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# An exceptional max-stable process fully parameterized by its extremal coefficients

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The extremal coefficient function (ECF) of a max-stable process  $X$  on some index set  $T$  assigns to each finite subset  $A \subset T$  the effective number of independent random variables among the collection  $\{X_t\}_{t \in A}$ . We introduce the class of Tawn–Molchanov processes that is in a 1:1 correspondence with the class of ECFs, thus also proving a complete characterization of the ECF in terms of negative definiteness. The corresponding Tawn–Molchanov process turns out to be exceptional among all max-stable processes sharing the same ECF in that its dependency set is maximal w.r.t. inclusion. This entails sharp lower bounds for the finite dimensional distributions of arbitrary max-stable processes in terms of its ECF. A spectral representation of the Tawn–Molchanov process and stochastic continuity are discussed. We also show how to build new valid ECFs from given ECFs by means of Bernstein functions.

*Keywords:* completely alternating; dependency set; extremal coefficient; max-linear model; max-stable process; negative definite; semigroup; spectrally discrete; Tawn–Molchanov process

## 1. Introduction

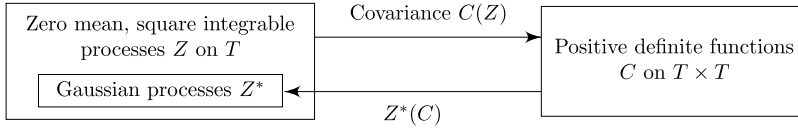
Besides the class of square integrable processes, the class of temporal or spatial *max-stable processes* is of common interest in stochastics and statistics, cf. [3, 4, 9, 13, 20, 31], for example. In spite of considerable differences between these two classes, for example, the non-existence of the first moments in case of max-stable processes with *unit Fréchet marginals*, connections between the two classes have been made for instance, by the *extremal Gaussian process* [25] and the *Brown–Resnick process* [16] that are parameterized by a correlation function and a variogram, respectively.

Naturally, extremal dependence measures such as the *extremal coefficients* [26, 28], the (*upper*) *tail dependence coefficients* [1, 5, 8, 11] or other special cases of the *extremogram* [8] are appropriate summary statistics for max-stable processes. In this article, we capture the full set of extremal coefficients of a max-stable process  $X = \{X_t\}_{t \in T}$  on some space  $T$  in the so-called *extremal coefficient function (ECF)*  $\theta$ , which assigns to each finite subset  $A$  of  $T$  the effective number of independent variables among the collection  $\{X_t\}_{t \in A}$ . We

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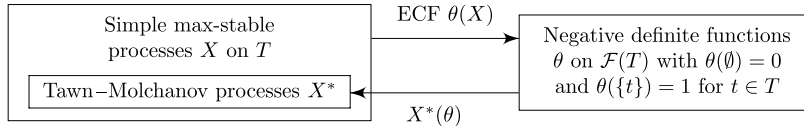
introduce a subclass of max-stable processes that is parameterized by the ECF, and thus reveal some analogies to *Gaussian processes* and *positive definite functions* as follows:

Among (zero mean) square integrable processes, the subclass of *Gaussian processes* takes a unique role, since it is in a 1–1 correspondence with the set of *covariance functions*, which are precisely the *positive definite functions*. This fact can be proven by means of Kolmogorov’s extension theorem and is illustrated in the following graph:

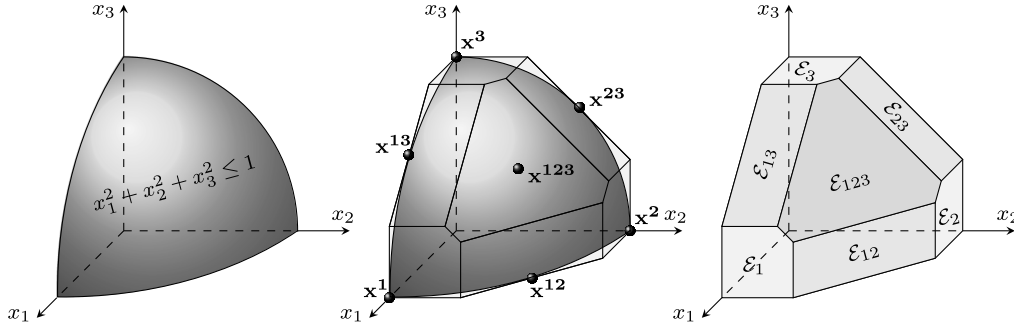


In case  $T$  is a metric space, the Gaussian process  $Z^*(C)$  is continuous in the mean square sense (and then also stochastically continuous) if and only if the covariance function  $C$  is continuous if and only if  $C$  is continuous on the diagonal (cf. [23], Theorem 5.3.3). Well-known operations on the set of positive definite functions  $C$ , and hence on the corresponding Gaussian processes  $Z^*(C)$ , include convex combinations and pointwise limits. Moreover, *Bernstein functions* play an important role for the construction of positive definite functions.

In our case, the crucial role of zero mean Gaussian processes is taken by the class of *Tawn–Molchanov processes (TM processes)*, which are in fact the spatial generalization of the multivariate *max-linear* model of [26]. Using Kolmogorov’s extension theorem, we shall see that each ECF  $\theta$  (of some max-stable process) uniquely determines a TM process  $X^*(\theta)$  having the same ECF (Theorem 8). Alongside, we generalize a multivariate result [19], Corollary 1, to the spatial setting, proving that the ECFs coincide with the functions  $\theta$  on  $\mathcal{F}(T)$  (the *set of finite subsets* of  $T$ ) that are normalized to  $\theta(\emptyset) = 0$  and  $\theta(\{t\}) = 1$  for  $t \in T$  and that are *negative definite* (or equivalently *completely alternating*) in a sense to be explained below (cf. Definition 6). This can be illustrated in analogy to the above sketch:



Having identified the ECF  $\theta$  as a negative definite quantity allows for several immediate consequences: First, we obtain an *integral representation* of  $\theta$  as a mixture of maps  $A \mapsto \mathbb{1}_{A \cap Q \neq \emptyset}$  (Corollary 16) and derive a *spectral representation* for the corresponding TM process  $X^*(\theta)$  (Theorem 18). Second, we consider operations on ECFs that allow to build new ECFs from given ones. We find that ECFs allow for convex combinations and pointwise limits (Corollaries 13 and 14) and that the class of *Bernstein functions* operates on ECFs (Corollary 20). We also recover the “triangle inequalities” for  $\theta$  from [7], Proposition 4, and see that the inequalities therein correspond to three specific choices of a Bernstein function, whereas we may plug in arbitrary Bernstein functions to obtain further “triangle inequalities” (Corollary 21).



