order are bounded. We say that a function $p(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{B}^{\infty}\left(\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}^n_{\boldsymbol{\xi}}\right)$, locally bounded, belongs to $\mathcal{S}^{\sigma}_{\varpi,\delta}$ if and only if for any multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\varsigma}$ there exists some positive constant $C_{\boldsymbol{\alpha},\boldsymbol{\varsigma}}$ such that

$$|D^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} D^{\boldsymbol{\varsigma}}_{\mathbf{x}} p(\mathbf{x},\boldsymbol{\xi})| \leq C_{\boldsymbol{\alpha},\boldsymbol{\varsigma}} \langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})-\varpi|\boldsymbol{\alpha}|+\delta|\boldsymbol{\varsigma}|}, \tag{2.1}$$

where $D^{\alpha}_{\boldsymbol{\xi}}$ and $D^{\varsigma}_{\mathbf{x}}$ respectively denote the derivatives with respect to $\boldsymbol{\xi}$ and $\mathbf{x}$, and $\langle \boldsymbol{\xi} \rangle = (1+|\boldsymbol{\xi}|^2)^{1/2}$. The following semi-norm is considered for the elements of $\mathcal{S}^{\sigma}_{\varpi,\delta}$:

$$|p|^{(\sigma)}_{l} = \max_{|\boldsymbol{\alpha}+\boldsymbol{\varsigma}|\leq l} \sup_{(\mathbf{x},\boldsymbol{\xi})\in \mathbb{R}^n\times\mathbb{R}^n} \left\{ |D^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} D^{\boldsymbol{\varsigma}}_{\mathbf{x}} p(\mathbf{x},\boldsymbol{\xi})| \langle \boldsymbol{\xi} \rangle^{-\sigma(\mathbf{x})+\varpi|\boldsymbol{\alpha}|-\delta|\boldsymbol{\varsigma}|} \right\}.$$

Definition **2.1**. (see Kikuchi and Negoro [21]) For $u \in \mathcal{S}(\mathbb{R}^n)$, the set of rapidly decreasing Schwartz functions, and let $P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, with symbol $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$, be defined as

$$Pu(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\boldsymbol{\xi}} p(\mathbf{x},\boldsymbol{\xi}) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.2}$$

where $\hat{u}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}\boldsymbol{\xi}} u(\mathbf{x}) d\mathbf{x}$, $\boldsymbol{\xi} \in \mathbb{R}^n$, is the Fourier transform of u, and $\exp(\pm i\mathbf{x}\boldsymbol{\xi}) = \exp\left(\pm i \sum_{i=1}^{n} \xi_i x_i\right)$. We refer to $P = p(\mathbf{x}, D_{\mathbf{x}})$ as a pseudodifferential operator of variable order with symbol $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$. The set of all pseudodifferential operators with symbol p in $\mathcal{S}^{\sigma}_{\varpi,\delta}$ is denoted by $\boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$.

The adjoint P^* of $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$ also belongs to $\boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$ (see, for instance, Lemma 6 in Kikuchi and Negoro [21]). A pseudodifferential operator $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$ is elliptic if there exist $C^* > 0$ and $M > 0$ such that

$$|p(\mathbf{x},\boldsymbol{\xi})| \geq C^* \langle \boldsymbol{\xi}\rangle^{\sigma(\mathbf{x})}, \quad |\boldsymbol{\xi}| \geq M, \quad \mathbf{x} \in \mathbb{R}^n, \ \forall \boldsymbol{\xi} \in \mathbb{R}^n. \tag{2.3}$$

Furthermore, $Q \in \boldsymbol{\mathcal{S}}^{\infty}_{\varpi,\delta} = \bigcup_{m\in\mathbb{R}} \boldsymbol{\mathcal{S}}^{m}_{\varpi,\delta}$ is said to be a left (resp. right) parametrix of P if there exists $R_L \in \boldsymbol{\mathcal{S}}^{-\infty}_{\varpi,\delta} = \bigcap_{m\in\mathbb{R}} \boldsymbol{\mathcal{S}}^{m}_{\varpi,\delta}$ (resp. $R_R \in \boldsymbol{\mathcal{S}}^{-\infty}_{\varpi,\delta} = \bigcap_{m\in\mathbb{R}} \boldsymbol{\mathcal{S}}^{m}_{\varpi,\delta}$) such that

$$QP = I + R_L \quad (\text{resp.} \quad PQ = I + R_R).$$

If P is elliptic, then there exists a parametrix of P in $\boldsymbol{\mathcal{S}}^{-\sigma}_{\varpi,\delta}$ (see, for example, Lemma 6 in Kikuchi and Negoro [21]). The fractional Sobolev spaces of variable order, having $\mathcal{S}(\mathbb{R}^n)$ as dense set, are now introduced, and their main properties are given in the subsequent lemmas.

Definition **2.2**. (see Kikuchi and Negoro [21]) Let $\sigma(\cdot)$ be a real-valued function in $\mathcal{B}^{\infty}(\mathbb{R}^n)$. Define the Sobolev space $H^{\sigma(\cdot)}(\mathbb{R}^n)$ of variable order $\sigma(\cdot)$ on $\mathbb{R}^n$ as

$$\begin{aligned} H^{\sigma(\cdot)}(\mathbb{R}^n) &= \left\{ u \in H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^n) : \ \langle D_{\cdot}\rangle^{\sigma(\cdot)} u \in L^2(\mathbb{R}^n) \right\}, \quad (2.4) \\ H^s(\mathbb{R}^n) &= \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \ \langle D_{\mathbf{x}}\rangle^{s} u \in L^2(\mathbb{R}^n) \right\}, \end{aligned}$$

where $\langle D_{\mathbf{x}}\rangle^{\sigma(\mathbf{x})} u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(i\mathbf{x}\boldsymbol{\xi}) \langle \boldsymbol{\xi}\rangle^{\sigma(\mathbf{x})} \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}$, with $\langle \boldsymbol{\xi}\rangle = (1+|\boldsymbol{\xi}|^2)^{1/2}$.

Lemma **2.1**. (see Lemma 7 in Kikuchi and Negoro [21]) *The above-introduced fractional Sobolev spaces of variable order satisfy:*

(i) If $u \in H^{\sigma(\cdot)}(\mathbb{R}^n)$, then, for $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$, $Pu \in L^2(\mathbb{R}^n)$.

(ii) Let σ_1 and σ_2 be functions in $\mathcal{B}^{\infty}(\mathbb{R}^n)$, with $\sigma_1(\mathbf{x}) \geq \sigma_2(\mathbf{x})$, for each $\mathbf{x} \in \mathbb{R}^n$. Then, $H^{\sigma_1(\cdot)}(\mathbb{R}^n) \subset H^{\sigma_2(\cdot)}(\mathbb{R}^n)$. In particular, $H^{\sigma(\cdot)}(\mathbb{R}^n) \subset H^{\underline{\sigma}(\cdot)}(\mathbb{R}^n)$.

(iii) $H^{\sigma(\cdot)}(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\begin{aligned} \langle u, v\rangle_{H^{\sigma(\cdot)}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (\langle D_{\mathbf{x}}\rangle^{\sigma(\mathbf{x})} u)(\mathbf{x}) \overline{\left(\langle D_{\mathbf{x}}\rangle^{\sigma(\mathbf{x})} v\right)(\mathbf{x})} d\mathbf{x} \\ &+ \int_{\mathbb{R}^n} \left(\langle D_{\mathbf{x}}\rangle^{\underline{\sigma}} u\right)(\mathbf{x}) \overline{\left(\langle D_{\mathbf{x}}\rangle^{\underline{\sigma}} v\right)(\mathbf{x})} d\mathbf{x}, \quad u, v \in H^{\sigma(\cdot)}(\mathbb{R}^n). \end{aligned} \tag{2.5}$$

Lemma **2.2**. (see Lemma 8 in Kikuchi and Negoro [22]) *Let $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\varpi,\delta}$ be elliptic. Then, $H^{\sigma(\cdot)}(\mathbb{R}^n) = \left\{ u \in H^{-\infty}(\mathbb{R}^n) : \ Pu \in L^2(\mathbb{R}^n)\right\}$ as a set. Moreover, the norm $\|u\|_{H^{\sigma(\cdot)}(\mathbb{R}^n)}$ is equivalent to the norm*

$$\|u\|_{H^{\sigma(\cdot),P}}(\mathbb{R}^n) := \left(\|Pu\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{H^{\underline{\sigma}}(\mathbb{R}^n)} \right)^{1/2}. \tag{2.6}$$

2.2. Mittag-Leffler function

The Mittag-Leffler function is here reminded (see Erdélyi *et al.* [12]; Haubold, Mathai and Saxena [16], for a more detailed description of this entire function and its properties).