**Figure 1.** Examples of dependency sets in a trivariate setting: a “typical” dependency set  $\mathcal{K}$  (left) and a dependency set  $\mathcal{K}^*$  stemming from a TM process (right). It is shown that  $\mathcal{K} \subset \mathcal{K}^*$  (middle). For further details, see the introduction, Example 31, Lemma 30 and Theorem 32.

For  $T$  being a metric space, we discuss *stochastic continuity*: The TM process  $X^*(\theta)$  is stochastically continuous if and only if  $\theta$  is continuous (cf. Definition 22) if and only if the bivariate map  $(s, t) \mapsto \theta(\{s, t\})$  is continuous if and only if the bivariate map  $(s, t) \mapsto \theta(\{s, t\})$  is continuous on the diagonal (Theorem 25).

Finally, we address the exceptional role of the TM processes among simple max-stable processes. To this end, Molchanov’s *dependency set*  $\mathcal{K}$  [19] is transferred to max-stable processes  $X$ . It comprises the finite dimensional distributions (f.d.d.) of  $X$  (Lemma 29). Now, let  $\mathcal{K}^*(\theta)$  denote the dependency set of the process  $X^*(\theta)$ . Then we identify  $\mathcal{K}^*(\theta)$  as intersection of halfspaces that are directly given by the ECF  $\theta$  (Theorem 32). It turns out that  $\mathcal{K}^*(\theta)$  is exceptional among the dependency sets  $\mathcal{K}$  of all max-stable processes sharing the same ECF  $\theta$ , since  $\mathcal{K}^*(\theta)$  is maximal w.r.t. inclusion as illustrated in Figure 1. Since inclusion of dependency sets corresponds to stochastic ordering, this observation leads to sharp inequalities for the f.d.d. of max-stable processes in terms of its ECF  $\theta$  (Corollary 33).

The text is structured as follows. After the introductory Section 2, the characterization of ECFs and the existence of TM processes is established in Section 3. Section 4 collects several immediate consequences and related results, while Section 5 exhibits the exceptional role of TM processes. Sections 4 and 5 can be read independently.

## 2. Foundations and definitions

### 2.1. Notation for max-stable processes and ECFs

A stochastic process  $X = \{X_t\}_{t \in T}$  on an arbitrary index set  $T$  is said to be *max-stable* if for each  $n \in \mathbb{N}$  and independent copies  $X^{(1)}, \dots, X^{(n)}$  of  $X$  the process of the maxima  $\{\bigvee_{i=1}^n X^{(i)}\}_{t \in T}$  has the same law as  $\{a_n(t)X_t + b_n(t)\}_{t \in T}$  for suitable norming functions  $a_n(t) > 0$  and  $b_n(t)$  on  $T$ . Without loss of generality, we shall deal with max-stable processes that have *standard Fréchet marginals*, that is,  $\mathbb{P}(X_t \leq x) = e^{-1/x}$  for  $t \in T$

and  $x \geq 0$ , and set  $a_n(t) = n$  and  $b_n(t) = 0$ . Such processes are called *simple* max-stable processes.

It has been shown (cf. [9, 15, 29]) that (simple) max-stable processes  $X = \{X_t\}_{t \in T}$  allow for a *spectral representation*  $(\Omega, \mathcal{A}, \nu, V)$ : there exists a (sufficiently rich) measure space  $(\Omega, \mathcal{A}, \nu)$  and measurable functions  $V_t : \Omega \rightarrow \mathbb{R}_+$  (with  $\int_{\Omega} V_t(\omega) \nu(d\omega) = 1$  for each  $t \in T$ ), such that the law of  $X = \{X_t\}_{t \in T}$  equals the law of

$$\left\{ \bigvee_{(U, \omega) \in \Pi} UV_t(\omega) \right\}_{t \in T}. \quad (1)$$

Here  $\Pi$  denotes a Poisson point process on  $\mathbb{R}_+ \times \Omega$  with intensity  $u^{-2} du \times \nu(d\omega)$ . The functions  $\{V_t\}_{t \in T}$  are called *spectral functions* and the measure  $\nu$  is called *spectral measure*.

In order to describe the *finite dimensional distributions* (f.d.d.) of  $X$ , we shall fix some convenient notation first: Let  $M \subset T$  be some non-empty finite subset of  $T$ . By  $\mathbb{R}^M$  (resp.  $[0, \infty]^M$ ) we denote the space of real-valued (resp.  $[0, \infty]$ -valued) functions on  $M$ . Elements of these spaces are denoted by  $x = (x_t)_{t \in M}$  where  $x_t = x(t)$ . The standard scalar product is given through  $\langle x, y \rangle = \sum_{t \in M} x_t y_t$ . For a subset  $L \subset M$ , we write  $\mathbf{1}_L$  for the indicator function of  $L$  in  $\mathbb{R}^M$  (resp.  $[0, \infty]^M$ ), such that  $\{\mathbf{1}_{\{t\}}\}_{t \in M}$  forms an orthonormal basis of  $\mathbb{R}^M$ . In this sense, the origin of  $\mathbb{R}^M$  equals  $\mathbf{1}_{\emptyset}$  being zero everywhere on  $M$ . Using this notation, we emphasize the fact that a multivariate distribution of a stochastic process is not any  $|M|$ -variate distribution, but it is bound to certain points (forming the set  $M$ ) in the space  $T$ . Finally, we consider some *reference norm*  $\|\cdot\|$  on  $\mathbb{R}^M$  (not necessarily the one from the scalar product) and denote the positive unit sphere  $S_M := \{a \in [0, \infty)^M : \|a\| = 1\}$ .

In terms of a spectral representation  $(\Omega, \mathcal{A}, \nu, V)$ , the f.d.d. of  $X$  are given through

$$-\log \mathbb{P}(X_t \leq x_t, t \in M) = \int_{\Omega} \left( \bigvee_{t \in M} \frac{V_t(\omega)}{x_t} \right) \nu(d\omega) \quad (2)$$

for  $x \in [0, \infty)^M \setminus \{\mathbf{1}_{\emptyset}\}$ . Alternatively, the f.d.d. of  $X$  for a finite subset  $\emptyset \neq M \subset T$  may be described by means of one of the following three quantities that are all equivalent to the knowledge of the f.d.d.:

- the (*finite dimensional*) *spectral measure*  $H_M$  (cf. [10, 22]), that is, the Radon measure on  $S_M$  such that for  $x \in [0, \infty)^M \setminus \{\mathbf{1}_{\emptyset}\}$

$$-\log \mathbb{P}(X_t \leq x_t, t \in M) = \int_{S_M} \left( \bigvee_{t \in M} \frac{a_t}{x_t} \right) H_M(da), \quad (3)$$

- the *stable tail dependence function*  $\ell_M$  (cf. [1]), that is, the function on  $[0, \infty)^M$  defined through

$$\ell_M(x) := -\log \mathbb{P}(X_t \leq 1/x_t, t \in M) = \int_{S_M} \left( \bigvee_{t \in M} a_t x_t \right) H_M(da), \quad (4)$$

- the (finite dimensional) dependency set  $\mathcal{K}_M$  (cf. [19]), that is, the largest compact convex set  $\mathcal{K}_M \subset [0, \infty)^M$  satisfying

$$\ell_M(x) = \sup\{\langle x, y \rangle : y \in \mathcal{K}_M\} \quad \forall x \in [0, \infty)^M. \quad (5)$$

In order to be a valid finite dimensional spectral measure of a simple max-stable random vector  $\{X_t\}_{t \in M}$ , the only constraint that a Radon measure  $H_M$  on  $S_M$  has to satisfy is that

$$\int_{S_M} a_t H_M(da) = 1$$

for each  $t \in M$ . This ensures standard Fréchet marginals.

Given a simple max-stable process  $X$  on  $T$ , we may assign to a non-empty finite subset  $A \subset T$  the *extremal coefficient*  $\theta(A)$  (cf. [26, 28]), that is

$$\theta(A) := \lim_{x \rightarrow \infty} \frac{\log \mathbb{P}(\bigvee_{t \in A} X_t \leq x)}{\log \mathbb{P}(X_t \leq x)} = \int_{S_M} \left( \bigvee_{t \in A} a_t \right) H_M(da) = \ell_M(\mathbf{1}_A). \quad (6)$$

Indeed, the expression  $\log \mathbb{P}(\bigvee_{t \in A} X_t \leq x) / \log \mathbb{P}(X_t \leq x)$  does not depend on  $x$  and equals the right-hand side (r.h.s.) for  $A \subset M$ . Observe that  $\theta(A)$  takes values in the interval  $[1, |A|]$ , where the value 1 corresponds to full dependence of the random variables  $\{X_t\}_{t \in A}$  and the value  $|A|$  (number of elements in  $A$ ) corresponds to independence. Roughly speaking, the extremal coefficient  $\theta(A)$  detects the effective number of independent variables among the random variables  $\{X_t\}_{t \in A}$ . It is coherent to set  $\theta(\emptyset) := 0$  to obtain a function  $\theta$  on  $\mathcal{F}(T)$ , the *set of finite subsets of  $T$* . We call the function

$$\theta : \mathcal{F}(T) \rightarrow [0, \infty)$$

*extremal coefficient function (ECF)* of the process  $X$ . The set of all ECFs of simple max-stable processes on a set  $T$  will be denoted by

$$\Theta(T) = \{\theta : \mathcal{F}(T) \rightarrow [0, \infty) : \theta \text{ is an ECF of a simple max-stable process on } T\}. \quad (7)$$

**Example 1.** The simplest ECFs are the functions  $\theta(A) = |A|$  corresponding to a process of independent random variables, and the indicator function  $\theta(A) = \mathbb{1}_{A \neq \emptyset}$  corresponding to a process of identical random variables.

Rather sophisticated examples of ECFs can be obtained using spectral representations  $(\Omega, \mathcal{A}, \nu, V)$  of processes  $X$ . In these terms the ECF  $\theta$  of a process  $X$  is given by

$$\theta(A) = \int_{\Omega} \left( \bigvee_{t \in A} V_t(\omega) \right) \nu(d\omega) \quad (8)$$

for  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$  and  $\theta(\emptyset) = 0$ .

**Example 2 (Mixed Moving Maxima – M3 process).** Consider the simple max-stable stationary process  $X$  on  $\mathbb{R}^d$  that is given through the following spectral representation  $(\Omega, \mathcal{A}, \nu, V)$ :

- $(\Omega, \mathcal{A}, \nu) = (\mathbb{F} \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d), \mu \otimes dz)$ , where  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dz)$  denotes the Lebesgue-measure on the Borel- $\sigma$ -algebra of  $\mathbb{R}^d$  and where  $(\mathbb{F}, \mathcal{F}, \mu)$  denotes a measure space of non-negative measurable functions on  $\mathbb{R}^d$  with  $\int_{\mathbb{F}} (\int_{\mathbb{R}^d} f(z) dz) \mu(df) = 1$ ,
- $V_t((f, z)) = f(t - z)$  for  $t \in \mathbb{R}^d$ ,

then we call  $X$  a *Mixed Moving Maxima process (M3 process)* (cf. also [17, 25, 29, 30]). Because of (8) the ECF  $\theta$  of a Mixed Moving Maxima process  $X$  can be computed as

$$\theta(A) = \int_{\mathbb{F}} \int_{\mathbb{R}^d} \left( \bigvee_{t \in A} f(t - z) \right) dz \mu(df)$$

for  $A \in \mathcal{F}(\mathbb{R}^d) \setminus \{\emptyset\}$  and  $\theta(\emptyset) = 0$ . In case  $\mu$  is a point mass at an indicator function  $f$  the bivariate coefficient  $\theta(\{s, t\})$  will be given by  $\theta(\{s, t\}) = 2 - f * \check{f}(s - t)$ , where  $f * \check{f}$  means the convolution of  $f$  with  $\check{f}$  and  $f(t) = f(-t)$ .