Definition **2.3**. The Mittag-Leffler function is defined by

$$E_\beta(\mathbf{z}) = \sum_{j=0}^{\infty} \frac{\mathbf{z}^j}{\Gamma(j\beta+1)}, \quad \mathbf{z} \in \mathbb{C}, \quad 0 < \beta \leq 1. \tag{2.7}$$

The following two lemmas play a crucial role in the derivation of the main results of this paper.

Lemma **2.3**. *For every $\beta \in (0,1)$, the uniform estimate*

$$\frac{1}{1+\Gamma(1-\beta)x} \leq E_\beta(-x) \leq \frac{1}{1+[\Gamma(1+\beta)]^{-1}x} \tag{2.8}$$

holds over $\mathbb{R}_+$ with optimal constants (see Simon [51]*, Theorem 4).*

Lemma **2.4**. *The solution to the eigenvalue equation*

$$\frac{d^\beta}{dt^\beta} T(t) = -\mu T(t), \quad 0 < t \leq T, \tag{2.9}$$

is given by the Mittag-Leffler function $E_\beta(-\mu t^\beta)$, for $\mu > 0$, and $\beta \in (0,1)$. Here, $\frac{d^\beta}{dt^\beta}$ denotes the fractional-in-time derivative in the Caputo-Djrbashian sense, given by

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \dfrac{\partial u}{\partial t}(t,\mathbf{x}), & \text{if } \beta = 1, \\ \dfrac{1}{\Gamma(1-\beta)} \dfrac{\partial}{\partial t} \displaystyle\int_0^t (t-\tau)^{-\beta} u(\tau,\mathbf{x})\, d\tau - \dfrac{u(0,\mathbf{x})}{t^\beta}, & \text{if } \beta \in (0,1),\ t \in (0,T], \end{cases} \tag{2.10}$$

(see, for example, Meerschaert and Sikorskii [35]*; Podlubny* [40]*).*

3. The model

In this paper, the following fractional-in-time and multifractional-in-space pseudodifferential model is considered:

$$\frac{\partial^\beta}{\partial t^\beta} c(t,\mathbf{x}) + Pc(t,\mathbf{x}) = I_t^{1-\beta} \dot{W}(t,\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^n, \ t > 0, \quad c(0,\mathbf{x}) = 0, \tag{3.1}$$

$$I_t^{1-\beta} \dot{W}(t,\mathbf{x}) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} \dot{W}(u,\mathbf{x}) du, \quad t \in \mathbb{R}_+, \ \forall \mathbf{x} \in \mathbb{R}^n, \tag{3.2}$$

where P is a pseudodifferential operator of variable order (see (2.2)), with positive symbol $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$, for $\underline{\sigma} > n$. In (3.2), the driven process is defined by fractional integration in time, in the mean-square sense, of the space-time Gaussian white noise $\dot{W}$ (see Samko, Kilbas and Marichev [47]). Process $\dot{W}$ is defined on the probability space $(\Omega, \mathcal{A}, P)$, and has covariance function $E[\dot{W}(t,\mathbf{x})\,\dot{W}(s,\mathbf{y})] = \delta(t-s)\delta(\mathbf{x}-\mathbf{y})$, for $t, s \in \mathbb{R}_+$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with δ denoting the Dirac Delta distribution.

Sufficient conditions for the trace property of the Green operator, and the weak-sense differentiability of its kernel, satisfying (3.1)–(3.2), are obtained in the following two propositions.

PROPOSITION **3.1**. *(i) Let $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$ be the symbol of a pseudodifferential operator P of variable order such that*

$$|p(\mathbf{x},\boldsymbol{\xi})| \geq C^{*}\langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n,\ \mathbf{x} \in \mathbb{R}^n, \tag{3.3}$$

for certain positive constant C^*. If $n < \underline{\sigma}$, then for every $\mathbf{x} \in \mathbb{R}^n$, and $t > 0$,

$$\int_{\mathbb{R}^n} E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right) d\boldsymbol{\xi} < \infty. \tag{3.4}$$

(ii) The same assertion holds for P being a pseudodifferential operator of variable order with symbol

$$p(\mathbf{x},\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{\sigma(\mathbf{x})}, \quad \forall \boldsymbol{\xi}, \mathbf{x} \in \mathbb{R}^n. \tag{3.5}$$

P r o o f. Since P satisfies (3.3), and $E_\beta(-x)$ is a monotone decreasing function with values in the interval $[0,1]$, for $t \in \mathbb{R}_+$, and $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
\int_{\mathbb{R}^n} E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right) d\boldsymbol{\xi} &\leq \int_{\mathbb{R}^n} E_\beta\left(-C^{*}\langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})} t^\beta\right) d\boldsymbol{\xi} \\
&= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty E_\beta\left(-C^{*}\langle \boldsymbol{\rho} \rangle^{\sigma(\mathbf{x})} t^\beta\right) \rho^{n-1} d\rho.
\end{aligned} \tag{3.6}$$

Considering the changes of variables $r = 1 + \rho^2$, and hence, $v = C^{*} r^{\sigma(\mathbf{x})/2} t^\beta$, applying Lemma **2.3**, we obtain, under the condition $\underline{\sigma} > n$, for each $t \in \mathbb{R}_+$, and for every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right) d\boldsymbol{\xi} \\
&\leq \frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left(\left[\frac{v}{C^{*}t^\beta}\right]^{\frac{2}{\sigma(\mathbf{x})}} - 1\right)^{\frac{n-2}{2}} \\
&\qquad \times E_\beta(-v) v^{\frac{2}{\sigma(\mathbf{x})}-1} (C^{*}t^\beta)^{-\frac{2}{\sigma(\mathbf{x})}} dv
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)} \left[\int_0^{C^*t^\beta} \left(\left[\frac{v}{C^*t^\beta}\right]^{\frac{2}{\sigma(\mathbf{x})}} - 1 \right)^{\frac{n-2}{2}} \right. \\
&\qquad\qquad \times E_\beta(-v) v^{\frac{2}{\sigma(\mathbf{x})}-1} (C^*t^\beta)^{-\frac{2}{\sigma(\mathbf{x})}} dv \\
&\left. + \int_{C^*t^\beta}^\infty \left(\left[\frac{v}{C^*t^\beta}\right]^{\frac{2}{\sigma(\mathbf{x})}} - 1 \right)^{\frac{n-2}{2}} E_\beta(-v) v^{\frac{2}{\sigma(\mathbf{x})}-1} (C^*t^\beta)^{-\frac{2}{\sigma(\mathbf{x})}} dv \right] \\
&= I_1 + I_2 < \infty. \qquad (3.7)
\end{aligned}
$$

Note that in (3.7), the integrand of I_1 is integrable on $[0, C^*t^\beta]$, for each $t > 0$ and $\mathbf{x} \in \mathbb{R}^n$. Moreover, from Lemma **2.3**, since $\underline{\sigma} > n$,

$$
\begin{aligned}
I_2 &= \int_{C^*t^\beta}^\infty \left(\left[\frac{v}{C^*t^\beta}\right]^{\frac{2}{\sigma(\mathbf{x})}} - 1 \right)^{\frac{n-2}{2}} E_\beta(-v) v^{\frac{2}{\sigma(\mathbf{x})}-1} (C^*t^\beta)^{-\frac{2}{\sigma(\mathbf{x})}} dv \\
&\leq (C^*t^\beta)^{-\frac{n}{\sigma(\mathbf{x})}} \int_{C^*t^\beta}^\infty E_\beta(-v) v^{\frac{n}{\sigma(\mathbf{x})}-1} dv \\
&\leq (C^*t^\beta)^{-\frac{n}{\sigma(\mathbf{x})}} \int_{C^*t^\beta}^\infty \frac{v^{\frac{n}{\sigma(\mathbf{x})}-1}}{(1 + [\Gamma(1+\beta)]^{-1} v)} dv < \infty.
\end{aligned}
$$