**Example 3 (Brown–Resnick process).** Consider the simple max-stable stationary process  $X$  on  $\mathbb{R}^d$  that is given through the following spectral representation  $(\Omega, \mathcal{A}, \nu, V)$ :

- $(\Omega, \mathcal{A}, \nu)$  denotes the probability space of a Gaussian process  $W$  on  $\mathbb{R}^d$  with stationary increments and variogram  $\gamma(z) = \mathbb{E}(W_z - W_o)^2$  for  $z \in \mathbb{R}^d$ .
- $V_t(\omega) = \exp(W_t(\omega) - \sigma^2(t)/2)$  for  $t \in \mathbb{R}^d$ , where  $\sigma^2(t)$  denotes the variance of  $W_t$ ,

then we call  $X$  a *Brown–Resnick process* (cf. [16]). Because of (8) the ECF  $\theta$  of a Brown–Resnick process  $X$  is

$$\theta(A) = \mathbb{E} \exp \left( \bigvee_{t \in A} W_t - \sigma^2(t)/2 \right)$$

for  $A \in \mathcal{F}(\mathbb{R}^d) \setminus \{\emptyset\}$  and  $\theta(\emptyset) = 0$ . Since the f.d.d. of  $X$  only depend on the variogram  $\gamma$ , the extremal coefficient  $\theta(A)$  will also depend only on the values  $\{\gamma(s - t)\}_{s, t \in A}$ . In particular, we have  $\theta(\{s, t\}) = 1 + \operatorname{erf}(\sqrt{\gamma(s - t)}/8)$  for the bivariate coefficient  $\theta(\{s, t\})$ , where  $\operatorname{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2} dt$  denotes the error function (cf. [16]). In case the variogram equals  $\gamma(z) = \lambda \|z\|^2$  for some  $\lambda > 0$ , explicit expressions for multivariate coefficients of higher orders up to  $d + 1$  can be found in [12].

## 2.2. A consistent max-linear model

A multivariate simple max-stable distribution is called *max-linear* (or *spectrally discrete*) if it arises as the distribution of a random vector  $X$  of the following form

$$X_i = \bigvee_{j=1}^q a_{ij} Z_j, \quad i = 1, \dots, p,$$

where  $Z = \{Z_j\}_{j=1}^q$  is a vector of i.i.d. unit Fréchet random variables and where  $\{a_{ij}\}_{p \times q}$  is a matrix of non-negative entries with  $\sum_{j=1}^q a_{ij} = 1$  for each row  $i = 1, \dots, p$ . This is equivalent to requiring the spectral measure  $H_M$  from (3) for  $M = \{1, \dots, \dots, p\}$  to be the following *discrete* measure on  $S_M$

$$H_M = \sum_{j=1}^q \|a_j\| \delta_{a_j/\|a_j\|},$$

where  $a_j$  denote the column vectors of the matrix  $\{a_{ij}\}_{p \times q}$ . Conversely, any discrete spectral measure of a simple max-stable random vector gives rise to such a matrix. Surely, the ECF of such a random vector  $X = \{X_i\}_{i \in M}$  is

$$\theta(A) = \sum_{j=1}^q \bigvee_{i \in A} a_{ij} \quad (9)$$

for  $\emptyset \neq A \subset M$  and  $\theta(\emptyset) = 0$  (cf. (6)).

In [26], the authors introduce a max-linear model for  $X^* = \{X_i^*\}_{i \in M}$  where the column index  $j$  ranges over all non-empty subsets  $L$  of  $M$  and where non-negative coefficients  $\tau_L$  are given for each column  $\emptyset \neq L \subset M$ , more precisely

$$X_i^* = \bigvee_{\emptyset \neq L \subset M} a_{i,L} Z_L, \quad i \in M, \text{ with } a_{i,L} = \tau_L \mathbb{1}_{i \in L},$$

which is equivalent to

$$X_i^* = \bigvee_{i \in L \subset M} \tau_L Z_L, \quad i \in M. \quad (10)$$

The model (10) is simple if and only if  $\sum_{\emptyset \neq L \subset M} a_{i,L} = \sum_{L \subset M: i \in L} \tau_L = 1$  for each  $i \in M$ . It follows from (9) that the ECF of model (10) is

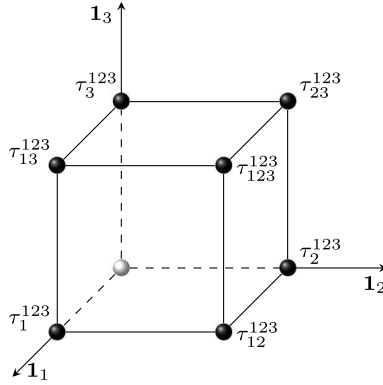
$$\theta(A) = \sum_{L \subset M: A \cap L \neq \emptyset} \tau_L$$

for  $\emptyset \neq A \subset M$  and  $\theta(\emptyset) = 0$ . Now, the interesting aspect of this model (10) with given coefficients  $\tau_L$  is that such models are in 1–1 correspondence with ECFs  $\theta$  on the finite set  $M$  (cf. [26], Theorems 3 and 4). Alongside, this leads to a set of inequalities which fully characterizes the set of ECFs  $\Theta(M)$  for finite sets  $M$  (cf. [26], Corollary 5). An alternative proof for these inequalities is offered in [19], Corollary 1, and it is noticed therein that they are equivalent to a property called complete alternation (see below).

As we seek a spatial generalization of these results, let us consider a max-stable processes  $X^* = \{X_t^*\}_{t \in T}$  on an arbitrary index set  $T$ , whose f.d.d. for a finite set  $M$  are precisely of the above form (10), where the coefficients  $\tau_L$  now additionally depend on  $M$ . That means we set the spectral measure  $H_M^*$  of the random vector  $\{X_t^*\}_{t \in M}$

$$H_M^* := \sum_{\emptyset \neq L \subset M} \tau_L^M \|\mathbf{1}_L\| \delta_{\mathbf{1}_L/\|\mathbf{1}_L\|}, \quad (11)$$





**Figure 2.** Spectral measure representation of  $\{X_t^*\}_{t \in M}$  for  $M = \{1, 2, 3\}$  if we choose the reference norm on  $\mathbb{R}^M$  to be the maximum norm. In this case, the spectral measure simplifies to a sum of weighted point masses on the vertices of a cube:  $H_M^* = \sum_{\emptyset \neq L \subset M} \tau_L^M \delta_{\mathbf{1}_L}$ .

such that the f.d.d. of the process  $X^*$  are given by (cf. (3))

$$-\log \mathbb{P}(X_t^* \leq x_t, t \in M) = \sum_{\emptyset \neq L \subset M} \tau_L^M \bigvee_{t \in L} \frac{1}{x_t}. \quad (12)$$

Here  $M$  ranges over all non-empty finite subsets of  $T$ , which we express as  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$ . Figure 2 illustrates this spectral measure for a trivariate distribution where  $M = \{1, 2, 3\}$  in case the reference norm is the maximum norm.

**Lemma 4.** *Let  $T$  be an arbitrary set and let coefficients  $\tau_L^M$  be given for  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$  and  $L \in \mathcal{F}(M) \setminus \{\emptyset\}$ , such that*

- (i)  $\tau_L^M \geq 0$  for all  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$  and  $L \in \mathcal{F}(M) \setminus \{\emptyset\}$ ,
- (ii)  $\tau_L^M = \tau_L^{M \cup \{t\}} + \tau_{L \cup \{t\}}^{M \cup \{t\}}$  for all  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$  and  $L \in \mathcal{F}(M) \setminus \{\emptyset\}$  and  $t \in T \setminus M$ ,
- (iii)  $\tau_{\{t\}}^{\{t\}} = 1$  for all  $t \in T$ .

Then the spectral measures  $\{H_M^*\}_{M \in \mathcal{F}(T) \setminus \{\emptyset\}}$  from (11) define a simple max-stable process  $X^* = \{X_t^*\}_{t \in T}$  on  $T$  with f.d.d. as in (12).

**Proof.** Condition (i) ensures that each spectral measure  $H_M^*$  defines a max-stable distribution with Fréchet marginals. Subsequently, condition (ii) ensures consistency of these distributions (i.e., the conditions for Kolmogorov's extension theorem are satisfied). Hence the spectral measures  $H_M^*$  define a max-stable process  $X^*$  on  $T$ . Finally, condition (iii) ensures that the process  $X^*$  has standard Fréchet marginals.  $\square$

**Remark 5.** Condition (ii) is equivalent to

$$\tau_K^A = \sum_{J \subset M \setminus A} \tau_{K \cup J}^M \quad \forall M \in \mathcal{F}(T) \setminus \{\emptyset\}, \emptyset \neq K \subset A \subset M. \quad (13)$$

### 3. The TM process and negative definiteness of ECFs

For the following characterization of the set of ECFs  $\Theta(T)$ , we use the fact that  $\mathcal{F}(T)$ , the set of finite subsets of  $T$ , forms a semigroup with respect to the union operation  $\cup$  and with neutral element the empty set  $\emptyset$ . The following notation is adopted from [18] and [2]. For a function  $f: \mathcal{F}(T) \rightarrow \mathbb{R}$  and elements  $K, L \in \mathcal{F}(T)$ , we set

$$(\Delta_K f)(L) := f(L) - f(L \cup K).$$

**Definition 6 (negative definiteness/ complete alternation).** A function  $\psi: \mathcal{F}(T) \rightarrow \mathbb{R}$  is called negative definite (in the semigroup sense) on  $\mathcal{F}(T)$  if for all  $n \geq 2$ ,  $\{K_1, \dots, K_n\} \subset \mathcal{F}(T)$  and  $\{a_1, \dots, a_n\} \subset \mathbb{R}$  with  $\sum_{j=1}^n a_j = 0$

$$\sum_{j=1}^n \sum_{k=1}^n a_j a_k \psi(K_j \cup K_k) \leq 0.$$

A function  $\psi: \mathcal{F}(T) \rightarrow \mathbb{R}$  is called completely alternating on  $\mathcal{F}(T)$  if for all  $n \geq 1$ ,  $\{K_1, \dots, K_n\} \subset \mathcal{F}(T)$  and  $K \in \mathcal{F}(T)$

$$(\Delta_{K_1} \Delta_{K_2} \cdots \Delta_{K_n} \psi)(K) = \sum_{I \subset \{1, \dots, n\}} (-1)^{|I|} \psi\left(K \cup \bigcup_{i \in I} K_i\right) \leq 0.$$

Because the semigroup  $(\mathcal{F}(T), \cup, \emptyset)$  is idempotent, these two terms coincide. That means  $\psi: \mathcal{F}(T) \rightarrow \mathbb{R}$  is completely alternating if and only if  $\psi$  is negative definite (in the semigroup sense), cf. [2], 4.4.16.

**Example 7 ([18], page 52).** An important example of a negative definite (completely alternating) function on  $\mathcal{F}(T)$  is the capacity functional  $C: \mathcal{F}(T) \rightarrow \mathbb{R}$  of a binary process  $Y = \{Y_t\}_{t \in T}$  with values in  $\{0, 1\}$ , which is given by  $C(\emptyset) = 0$  and

$$C(A) = \mathbb{P}(\exists t \in A \text{ such that } Y_t = 1).$$

Now, we are in position to characterize the set  $\Theta(T)$  of possible ECFs on  $\mathcal{F}(T)$  and to define a corresponding max-linear process  $X^*$ .