(ii) In a similar way, considering polar coordinates, and the change of variables $v = r^{\sigma(\mathbf{x})} t^\beta$, applying Lemma **2.3**, we obtain, for each $t \in \mathbb{R}_+$, and $\mathbf{x} \in \mathbb{R}^n$, since $n < \underline{\sigma}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} E_\beta \left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})} t^\beta \right) d\boldsymbol{\xi} \\
&= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty r^{n-1} E_\beta \left(-r^{\sigma(\mathbf{x})} t^\beta \right) dr \\
&= \frac{2\pi^{n/2} t^{-\frac{n\beta}{\sigma(\mathbf{x})}}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)} \int_0^\infty z^{\frac{n}{\sigma(\mathbf{x})}-1} E_\beta(-z)\, dz \\
&\leq \frac{2\pi^{n/2} t^{-\frac{n\beta}{\sigma(\mathbf{x})}}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{z^{\frac{n}{\sigma(\mathbf{x})}-1}}{1 + [\Gamma(1+\beta)]^{-1} z} dz < \infty. \qquad (3.8)
\end{aligned}
$$

□

Remark **3.1**. Under (2.3), Proposition **3.1**(i) also holds, since for C^* and M in (2.3),

$$
\begin{aligned}
&\int_{\mathbb{R}^n} E_\beta \left(-p(\mathbf{x}, \boldsymbol{\xi}) t^\beta \right) d\boldsymbol{\xi} \\
&\leq \int_{\{\boldsymbol{\xi} \in \mathbb{R}^n; |\boldsymbol{\xi}| \leq \max(M, C^*t^\beta)\}} \frac{1}{1 + [\Gamma(1+\beta)]^{-1} t^\beta p(\mathbf{x}, \boldsymbol{\xi})} d\boldsymbol{\xi}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}\int_{\max(M,C^*t^\beta)}^{\infty}\left(\left[\frac{\rho}{C^*t^\beta}\right]^{\frac{2}{\sigma(\mathbf{x})}}-1\right)^{\frac{n-2}{2}}\\
&\qquad\qquad\times E_\beta(-\rho)\rho^{\frac{2}{\sigma(\mathbf{x})}-1}(C^*t^\beta)^{-\frac{2}{\sigma(\mathbf{x})}}d\rho\\
&\leq\int_{\{\boldsymbol{\xi}\in\mathbb{R}^n;|\boldsymbol{\xi}|\leq\max(M,C^*t^\beta)\}}d\boldsymbol{\xi}\\
&+\frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}(C^*t^\beta)^{-\frac{n}{\sigma(\mathbf{x})}}\int_{\max(M,C^*t^\beta)}^{\infty}\frac{\rho^{\frac{n}{\sigma(\mathbf{x})}-1}}{(1+[\Gamma(1+\beta)]^{-1}\rho)}d\rho,
\end{aligned}
$$

and the rest of the proof follows as in Proposition **3.1**(i).

PROPOSITION **3.2**. *(i) Let P be a pseudodifferential operator of variable order with symbol $p\in\mathcal{S}^{\sigma}_{\varpi,\delta}$ satisfying (3.3). For $\underline{\sigma}>n$,*

$$
\left|\int_{\mathbb{R}^n}p(\mathbf{x},\boldsymbol{\xi})E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right)\hat{\psi}(\boldsymbol{\xi})d\boldsymbol{\xi}\right|<\infty,\quad \psi\in H^{\sigma(\cdot)}(\mathbb{R}^n), \tag{3.9}
$$

where, as before, $\hat{\psi}$ denotes the Fourier transform of ψ.
(ii) Equation (3.9) also holds for P being a pseudodifferential operator of variable order with symbol given in (3.5).

P r o o f. (i) From the Cauchy-Schwartz inequality, Lemma **2.2** and Proposition **3.1**, we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}p(\mathbf{x},\boldsymbol{\xi})E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right)\hat{\psi}(\boldsymbol{\xi})d\boldsymbol{\xi}\right|\\
&\leq\sqrt{\int_{\mathbb{R}^n}[p(\mathbf{x},\boldsymbol{\xi})]^2|\hat{\psi}(\boldsymbol{\xi})|^2d\boldsymbol{\xi}}\sqrt{\int_{\mathbb{R}^n}\left[E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right)\right]^2d\boldsymbol{\xi}}\\
&\leq\|P\psi\|_{L^2(\mathbb{R}^n)}\sqrt{\int_{\mathbb{R}^n}E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right)d\boldsymbol{\xi}}<\infty,\quad\forall\psi\in H^{\sigma(\cdot)}(\mathbb{R}^n).
\end{aligned}
\tag{3.10}
$$

(ii) In a similar way to equation (3.10),

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}p(\mathbf{x},\boldsymbol{\xi})E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})t^\beta\right)\hat{\psi}(\boldsymbol{\xi})d\boldsymbol{\xi}\right|\\
&\leq\sqrt{\int_{\mathbb{R}^n}\left[|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}\right]^2|\hat{\psi}(\boldsymbol{\xi})|^2d\boldsymbol{\xi}}\sqrt{\int_{\mathbb{R}^n}\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}t^\beta\right)\right]^2d\boldsymbol{\xi}}
\end{aligned}
$$

$$\leq \sqrt{\int_{\mathbb{R}^n} [1+|\boldsymbol{\xi}|^2]^{\sigma(\mathbf{x})} \, |\hat{\psi}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}} \sqrt{\int_{\mathbb{R}^n} \left[E_\beta \left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})} t^\beta\right)\right]^2 d\boldsymbol{\xi}}$$

$$\leq \|P\psi\|_{L^2(\mathbb{R}^n)} \sqrt{\int_{\mathbb{R}^n} E_\beta \left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})} t^\beta\right) d\boldsymbol{\xi}} < \infty, \quad \forall \psi \in H^{\sigma(\cdot)}(\mathbb{R}^n).$$

□

In the next result, a zero-mean Gaussian solution to equations (3.1)–(3.2) is obtained, in the weak sense, on the space $H^{\sigma(\cdot)}(\mathbb{R}^n)$.

Proposition **3.3**. *(i) Let P be the pseudodifferential operator of variable order with symbol $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$. Assume that p satisfies (3.3). Then, for $\underline{\sigma} > n$, the weak-sense Gaussian solution c to (3.1)–(3.2) is given by*

$$c(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^n} G(t,\mathbf{x};s,\mathbf{y}) \dot{W}(s,\mathbf{y}) ds d\mathbf{y} = \int_0^t \mathcal{G}_{t-s}(\dot{W}_s)(\mathbf{x}) ds,$$

$$\mathcal{G}_{t-s}(u)(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\boldsymbol{\xi}} E_\beta \left(-p(\mathbf{x},\boldsymbol{\xi})(t-s)^\beta\right) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}$$
$$\forall \mathbf{x} \in \mathbb{R}^n, \quad \forall u \in L^2(\mathbb{R}^n), \quad t \geq s,$$
$$\mathcal{G}_{t-s}(u)(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall u \in L^2(\mathbb{R}^n), \quad s > t, \tag{3.11}$$

satisfying, for all $\psi \in H^{\sigma(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left[\frac{\partial^\beta}{\partial t^\beta} c\,(t,\mathbf{x}) + Pc\,(t,\mathbf{x})\right] \psi(\mathbf{x}) d\mathbf{x}$$
$$\underset{m.s.}{=} \int_{\mathbb{R}^n} I_t^{1-\beta} \dot{W}\,(t,\mathbf{x})\, \psi(\mathbf{x}) d\mathbf{x}. \tag{3.12}$$

(ii) The solution is likewise defined for pseudodifferential operator P with symbol p given in equation (3.5).
(iii) For each $t, s \in \mathbb{R}_+$, with $s < t$, and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the covariance function of c is given by

$$E[c(t,\mathbf{x})c(s,\mathbf{y})] = \int_0^{t\wedge s} \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}(\mathbf{x}-\mathbf{y})} E_\beta \left(-p(\mathbf{x},\boldsymbol{\xi})(t-u)^\beta\right)$$
$$\times E_\beta \left(-p(\mathbf{y},\boldsymbol{\xi})(s-u)^\beta\right) d\boldsymbol{\xi} du. \tag{3.13}$$

P r o o f. (i) From Lemma **2.4**, and Proposition **3.2**, for $\psi \in H^{\sigma(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{\partial^\beta}{\partial t^\beta} E_\beta \left(-p(\mathbf{x},\boldsymbol{\xi}) t^\beta\right) \hat{\psi}(\boldsymbol{\xi}) d\boldsymbol{\xi}$$
$$= -\int_{\mathbb{R}^n} p(\mathbf{x},\boldsymbol{\xi}) E_\beta \left(-p(\mathbf{x},\boldsymbol{\xi}) t^\beta\right) \hat{\psi}(\boldsymbol{\xi}) d\boldsymbol{\xi} < \infty. \tag{3.14}$$