**Theorem 8.**

- (a) The function  $\theta: \mathcal{F}(T) \rightarrow \mathbb{R}$  is the ECF of a simple max-stable process on  $T$  if and only if the following conditions are satisfied:

- (i)  $\theta$  is negative definite,
  - (ii)  $\theta(\emptyset) = 0$ ,
  - (iii)  $\theta(\{t\}) = 1$  for all  $t \in T$ .
- (b) If these conditions are satisfied, the following choice of coefficients

$$\tau_L^M := -\Delta_{\{t_1\}} \cdots \Delta_{\{t_l\}} \theta(M \setminus L) = \sum_{I \subset L} (-1)^{|I|+1} \theta((M \setminus L) \cup I)$$

$$\forall M \in \mathcal{F}(T) \setminus \{\emptyset\}, \emptyset \neq L = \{t_1, \dots, t_l\} \subset M$$

for model (11) defines a simple max-stable process  $X^*$  on  $T$  which realizes  $\theta$  as its own ECF  $\theta^*$ .

**Definition 9 (Tawn–Molchanov process (TM process)).** Referring to the previous work in [6, 19, 26], we will call the simple max-stable process  $X^*$  from Theorem 8(b) Tawn–Molchanov process (TM process) henceforth.

**Proof of Theorem 8.** If  $\theta$  is an ECF of a simple max-stable process  $X$  on  $T$ , then necessarily  $\theta(\emptyset) = 0$  and  $\theta(\{t\}) = 1$  for all  $t \in T$  (cf. (6)). Further, it is an application of l'Hôpitals rule that for  $A \subset \mathcal{F}(T) \setminus \{\emptyset\}$

$$\begin{aligned} \theta(A) &= \lim_{x \rightarrow \infty} \frac{-\log \mathbb{P}(\bigvee_{t \in A} X_t \leq x)}{-\log \mathbb{P}(X_t \leq x)} = \lim_{x \rightarrow \infty} \frac{1 - \mathbb{P}(\bigvee_{t \in A} X_t \leq x)}{1 - \mathbb{P}(X_t \leq x)} \\ &= \lim_{x \rightarrow \infty} \frac{\mathbb{P}(\exists t \in A \text{ such that } X_t \geq x)}{\mathbb{P}(X_t \geq x)} = \lim_{x \rightarrow \infty} \frac{C^{(x)}(A)}{p^{(x)}}, \end{aligned} \tag{14}$$

where  $C^{(x)}$  denotes the capacity functional for the binary process  $Y_t = \mathbb{1}_{X_t \geq x}$  and  $p^{(x)} = \mathbb{E}Y_t = 1 - e^{-1/x}$ . Since negative definiteness respects scaling and pointwise limits, negative definiteness of  $\theta$  follows from Example 7. This shows the necessity of (i)–(iii).

Now, let  $\theta: \mathcal{F}(T) \rightarrow \mathbb{R}$  be a function satisfying conditions (i)–(iii) and let the coefficients  $\tau_L^M$  be given as above. We need to check that they fulfill the (in)equalities from Lemma 4. Indeed we have:

- The inequalities  $\tau_L^M = -\Delta_{\{t_1\}} \cdots \Delta_{\{t_l\}} \theta(M \setminus L) \geq 0$  follow directly from the complete alternation of  $\theta$  that is equivalent to (i).
- From the definition of  $\Delta_{\{t\}}$  we observe

$$\begin{aligned} \tau_{L \cup \{t\}}^{M \cup \{t\}} &= -\Delta_{\{t\}} \Delta_{\{t_1\}} \cdots \Delta_{\{t_l\}} \theta((M \cup \{t\}) \setminus (L \cup \{t\})) \\ &= -\Delta_{\{t_1\}} \cdots \Delta_{\{t_l\}} \theta(M \setminus L) + \Delta_{\{t_1\}} \cdots \Delta_{\{t_l\}} \theta(M \cup \{t\} \setminus L) \\ &= \tau_L^M - \tau_L^{M \cup \{t\}}. \end{aligned}$$

- For  $t \in T$ , we have  $\tau_{\{t\}}^{\{t\}} = \theta(\{t\}) = 1$  because of (iii).

Thus, the coefficients  $\tau_L^M$  define a simple max-stable process  $X^*$  on  $T$  as given by model (11). Finally, we compute the ECF  $\theta^*$  of  $X^*$  and see that it coincides with  $\theta$ : For the empty set, we have  $\theta^*(\emptyset) = 0 = \theta(\emptyset)$  because of (ii); otherwise we compute for  $A \subset \mathcal{F}(T) \setminus \{\emptyset\}$  that

$$\begin{aligned} \theta^*(A) &\stackrel{(6),(11)}{=} \sum_{\emptyset \neq L \subset A} \tau_L^A = \sum_{\emptyset \neq L \subset A} \sum_{I \subset L} (-1)^{|I|+1} \theta((A \setminus L) \cup I) \\ &= \sum_{\emptyset \neq K \subset A} \theta(K) \sum_{\substack{\emptyset \neq L \subset A \\ A \setminus L \subset K}} (-1)^{|K \cap L|+1} = \sum_{\emptyset \neq K \subset A} \theta(K) (-(-\mathbb{1}_{K=A})) = \theta(A). \end{aligned}$$

This shows sufficiency of (i)–(iii) and part (b).  $\square$

Theorem 8 is in analogy to the following standard result for Gaussian processes (as illustrated in the sketches in the [Introduction](#)):

- (a) A function  $C: T \times T \rightarrow \mathbb{R}$  is a covariance function if and only if it is positive definite.
- (b) If  $C: T \times T \rightarrow \mathbb{R}$  is positive definite, we may choose a (zero mean) Gaussian process which realizes  $C$  as its own covariance function.

Both statements are intrinsically tied together. When proving them by means of Kolmogorov’s extension theorem, one proceeds in the same manner as we did for Theorem 8. The necessity of positive definiteness of covariance functions is easily derived even for the bigger class of square-integrable processes, whilst sufficiency can be established by showing that Gaussian processes can realize any positive definite function as covariance function. In some points (such as continuity relations), this analogy will be deepened. Other aspects (such as the exceptional role of dependency sets in Section 5) seem unsuitable for a direct comparison.

**Remark 10.** In order to incorporate stationarity w.r.t. some group  $G$  acting on  $T$  (for example,  $\mathbb{R}^d$  acting on  $\mathbb{R}^d$  by translation), we just have to add the following condition (iv)  $\theta(gA) = \theta(A)$  for all  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$  and for all  $g \in G$ . Then the process  $X^*$  will be stationary w.r.t. this group action.