Therefore, for every $\psi \in H^{\sigma(\cdot)}(\mathbb{R}^n)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} \int_0^\tau \int_{\mathbb{R}^n} G(\tau, \mathbf{x}; s, \mathbf{y}) \\
&\qquad \times \dot{W}(s, \mathbf{y}) \psi(\mathbf{x}) d\mathbf{y} ds d\tau d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t u^{-\beta} \int_0^{t-u} \int_D G(t-u, \mathbf{x}; s, \mathbf{y}) \\
&\qquad \times \dot{W}(s, \mathbf{y}) \psi(\mathbf{x}) d\mathbf{y} ds du d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{\Gamma(1-\beta)} \int_0^t u^{-\beta} \left[\int_{\mathbb{R}^n} G(t-u, \mathbf{x}; t-u, \mathbf{y}) \right.\right. \\
&\qquad \left.\left. \times \dot{W}(t-u, \mathbf{y}) d\mathbf{y} \right] du \right) \psi(\mathbf{x}) d\mathbf{x} \\
&+ \int_{\mathbb{R}^n} \left[\int_0^{t-u} \frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}^n} \left[\frac{d}{dt} \int_0^t u^{-\beta} G(t-u, \mathbf{x}; s, \mathbf{y}) \, du \right] \right. \\
&\qquad \left. \times \dot{W}(s, \mathbf{y}) d\mathbf{y} ds \right] \psi(\mathbf{x}) d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \left[\frac{1}{\Gamma(1-\beta)} \int_0^t u^{-\beta} \dot{W}(t-u, \mathbf{x}) du \right] \psi(\mathbf{x}) d\mathbf{x} \\
&+ \int_{\mathbb{R}^n} \left[\int_0^t \int_{\mathbb{R}^n} \frac{\partial^\beta}{\partial t^\beta} G(t, \mathbf{x}; s, \mathbf{y}) \, \dot{W}(s, \mathbf{y}) d\mathbf{y} ds \right] \psi(\mathbf{x}) d\mathbf{x} \\
&= \int_{\mathbb{R}^n} I_t^{1-\beta} \dot{W}(t, \mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}^n} P \left[\int_0^t \int_{\mathbb{R}^n} G(t, \mathbf{x}; s, \mathbf{y}) \right. \\
&\qquad \left. \times \dot{W}(s, \mathbf{y}) d\mathbf{y} ds \right] \psi(\mathbf{x}) d\mathbf{x}. \qquad (3.15)
\end{aligned}$$

(ii) For the solution (3.11) defined in terms of symbol p, in equation (3.5), in a similar way to equations (3.14)–(3.15), from Lemma **2.4**, and Proposition **3.2**(ii), the same assertion holds.

(iii) Equation (3.13) follows straightforwardly from equation (3.11), since

$$\begin{aligned}
E[c(t, \mathbf{x}) c(s, \mathbf{y})] &= \int_0^t \int_0^s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(\mathbf{x}\boldsymbol{\xi} - \mathbf{y}\boldsymbol{\omega})}}{(2\pi)^{2n}} E_\beta \left(-p(\mathbf{x}, \boldsymbol{\xi})(t-u)^\beta \right) \\
&\qquad \times E_\beta \left(-p(\mathbf{y}, \boldsymbol{\omega})(s-v)^\beta \right) E[\dot{W}(u, \boldsymbol{\xi}) \dot{W}(v, \boldsymbol{\omega})] d\boldsymbol{\xi} d\boldsymbol{\omega} du dv \\
= \frac{1}{(2\pi)^{2n}} &\int_0^t \int_0^s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\mathbf{x}\boldsymbol{\xi} - \mathbf{y}\boldsymbol{\omega})} E_\beta \left(-p(\mathbf{x}, \boldsymbol{\xi})(t-u)^\beta \right) \\
&\qquad \times E_\beta \left(-p(\mathbf{y}, \boldsymbol{\omega})(s-v)^\beta \right) \delta(u-v) \delta(\boldsymbol{\xi} - \boldsymbol{\omega}) d\boldsymbol{\xi} d\boldsymbol{\omega} du dv
\end{aligned}$$

$$= \frac{1}{(2\pi)^{2n}} \int_0^{t\wedge s} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}(\mathbf{x}-\mathbf{y})} E_\beta\left(-p(\mathbf{x},\boldsymbol{\xi})(t-u)^\beta\right) \times E_\beta\left(-p(\mathbf{y},\boldsymbol{\xi})(s-u)^\beta\right) d\boldsymbol{\xi} du. \quad (3.16)$$

□

EXAMPLE 1. Let P be a pseudodifferential operator with symbol $p \in \mathcal{S}^{\sigma}_{\varpi,\delta}$, such that $C_{\mathbf{0},\mathbf{0}} = 1$, for $\boldsymbol{\alpha} = \boldsymbol{\varsigma} = \mathbf{0}$ in equation (2.1), and $C^* = 1$ in equation (3.3). Then,

$$p(\mathbf{x},\boldsymbol{\xi}) = \left[1 + |\boldsymbol{\xi}|^2\right]^{\sigma(\mathbf{x})/2}, \quad \forall \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^n. \quad (3.17)$$

4. Hölder continuity in time

In this section, we study the conditions under which the weak-sense Gaussian solution c, defined in Proposition **3.3**, is Hölder continuous in time, in the $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ sense.

THEOREM **4.1**. *Let c be defined as in Proposition* **3.3***(ii). Assume that $\underline{\sigma} > \max\left(\beta n, \frac{n}{2}\right)$. Then, for every $\mathbf{x} \in \mathbb{R}^n$, as $s \to t$ $(0 < s < t)$*

$$E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 \leq g_{\mathbf{x}}(t-s) \quad (4.1)$$

$$g_{\mathbf{x}}(t-s) = \mathcal{O}\left((t-s)^{1-\frac{\beta n}{\sigma(\mathbf{x})}}\right), \quad s \to t, \ 0 < s < t. \quad (4.2)$$

P r o o f. From equation (3.13), for each $\mathbf{x} \in \mathbb{R}^n$, and for $0 < s < t$,

$$\begin{aligned}
&E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 \\
&= (2\pi)^{-2n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&+ (2\pi)^{-2n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(s-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&- \frac{2}{(2\pi)^{2n}} \int_0^s \int_{\mathbb{R}^n} E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right) E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(s-u)^\beta\right) d\boldsymbol{\xi} du \\
&+ (2\pi)^{-2n} \int_s^t \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&\leq (2\pi)^{-2n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(s-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&- (2\pi)^{-2n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du
\end{aligned}$$

$$+(2\pi)^{-2n}\int_s^t\int_{\mathbb{R}^n}\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi}du. \tag{4.3}$$

In a similar way to equation (3.8), for every $\mathbf{x}\in\mathbb{R}^n$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi}\\
&=\frac{2\pi^{n/2}(t-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}\int_0^\infty z^{\frac{n}{\sigma(\mathbf{x})}-1}[E_\beta(-z)]^2dz\\
&\leq\frac{2\pi^{n/2}(t-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{z^{\frac{n}{\sigma(\mathbf{x})}-1}}{(1+[\Gamma(1+\beta)]^{-1}z)^2}dz\\
&=M(t-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}},
\end{aligned}\tag{4.4}$$

with

$$M=\frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{z^{\frac{n}{\sigma(\mathbf{x})}-1}}{(1+[\Gamma(1+\beta)]^{-1}z)^2}dz, \tag{4.5}$$

which is finite for $n/2<\underline{\sigma}$. From equations (4.3) and (4.4), for every $\mathbf{x}\in\mathbb{R}^n$,

$$\begin{aligned}
&E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2\leq K(\mathbf{x})\int_0^s(s-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}}du\\
&-K(\mathbf{x})\int_0^s(t-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}}du\\
&+K(\mathbf{x})\int_s^t(t-u)^{-\frac{n\beta}{\sigma(\mathbf{x})}}du\\
&=K(\mathbf{x})\left[\frac{s^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}}+\frac{(t-s)^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}}-\frac{t^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}}\right]\\
&+K(\mathbf{x})\frac{(t-s)^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}}\leq 2K(\mathbf{x})\frac{(t-s)^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}},
\end{aligned}\tag{4.6}$$

for $\underline{\sigma}>n\beta$, where $K(\mathbf{x})=(2\pi)^{-2n}\frac{2\pi^{n/2}}{\sigma(\mathbf{x})\Gamma\left(\frac{n}{2}\right)}\int_0^\infty z^{\frac{n}{\sigma(\mathbf{x})}-1}[E_\beta(-z)]^2dz$.

Thus, in (4.2),

$$g_{\mathbf{x}}(t-s)=2K(\mathbf{x})\frac{(t-s)^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{1-\frac{n\beta}{\sigma(\mathbf{x})}}=\Gamma(\mathbf{x})(t-s)^{1-\frac{n\beta}{\sigma(\mathbf{x})}}. \tag{4.7}$$

□

Corollary **4.1**. *Let p be the symbol defining the spatial pseudodifferential operator P in equations (3.1)–(3.2). Assume that*

$$p(\mathbf{x},\boldsymbol{\xi}) = [f(|\boldsymbol{\xi}|)]^{\sigma(\mathbf{x})}, \quad \forall \mathbf{x}\in\mathbb{R}^n, \tag{4.8}$$

where $\sigma\in\mathcal{B}^{\infty}(\mathbb{R}^n)$, and f is continuous and such that

$$\begin{aligned} f^{-1}(0) &= \tau \geq 0 \\ f(r) &= \mathcal{O}(r^{\gamma}), \quad r\to\infty \\ f'(r) = \frac{df}{dr}(r) &= \mathcal{O}(r^{\varrho}), \quad r\to\infty. \end{aligned} \tag{4.9}$$

Then, for $\underline{\sigma} > \max\left(\beta n, \frac{n}{2\gamma-\varrho}\right)$, for every $\mathbf{x}\in\mathbb{R}^n$,

$$E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 \leq h_{\mathbf{x}}(t-s), \tag{4.10}$$

where

$$h_{\mathbf{x}}(t-s) = \mathcal{O}\left((t-s)^{1-\frac{\beta n}{\sigma(\mathbf{x})}}\right), \quad s\to t,\ 0<s<t. \tag{4.11}$$

P r o o f. The proof follows in a similar way to Theorem **4.1**, considering the change of variable $z = f(\omega)$, in equation (4.4), leading to the following expression of the integral (4.5):

$$\int_{\tau}^{\infty} \frac{f(\omega)^{\frac{n}{\sigma(\mathbf{x})}-1} f'(\omega)}{(1+[\Gamma(1+\beta)]^{-1}f(\omega))^2} d\omega,$$

which is finite for $\underline{\sigma} > \frac{n}{2\gamma-\varrho}$. □

Example 2. In Corollary **4.1**, we can consider

$$f(\omega) = \omega^2 - 1, \quad \omega\in\mathbb{R}_+.$$

Hence,

$$p(\mathbf{x},\boldsymbol{\xi}) = (|\boldsymbol{\xi}|^2-1)^{\sigma(\mathbf{x})}, \quad \forall\boldsymbol{\xi}\in\mathbb{R}^n,\ \mathbf{x}\in\mathbb{R}^n, \tag{4.12}$$

and equations (4.10)–(4.11) hold for $\underline{\sigma} > \max\left(\beta n, \frac{n}{3}\right)$.

Corollary **4.2**. *Let p be the symbol defining the spatial pseudodifferential operator P in equations (3.1)–(3.2). Assume that*

$$p(\mathbf{x},\boldsymbol{\xi}) = \psi(\mathbf{x},|\boldsymbol{\xi}|), \quad \forall \mathbf{x},\boldsymbol{\xi}\in\mathbb{R}^n, \tag{4.13}$$

with ψ being such that, for each fixed $\mathbf{x}\in\mathbb{R}^n$,

(i) there exist $\psi^{-1}(\mathbf{x},\cdot) = \psi_{\mathbf{x}}^{-1}(\cdot)$ and $\frac{d\psi_{\mathbf{x}}^{-1}}{dr}(\cdot)$, with $\psi^{-1}(\mathbf{x},0)=\tau$, $\tau\geq 0$,

(ii) for $\lambda > 0$, $\psi^{-1}(\mathbf{x}, \lambda r) = \lambda^{\theta(\mathbf{x})}\psi^{-1}(\mathbf{x}, r)$, *for* $r \in \mathbb{R}_+$, *and for* $r \to \infty$, $\psi^{-1}(\mathbf{x}, r) = \mathcal{O}\left(r^{\phi(\sigma(\mathbf{x}))}\right)$, *and* $\frac{\partial \psi^{-1}}{\partial r}(r) = \mathcal{O}(r^{\nu(\sigma(\mathbf{x}))})$, *where* θ, ϕ *and* ν *are positive bounded functions.*

Then, for $n - 1 < \inf_{\mathbf{x}\in\mathbb{R}^n} \frac{1-\nu(\sigma(\mathbf{x}))}{\phi(\sigma(\mathbf{x}))}$, *and for* $\beta n < \inf_{\mathbf{x}\in\mathbb{R}^n} 1/\theta(\mathbf{x})$,

$$E[c(t, \mathbf{x}) - c(s, \mathbf{x})]^2 \leq \varphi_{\mathbf{x}}(t - s), \tag{4.14}$$

for every $\mathbf{x} \in \mathbb{R}^n$, *where*

$$\varphi_{\mathbf{x}}(t - s) = \mathcal{O}\left((t - s)^{1-\theta(\mathbf{x})\beta n}\right), \quad s \to t,\ 0 < s < t. \tag{4.15}$$

P r o o f. Without loss of generality, in the proof of this result, we consider $\tau = 0$, in Condition (i). Under Conditions (i)-(ii) assumed on ψ in equation (4.13), for each $\mathbf{x} \in \mathbb{R}^n$ and $0 < s < t$, in a similar way to equation (4.3)

$$\begin{aligned}
&E[c(t, \mathbf{x}) - c(s, \mathbf{x})]^2 \\
&\leq \frac{1}{(2\pi)^n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-p(\mathbf{x}, |\boldsymbol{\xi}|)(s - u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&- \frac{1}{(2\pi)^n} \int_0^s \int_{\mathbb{R}^n} \left[E_\beta\left(-p(\mathbf{x}, |\boldsymbol{\xi}|)(t - u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&+ \frac{1}{(2\pi)^n} \int_s^t \int_{\mathbb{R}^n} \left[E_\beta\left(-p(\mathbf{x}, |\boldsymbol{\xi}|)(t - u)^\beta\right)\right]^2 d\boldsymbol{\xi} du. \\
&= \frac{2\pi^{n/2}}{(2\pi)^n \Gamma\left(\frac{n}{2}\right)} \int_0^s (s - u)^{-\theta(\mathbf{x})\beta n} du \\
&\qquad \times \int_0^\infty \left[\psi^{-1}(\mathbf{x}, z)\right]^{n-1} \left[E_\beta\left(-z\right)\right]^2 \left[\frac{d}{dz}\psi^{-1}(\mathbf{x}, z)\right] dz \\
&- \frac{2\pi^{n/2}}{(2\pi)^n \Gamma\left(\frac{n}{2}\right)} \int_0^s (t - u)^{-\theta(\mathbf{x})\beta n} du \\
&\qquad \times \int_0^\infty \left[\psi^{-1}(\mathbf{x}, z)\right]^{n-1} \left[E_\beta\left(-z\right)\right]^2 \left[\frac{d}{dz}\psi^{-1}(\mathbf{x}, z)\right] dz \\
&+ \frac{2\pi^{n/2}}{(2\pi)^n \Gamma\left(\frac{n}{2}\right)} \int_s^t (t - u)^{-\theta(\mathbf{x})\beta n} du \\
&\qquad \times \int_0^\infty \left[\psi^{-1}(\mathbf{x}, z)\right]^{n-1} \left[E_\beta\left(-z\right)\right]^2 \left[\frac{d}{dz}\psi^{-1}(\mathbf{x}, z)\right] dz.
\end{aligned} \tag{4.16}$$

From Lemma **2.3**, under Condition (ii),

$$I(\mathbf{x}) = \int_0^\infty \left[\psi^{-1}(\mathbf{x},z)\right]^{n-1} \left[E_\beta\left(-z\right)\right]^2 \left[\frac{d}{dz}\psi^{-1}(\mathbf{x},z)\right] dz \tag{4.17}$$
$$\leq \int_0^\infty \frac{\left[\psi^{-1}(\mathbf{x},z)\right]^{n-1}\left[\frac{d}{dz}\psi^{-1}(\mathbf{x},z)\right]}{(1+[\Gamma(1+\beta)]^{-1}z)^2} < \infty,$$

for $n-1 < \inf_{\mathbf{x}\in\mathbb{R}^n} \frac{1-\nu(\sigma(\mathbf{x}))}{\phi(\sigma(\mathbf{x}))}$.