**Remark 11.** Instead of requiring the max-stable processes in Theorem 8 to have *standard* Fréchet marginals everywhere, we can admit a different scale at different locations, that is,  $\mathbb{P}(X_t \leq x) = \exp(-s_t/x)$  for a positive scaling parameter  $s_t$  for  $t \in T$ . In that case Theorem 8 holds true without condition (iii) and the word “simple”. To make sense of the ECF as in (6) in this case, either use a reference point  $t \in T$  or set  $\log \mathbb{P}(X_t \leq x) = -1/x$  in the denominator. Beware of that the ECF  $\theta$  cannot be interpreted as the number of independent variables anymore in this case.

**Remark 12.** In [26], the last issue of the proof is derived for finite sets  $T$  by a Moebius inversion. The relation to the proof therein becomes more transparent if we compute

$\theta^*(A)$  for  $A \subset M$  from the coefficients  $\{\tau_L^M\}_{\emptyset \neq L \subset M}$  for arbitrary  $M \supset A$  instead of  $M = A$ :

$$\theta^*(A) \stackrel{(6),(11)}{=} \sum_{\emptyset \neq K \subset A} \tau_K^A \stackrel{(13)}{=} \sum_{\emptyset \neq K \subset A} \sum_{J \subset M \setminus A} \tau_{K \cup J}^M = \sum_{L \subset M: L \cap A \neq \emptyset} \tau_L^M. \quad (15)$$

## 4. Direct consequences of Theorem 8

Here, we collect some direct consequences of the above Theorem 8. Therefore, note that the first part of Theorem 8 can also be expressed as (cf. (7))

$$\Theta(T) = \{\theta: \mathcal{F}(T) \rightarrow [0, \infty) : \theta \text{ is negative definite, } \theta(\emptyset) = 0, \theta(\{t\}) = 1 \text{ for } t \in T\}. \quad (16)$$

### 4.1. Convexity and compactness

**Corollary 13.** *The set of ECFs  $\Theta(T)$  is convex.*

**Proof.** This can be seen directly from (16) since all involved properties are compatible with convex combinations. As a constructive argument, use the fact that the ECF of the max-combination  $\alpha X \vee (1 - \alpha)Y$  of two independent simple max-stable processes  $X$  and  $Y$  on  $T$  is the convex combination of their ECFs for  $\alpha \in (0, 1)$ .  $\square$

**Corollary 14.** *The set of ECFs  $\Theta(T)$  is compact w.r.t. the topology of pointwise convergence.*

**Proof.** The topology of pointwise convergence on  $\mathbb{R}^{\mathcal{F}(T)}$  is the product topology. Since  $\theta(\emptyset) = 0$  and  $\theta(A) \in [1, |A|]$  for  $\theta \in \Theta(T)$  and  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$ , the set  $\Theta(T)$  is a subset of the product space

$$\{0\} \times \prod_{A \in \mathcal{F}(T) \setminus \{\emptyset\}} [1, |A|],$$

which is compact by Tychonoff's theorem. Moreover, since elements of  $\Theta(T)$  are completely characterized by finite dimensional equalities and inequalities involving  $\leq$  only (stemming from (16)), the set  $\Theta(T)$  is closed. Hence,  $\Theta(T)$  is compact.  $\square$

**Remark 15.** Note that even though we say “the topology of pointwise convergence”, the “points” meant here are indeed elements of  $\mathcal{F}(T)$ , that is, finite subsets of  $T$ . In particular it follows from the compactness of  $\Theta(T)$  that  $\Theta(T)$  is sequentially closed. That means if  $(\theta_n)_{n \in \mathbb{N}}$  is a sequence of ECFs such that  $\theta_n(A)$  converges to some value  $f(A)$  for each  $A \in \mathcal{F}(T)$ , then  $f$  is an ECF.

## 4.2. Spectral representation of the TM process

Another consequence of Theorem 8 is that ECFs allow for an *integral representation* as a mixture of functions  $A \mapsto \mathbb{1}_{A \cap Q \neq \emptyset}$ , where  $Q$  is from the power set of  $T$ . To be more precise, let us denote the power set of  $T$  by  $\mathcal{P}(T)$  and consider the topology on  $\mathcal{P}(T)$  that is generated by the maps  $Q \mapsto \mathbb{1}_{A \cap Q \neq \emptyset}$  for  $A \in \mathcal{F}(T)$  or equivalently (since  $\mathcal{F}(T)$  is generated by the singletons  $\{\{t\}\}_{t \in T}$ ) the topology on  $\mathcal{P}(T)$  that is generated by the maps  $Q \mapsto \mathbb{1}_{t \in Q}$  for  $t \in T$ . Identifying  $\mathcal{P}(T)$  with  $\{0, 1\}^T$ , this space is also known as *Cantor cube*. As in [2], Definition 2.1.1, a measure  $\mu$  on the Borel- $\sigma$ -algebra of  $\mathcal{P}(T)$  w.r.t. this topology will be called *Radon measure* if  $\mu$  is finite on compact sets and  $\mu$  is inner regular.

**Corollary 16.** *Let  $\theta \in \Theta(T)$  be an ECF. Then  $\theta$  uniquely determines a positive Radon measure  $\mu$  on  $\mathcal{P}(T) \setminus \{\emptyset\}$  such that*

$$\theta(A) = \mu(\{Q \in \mathcal{P}(T) \setminus \{\emptyset\} : A \cap Q \neq \emptyset\}) = \int_{\mathcal{P}(T) \setminus \{\emptyset\}} \mathbb{1}_{A \cap Q \neq \emptyset} \mu(dQ),$$

where  $\theta(\{t\}) = 1$  for  $t \in T$ . The function  $\theta$  is bounded if and only if  $\mu(\mathcal{P}(T) \setminus \{\emptyset\}) < \infty$ .