Thus, from (4.16), for each $\mathbf{x}\in\mathbb{R}^n$, with $\theta(\mathbf{x})\beta n < 1$, and $I(\mathbf{x})$ being given in (4.17).

$$E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2$$
$$\leq \frac{2\pi^{n/2}(2\pi)^{-n}}{\Gamma\left(\frac{n}{2}\right)} I(\mathbf{x}) \left(s^{1-\theta(\mathbf{x})\beta n} - t^{1-\theta(\mathbf{x})\beta n} + 2(t-s)^{1-\theta(\mathbf{x})\beta n}\right)$$
$$\leq \frac{4\pi^{n/2}(2\pi)^{-n}}{\Gamma\left(\frac{n}{2}\right)} I(\mathbf{x}) \left((t-s)^{1-\theta(\mathbf{x})\beta n}\right). \tag{4.18}$$

□

EXAMPLE 3. Let us consider the pseudodifferential operator of variable order P with symbol

$$p(\mathbf{x},\boldsymbol{\xi}) = L(\mathbf{x},|\boldsymbol{\xi}|)\left(1+|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}\right)^{1/\gamma}, \quad \gamma>0, \quad \forall\boldsymbol{\xi},\mathbf{x}\in\mathbb{R}^n,$$

where, for each $\mathbf{x}\in\mathbb{R}^n$, $L(\mathbf{x},\cdot)$ is a positive slowly varying function at infinity. In particular, we consider

$$L(\mathbf{x},|\boldsymbol{\xi}|) = \frac{|\boldsymbol{\xi}|^{\frac{\sigma(\mathbf{x})}{\gamma}}}{\left(1+|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}\right)^{1/\gamma}}, \quad \gamma>0, \quad \forall\boldsymbol{\xi},\mathbf{x}\in\mathbb{R}^n.$$

Then, for each $\mathbf{x}\in\mathbb{R}^n$,

$$\psi^{-1}(\mathbf{x},r) = r^{\frac{\gamma}{\sigma(\mathbf{x})}}, \quad \forall r\in\mathbb{R}_+.$$

Thus, Conditions (i)–(ii) are satisfied with

$$\phi(\sigma(\mathbf{x})) = \theta(\mathbf{x}) = \frac{\gamma}{\sigma(\mathbf{x})}, \quad \nu(\sigma(\mathbf{x})) = \frac{\gamma}{\sigma(\mathbf{x})} - 1,\ \forall\mathbf{x}\in\mathbb{R}^n.$$

Equations (4.14)–(4.15) in Corollary **4.2** then hold for $\frac{\underline{\sigma}}{\gamma} > \max\left(\beta n, \frac{n}{2}\right)$.

5. Mean-quadratic local variation in space

The mean quadratic fractional local exponent, in space, of the increments of the solution c, in Proposition **3.3**(ii), is now derived.

THEOREM **5.1**. *Let c be defined as in Proposition* **3.3***(ii), with* $\underline{\sigma} > \max\left(\beta n, \frac{n}{2}\right)$, *and $\sigma(\cdot)$ being Hölder continuous such that, for $\|\mathbf{x}-\mathbf{y}\|\to 0$,*

$$|\sigma(\mathbf{x})-\sigma(\mathbf{y})| = \mathcal{O}(\|\mathbf{x}-\mathbf{y}\|^\eta), \quad \eta>1, \quad \forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n. \tag{5.1}$$

Then, for $\|\mathbf{x}-\mathbf{y}\|\to 0$,

$$
\begin{aligned}
&-\widetilde{K}_t\|\mathbf{x}-\mathbf{y}\|^{\eta}+\widetilde{C}_t\,\|\mathbf{x}-\mathbf{y}\|^2\leq E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2\\
&\leq \widetilde{L}_t\|\mathbf{x}-\mathbf{y}\|^{\eta}+C_t\|\mathbf{x}-\mathbf{y}\|^{\min\left(2,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}\right)},
\end{aligned}
\tag{5.2}
$$

for certain positive constants $\widetilde{K}_t, \widetilde{C}_t$, $\widetilde{L}_t$ *and* C_t. *In particular,*

$$
\begin{aligned}
&E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2\leq g(t,\mathbf{x},\mathbf{y})\\
&g(t,\mathbf{x},\mathbf{y})=\mathcal{O}\left(\|\mathbf{x}-\mathbf{y}\|^{\min\left(\eta,2,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}\right)}\right),\ \|\mathbf{x}-\mathbf{y}\|\to 0.
\end{aligned}
$$

P r o o f. For $\|\mathbf{x}-\mathbf{y}\|\to 0$, from Proposition **3.3**(iii),

$$
\begin{aligned}
&E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2\\
&=\frac{1}{(2\pi)^{2n}}\left[\int_0^t\int_{\mathbb{R}^n}\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^{\beta}\right)\right]^2 d\boldsymbol{\xi}du\right.\\
&+\int_0^t\int_{\mathbb{R}^n}\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{y})}(t-u)^{\beta}\right)\right]^2 d\boldsymbol{\xi}du\\
&-2\int_0^t\int_{\mathbb{R}^n}e^{i\boldsymbol{\xi}(\mathbf{x}-\mathbf{y})}E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^{\beta}\right)\\
&\qquad\left.\times E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{y})}(t-u)^{\beta}\right)d\boldsymbol{\xi}du\right]\\
&=\frac{1}{(2\pi)^{2n}}\int_0^t\int_{\mathbb{R}^n}\left|1-L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2\\
&\qquad\times\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^{\beta}\right)\right]^2 d\boldsymbol{\xi}du,
\end{aligned}
\tag{5.3}
$$

where

$$
L(\mathbf{x},\mathbf{y})=\frac{E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{y})}(t-u)^{\beta}\right)}{E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^{\beta}\right)},\tag{5.4}
$$

with, under Condition (5.1),

$$
\log(L(\mathbf{x},\mathbf{y}))=\mathcal{O}(\|\mathbf{x}-\mathbf{y}\|^{\eta}),\quad \|\mathbf{x}-\mathbf{y}\|\to 0,\quad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^n.\tag{5.5}
$$

Thus,

$$
\left|1-L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2=1+L(\mathbf{x},\mathbf{y})\left[L(\mathbf{x},\mathbf{y})-2\cos\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right)\right],
$$

converges, as $\|\mathbf{x}-\mathbf{y}\|\to 0$, to

$$
\left|1-e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2=4\sin^2\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right)=2\left(1-\cos\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right)\right),
$$

since, according to (5.5), $L(\mathbf{x},\mathbf{y})$ converges to one, when $\|\mathbf{x}-\mathbf{y}\| \to 0$. Hence, for $\|\mathbf{x}-\mathbf{y}\|$ sufficiently close to zero,

$$\left|1 - L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 \leq \varepsilon + 4\sin^2\left(\frac{\langle \mathbf{x}-\mathbf{y}, \boldsymbol{\xi}\rangle}{2}\right) \leq \varepsilon + \|\boldsymbol{\xi}\|^2\|\mathbf{x}-\mathbf{y}\|^2, \quad (5.6)$$

where

$$\varepsilon = \mathcal{O}(\|\mathbf{x}-\mathbf{y}\|^{\eta}), \quad \|\mathbf{x}-\mathbf{y}\| \to 0. \quad (5.7)$$

Consider first $n+2 < \min\left(2, \beta^{-1}\right)\underline{\sigma}$, from equations (5.3) and (5.6), as $\|\mathbf{x}-\mathbf{y}\| \to 0$, for certain positive constant K in (5.7),

$$\begin{aligned}
E[c(t,\mathbf{x}) - c(t,\mathbf{y})]^2 &\leq \frac{1}{(2\pi)^{2n}} \int_0^t \int_{\mathbb{R}^n} \left[\varepsilon + \|\boldsymbol{\xi}\|^2 \|\mathbf{x}-\mathbf{y}\|^2\right] \\
&\qquad \times \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&= \frac{2\pi^{n/2}\varepsilon}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)} \int_0^t \int_0^\infty \rho^{n-1}\left[E_\beta\left(-\rho^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\rho du \\
&+ \frac{\|\mathbf{x}-\mathbf{y}\|^2 2\pi^{n/2}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)} \int_0^t \int_0^\infty \rho^{n+1}\left[E_\beta\left(-\rho^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\rho du \\
&\leq \frac{2\pi^{n/2}\varepsilon}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)} \int_0^t \int_0^\infty \frac{\rho^{n-1}}{[1+[\Gamma(1+\beta)]^{-1}\rho^{\sigma(\mathbf{x})}(t-u)^\beta]^2} d\rho du \\
&+ \frac{\|\mathbf{x}-\mathbf{y}\|^2 2\pi^{n/2}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)} \int_0^t \int_0^\infty \frac{\rho^{n+1}}{[1+[\Gamma(1+\beta)]^{-1}\rho^{\sigma(\mathbf{x})}(t-u)^\beta]^2} d\rho du \\
&\leq K \frac{2\pi^{n/2} B\left(\frac{n}{\sigma(\mathbf{x})}, 2 - \frac{n}{\sigma(\mathbf{x})}\right) t^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)\sigma(\mathbf{x})\left(1-\frac{n\beta}{\sigma(\mathbf{x})}\right)[\Gamma(1+\beta)]^{-n/\sigma(\mathbf{x})}} \|\mathbf{x}-\mathbf{y}\|^\eta \\
&+ \frac{2\pi^{n/2} B\left(\frac{n+2}{\sigma(\mathbf{x})}, 2 - \frac{n+2}{\sigma(\mathbf{x})}\right) t^{1-\frac{(n+2)\beta}{\sigma(\mathbf{x})}}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)\sigma(\mathbf{x})\left(1-\frac{(n+2)\beta}{\sigma(\mathbf{x})}\right)[\Gamma(1+\beta)]^{-(n+2)/\sigma(\mathbf{x})}} \|\mathbf{x}-\mathbf{y}\|^2.
\end{aligned} \quad (5.8)$$

Secondly, for the case $n+2 \geq \min\left(2, \beta^{-1}\right)\underline{\sigma}$, there exists $\widetilde{\varepsilon} < 1$ such that $n + 2\widetilde{\varepsilon} < \min\left(2, \beta^{-1}\right)\underline{\sigma}$, since $\underline{\sigma} > \max\left(\beta n, \frac{n}{2}\right)$. In particular, $2\widetilde{\varepsilon}$ must satisfy $2\widetilde{\varepsilon} < \frac{\sigma(\mathbf{x})-\beta n}{\beta}$. From (5.6), we obtain

$$\left|1 - L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 \leq \varepsilon + 4^{1-\widetilde{\varepsilon}} \|\boldsymbol{\xi}\|^{2\widetilde{\varepsilon}} \|\mathbf{x}-\mathbf{y}\|^{2\widetilde{\varepsilon}}. \quad (5.9)$$

In a similar way to equation (5.8), from equation (5.9), we then have

$$
\begin{aligned}
E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2 &\leq \frac{1}{(2\pi)^{2n}}\int_0^t\int_{\mathbb{R}^n}\left[\varepsilon+4^{1-\widetilde{\varepsilon}}\|\boldsymbol{\xi}\|^{2\widetilde{\varepsilon}}\|\mathbf{x}-\mathbf{y}\|^{2\widetilde{\varepsilon}}\right] \\
&\qquad\qquad \times\left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi}du \\
&= \frac{2\pi^{n/2}\varepsilon}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)}\int_0^t\int_0^\infty \rho^{n-1}\left[E_\beta\left(-\rho^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\rho du \\
&+ \int_0^t\int_0^\infty \frac{\|\mathbf{x}-\mathbf{y}\|^{2\widetilde{\varepsilon}}4^{1-\widetilde{\varepsilon}}\rho^{n+2\widetilde{\varepsilon}-1}\left[E_\beta\left(-\rho^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\rho du}{\Gamma\left(\frac{n}{2}\right)2^{2n-1}\pi^{2n-n/2}} \\
&\leq \frac{2\pi^{n/2}\varepsilon}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)}\int_0^t\int_0^\infty \frac{\rho^{n-1}d\rho du}{[1+[\Gamma(1+\beta)]^{-1}\rho^{\sigma(\mathbf{x})}(t-u)^\beta]^2} \\
&+ \int_0^t\int_0^\infty \frac{\|\mathbf{x}-\mathbf{y}\|^{2\widetilde{\varepsilon}}4^{1-\widetilde{\varepsilon}}2\pi^{n/2}\rho^{n+2\widetilde{\varepsilon}-1}d\rho du}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)[1+[\Gamma(1+\beta)]^{-1}\rho^{\sigma(\mathbf{x})}(t-u)^\beta]^2} \\
&\leq K\frac{2\pi^{n/2}B\left(\frac{n}{\sigma(\mathbf{x})},2-\frac{n}{\sigma(\mathbf{x})}\right)t^{1-\frac{n\beta}{\sigma(\mathbf{x})}}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)\sigma(\mathbf{x})\left(1-\frac{n\beta}{\sigma(\mathbf{x})}\right)[\Gamma(1+\beta)]^{-n/\sigma(\mathbf{x})}}\|\mathbf{x}-\mathbf{y}\|^\eta \\
&+ \frac{4^{1-\widetilde{\varepsilon}}2\pi^{n/2}B\left(\frac{n+2\widetilde{\varepsilon}}{\sigma(\mathbf{x})},2-\frac{n+2\widetilde{\varepsilon}}{\sigma(\mathbf{x})}\right)t^{1-\frac{(n+2\widetilde{\varepsilon})\beta}{\sigma(\mathbf{x})}}}{(2\pi)^{2n}\Gamma\left(\frac{n}{2}\right)\sigma(\mathbf{x})\left(1-\frac{(n+2\widetilde{\varepsilon})\beta}{\sigma(\mathbf{x})}\right)[\Gamma(1+\beta)]^{-(n+2\widetilde{\varepsilon})/\sigma(\mathbf{x})}}\|\mathbf{x}-\mathbf{y}\|^{2\widetilde{\varepsilon}}.
\end{aligned}
\tag{5.10}
$$

From equations (5.8) and (5.10), for each $t>0$, as $\|\mathbf{x}-\mathbf{y}\|\to 0$,

$$
E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2\leq g(t,\mathbf{x},\mathbf{y}),\quad g(t,\mathbf{x},\mathbf{y})=\mathcal{O}\left(\|\mathbf{x}-\mathbf{y}\|^{\min\left(\eta,2,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^-\right)}\right).
$$

Let us now derive the lower bound in equation (5.2). The convergence, as $\|\mathbf{x}-\mathbf{y}\|\to 0$, of

$$
\left|1-L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 = 1+L(\mathbf{x},\mathbf{y})\left[L(\mathbf{x},\mathbf{y})-2\cos\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right)\right]
$$

to

$$
\left|1-e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 = 4\sin^2\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right) = 2\left(1-\cos\left(\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\xi}\rangle}{2}\right)\right)
$$

implies, for $\|\mathbf{x}-\mathbf{y}\|$ sufficiently small,

$$
\left|1-L(\mathbf{x},\mathbf{y})e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 \geq -\varepsilon+\left|1-e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\right|^2 > 0. \tag{5.11}
$$

From equations (5.3) and (5.11), applying the inequality $|e^z - 1| > \frac{1}{4}|z|$, for $|z| < 1$, we obtain, considering the set $A = \left\{\boldsymbol{\xi} \in \mathbb{R}^n_+ : \ |\boldsymbol{\xi}| \leq x_0 < \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right\}$,

$$\begin{aligned}
&E[c(t,\mathbf{x}) - c(t,\mathbf{y})]^2 \\
&\geq \frac{-\varepsilon}{(2\pi)^{2n}} \int_0^t \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&+ \frac{1}{4(2\pi)^{2n}} \int_0^t \int_A |\boldsymbol{\xi}|^2 \|\mathbf{x}-\mathbf{y}\|^2 \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&\geq -\widetilde{K}_t \|\mathbf{x}-\mathbf{y}\|^\eta + \widetilde{C}_t \|\mathbf{x}-\mathbf{y}\|^2,
\end{aligned}$$

for $\widetilde{K}_t > 0$, computed from (5.7) and $\int_0^t \int_{\mathbb{R}^n} \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi}$, which is finite for $\underline{\sigma} > \max\left(\frac{n}{2}, \beta n\right)$. Here, $\widetilde{C}_t$ is also a positive constant depending on t, since $E_\beta(-x) < 1$, for $x \in \mathbb{R}_+$ and $\beta \in (0,1)$, and

$$\begin{aligned}
&\int_0^t \int_A |\boldsymbol{\xi}|^2 \left[E_\beta\left(-|\boldsymbol{\xi}|^{\sigma(\mathbf{x})}(t-u)^\beta\right)\right]^2 d\boldsymbol{\xi} du \\
&\leq \int_0^t \int_A |\boldsymbol{\xi}|^2 d\boldsymbol{\xi} du \leq x_0^2 t |A| < \infty,
\end{aligned}$$

where $|A|$ is the n-dimensional volume of A. □

Corollary **5.1**. *Assume that the conditions of Corollary* **4.1** *hold, and that $\sigma(\cdot)$ is Hölder continuous such that, for $\|\mathbf{x}-\mathbf{y}\| \to 0$,*

$$|\sigma(\mathbf{x}) - \sigma(\mathbf{y})| = \mathcal{O}(\|\mathbf{x}-\mathbf{y}\|^\eta), \quad \eta > 1, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{5.12}$$

Then, equation (5.2) holds for c defined from pseudodifferential operator P with symbol (4.8), satisfying the conditions given in Corollary **4.1**.

P r o o f. The proof follows in a similar way to Theorem **5.1** from the conditions assumed in Corollary **4.1** (see also the proof of such a corollary). □

6. Mean quadratic local variation in time and space

We apply here the results derived in the previous sections to obtain the mean-quadratic local variation properties of the spatiotemporal increments of the weak-sense solution c given in Proposition **3.3**(ii).

Theorem **6.1**. *Assume that the conditions of Theorems* **4.1** *and* **5.1** *hold. Then, the solution c to equations (3.1)–(3.2), defined in Proposition* **3.3***(ii), satisfies, for $s, t \in (0, T]$, with $0 < s < t$, and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, as $s \to t$ and $\|\mathbf{x}-\mathbf{y}\| \to 0$,*

$$E[c(t,\mathbf{x})-c(s,\mathbf{y})]^2 \quad \leq \quad \chi(t,s,\mathbf{x},\mathbf{y})$$

with $\chi(t,s,\mathbf{x},\mathbf{y}) = \mathcal{O}\left(\|(t,\mathbf{x})-(s,\mathbf{y})\|^{\left(1-\frac{n\beta}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\right)$, *as* $\|(t,\mathbf{x})-(s,\mathbf{y})\| \to 0$, *where* $x\wedge y$ *denotes the minimum of* x *and* y.

P r o o f. Under the conditions of Theorems **4.1** and **5.1**, for $0<s<t$, and $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, applying Cauchy–Schwarz inequality, when $(t-s)\to 0$, and $\|\mathbf{x}-\mathbf{y}\|\to 0$,

$$\begin{aligned}
&E[c(t,\mathbf{x})-c(s,\mathbf{y})]^2 = E[c(t,\mathbf{x})-c(s,\mathbf{x})+c(s,\mathbf{x})-c(s,\mathbf{y})]^2\\
&= E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x})-c(s,\mathbf{y})]^2\\
&\qquad +2E\left[(c(t,\mathbf{x})-c(s,\mathbf{x}))(c(s,\mathbf{x})-c(s,\mathbf{y}))\right]\\
&\leq E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x})-c(s,\mathbf{y})]^2\\
&\qquad +2\left|E\left[(c(t,\mathbf{x})-c(s,\mathbf{x}))(c(s,\mathbf{x})-c(s,\mathbf{y}))\right]\right|\\
&\leq E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x})-c(s,\mathbf{y})]^2\\
&\qquad +2\sqrt{E\left[c(t,\mathbf{x})-c(s,\mathbf{x})\right]^2}\sqrt{E\left[c(s,\mathbf{x})-c(s,\mathbf{y})\right]^2}\\
&\leq \Gamma(\mathbf{x})(t-s)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)} + \max\left(\widetilde{L}_s, C_s\right)\|\mathbf{x}-\mathbf{y}\|^{\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\\
&+2\sqrt{\Gamma(\mathbf{x})(t-s)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)}}\\
&\qquad \times\sqrt{\max\left(\widetilde{L}_s, C_s\right)\|\mathbf{x}-\mathbf{y}\|^{\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}}\\
&\leq 4\max\left(\Gamma(\mathbf{x}),\widetilde{L}_s, C_s\right)\left[\frac{1}{2}\left((t-s)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\right.\right.\\
&\qquad\qquad \left.+\|\mathbf{x}-\mathbf{y}\|^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\right)\\
&\qquad\qquad +\frac{1}{2}\left(\left((t-s)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\right)^2\right.\\
&\qquad\qquad \left.\left.+\left(\|\mathbf{x}-\mathbf{y}\|^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge\min\left(\eta,\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-},2\right)}\right)^2\right)^{1/2}\right],
\end{aligned}$$

where, for each $\mathbf{x} \in \mathbb{R}^n$, $\Gamma(\mathbf{x})$ has been introduced in equation (4.7), and, for each $s \in \mathbb{R}_+$, $\widetilde{L}_s$ and C_s appear in equation (5.2) in Theorem **5.1**.

Since $0 < \left(1 - \frac{\beta n}{\sigma(\mathbf{x})}\right) \wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right) < 1$, hence, we can apply Jensen's inequality for concave function x^{ξ}, $0 < \xi < 1$, for $s, t \in (0, T]$,

$$
\begin{aligned}
& E[c(t,\mathbf{x}) - c(s,\mathbf{y})]^2 \\
& \leq 4 \max\left(\Gamma(\mathbf{x}), \widetilde{L}_T, C_T\right) \left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right)\right]} \\
& \times \Bigg[\left((t-s)^2 + \|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2\left[\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right)\right]} \\
& + \sqrt{\left((t-s)^2 + \|\mathbf{x}-\mathbf{y}\|^2\right)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right)}} \Bigg] \\
& = 2K^{\star\star}_{T,\mathbf{x}} \|(t,\mathbf{x}) - (s,\mathbf{y})\|^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right)}, \qquad (6.1)
\end{aligned}
$$

where

$$
K^{\star\star}_{T,\mathbf{x}} = 4 \max\left(\Gamma(\mathbf{x}), \widetilde{L}_T, C_T\right) \left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\wedge \min\left(\eta, \left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, 2\right)\right]}.
$$

□

7. Final comments

In the case of considering fractional-in-time and multifractional-in-space stochastic partial differential equations, with fractional integrated-in-time non-linear random external force, our guess is that, for the spatial multifractional pseudodifferential operators considered in Theorems **4.1**, **5.1**, and Corollary **4.1**, the k-order moments of the temporal and spatial increments of the solution satisfy

$$
\begin{aligned}
& E|c(t,\mathbf{x}) - c(s,\mathbf{x})|^k = \mathcal{O}\left((t-s)^{\left(1-\frac{\beta n}{\sigma(\mathbf{x})}\right)\frac{k}{2}}\right), \ 0 < s < t, \ s \to t, \\
& \widetilde{H}_t \|\mathbf{x}-\mathbf{y}\|^{\eta} + C^{\star}_t \|\mathbf{x}-\mathbf{y}\|^k \leq E|c(t,\mathbf{x}) - c(t,\mathbf{y})|^k \\
& \leq H^{\star}_t \|\mathbf{x}-\mathbf{y}\|^{\eta} + C^{\star\star}_t \|\mathbf{x}-\mathbf{y}\|^{\min\left(\left[\frac{\sigma(\mathbf{x})-\beta n}{\beta}\right]^{-}, \eta, 2\right)\frac{k}{2}}, \\
& \qquad \|\mathbf{x}-\mathbf{y}\| \to 0,
\end{aligned}
$$

for certain positive constants $\widetilde{H}_t$, $C^{\star}_t$, $H^{\star}_t$ and $C^{\star\star}_t$. These results constitute the subject of a subsequent paper (see Proposition 2 in Mijena and Nane

[38], for the factional fixed order case in space and time). The bounded domain case requires further research (see Anh, Leonenko and Ruiz-Medina [2], and Foondun, Mijena and Nane [13]).

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