**Proof.** Since  $\theta$  is negative definite (Theorem 8) and  $\mathcal{F}(T)$  is idempotent, we may apply [2], Proposition 4.4.17. It says that  $\theta$  uniquely determines a positive Radon measure  $\tilde{\mu}$  on  $\widehat{\mathcal{F}(T)} \setminus \{1\}$ , where  $\widehat{\mathcal{F}(T)}$  denotes the dual semigroup of  $\mathcal{F}(T)$  (cf. [2], 4.2.1 and 4.4.16), such that  $\theta(A) = \tilde{\mu}(\{\rho \in \widehat{\mathcal{F}(T)} \setminus \{1\} \mid \rho(A) = 0\})$ . The function  $\theta$  is bounded if and only if  $\tilde{\mu}(\widehat{\mathcal{F}(T)} \setminus \{1\}) < \infty$ .

Now, it can be easily seen that semicharacters on  $\mathcal{F}(T)$  are in a 1–1 correspondence with subsets of  $T$  via  $\widehat{\mathcal{F}(T)} \ni \rho \rightarrow \{t \in T : \rho(\{t\}) = 0\} \in \mathcal{P}(T)$  and  $\mathcal{P}(T) \ni Q \rightarrow \mathbb{1}_{(\cdot) \cap Q = \emptyset} \in \widehat{\mathcal{F}(T)}$ . Here the constant function 1 corresponds to the empty set. Moreover, the topology considered on  $\widehat{\mathcal{F}(T)}$  is the topology of pointwise convergence. Transported to  $\mathcal{P}(T)$  this is the topology generated by the maps  $Q \mapsto \mathbb{1}_{A \cap Q \neq \emptyset}$  for  $A \in \mathcal{F}(T)$ . Let  $\mu$  denote the Radon measure  $\tilde{\mu}$  transported to  $\mathcal{P}(T) \setminus \{\emptyset\}$ . Then the corollary follows.  $\square$

**Remark 17.** In case  $T = M$  is finite, we have that  $\mathcal{P}(M) = \mathcal{F}(M)$  carries the discrete topology and

$$\theta(A) = \mu(\{Q \in \mathcal{F}(M) \setminus \{\emptyset\} : A \cap Q \neq \emptyset\}) = \sum_{Q \in \mathcal{F}(M) \setminus \{\emptyset\}} \mu(\{Q\}) \mathbb{1}_{A \cap Q \neq \emptyset}.$$

A comparison with (15) reveals that  $\mu(\{Q\}) = \tau_Q^M$ . In this sense, the coefficients  $\tau_Q^M$  of the max-linear model (11) can be interpreted as finite dimensional “Fourier coefficients” of the negative definite function  $\theta$ .

The integral representation of the ECF  $\theta$  also yields a spectral representation for the corresponding TM process  $X^*$ .

**Theorem 18.** *The TM process  $X^* = \{X_t^*\}_{t \in T}$  with ECF  $\theta$  has the following spectral representation  $(\Omega, \mathcal{A}, \nu, V)$  (cf. (1)):*

- $(\Omega, \mathcal{A}, \nu)$  is the measure space  $(\mathcal{P}(T), \mathcal{B}(\mathcal{P}(T)), \mu)$  from Corollary 16,
- $V_t(Q) = \mathbb{1}_{t \in Q}$ .

**Proof.** We need to check that the f.d.d. of  $X^*$  satisfy (2). The f.d.d. of  $X^*$  are given by (cf. (12))

$$-\log \mathbb{P}(X_t^* \leq x_t, t \in M) = \sum_{\emptyset \neq L \subset M} \tau_L^M \bigvee_{t \in L} \frac{1}{x_t},$$

where the coefficients  $\tau_L^M$  can be computed from the ECF  $\theta$  as in Theorem 8(b) and  $\theta$  satisfies the integral representation from Corollary 16, that is,

$$\begin{aligned} \tau_L^M &= \sum_{I \subset L} (-1)^{|I|+1} \theta((M \setminus L) \cup I) \\ &= \sum_{I \subset L} (-1)^{|I|+1} \int_{\mathcal{P}(T) \setminus \{\emptyset\}} \mathbb{1}_{((M \setminus L) \cup I) \cap Q \neq \emptyset} \mu(dQ). \end{aligned}$$

Using the identity

$$\begin{aligned} &\sum_{I \subset L} (-1)^{|I|+1} \mathbb{1}_{((M \setminus L) \cup I) \cap Q \neq \emptyset} \\ &= \sum_{I \subset L} (-1)^{|I|+1} (\mathbb{1}_{(M \setminus L) \cap Q \neq \emptyset} + \mathbb{1}_{I \cap Q \neq \emptyset} - \mathbb{1}_{(M \setminus L) \cap Q \neq \emptyset} \mathbb{1}_{I \cap Q \neq \emptyset}) \\ &= 0 \cdot \mathbb{1}_{(M \setminus L) \cap Q \neq \emptyset} + (1 - \mathbb{1}_{(M \setminus L) \cap Q \neq \emptyset}) \sum_{I \subset L} (-1)^{|I|+1} \mathbb{1}_{I \cap Q \neq \emptyset} \\ &= \mathbb{1}_{(M \setminus L) \cap Q = \emptyset} \mathbb{1}_{L \subset Q} = \mathbb{1}_{L=M \cap Q}, \end{aligned}$$

we obtain that

$$\tau_L^M = \int_{\mathcal{P}(T) \setminus \{\emptyset\}} \mathbb{1}_{L=M \cap Q} \mu(dQ).$$

It follows that the f.d.d. of  $X^*$  satisfy

$$\begin{aligned} -\log \mathbb{P}(X_t^* \leq x_t, t \in M) &= \int_{\mathcal{P}(T) \setminus \{\emptyset\}} \sum_{\emptyset \neq L \subset M} \mathbb{1}_{L=M \cap Q} \bigvee_{t \in L} \frac{1}{x_t} \mu(dQ) \\ &= \int_{\mathcal{P}(T) \setminus \{\emptyset\}} \bigvee_{t \in M} \frac{\mathbb{1}_{t \in Q}}{x_t} \mu(dQ) = \int_{\Omega} \left( \bigvee_{t \in M} \frac{V_t(\omega)}{x_t} \right) \nu(d\omega) \end{aligned}$$

as desired. This finishes the proof.  $\square$

### 4.3. Triangle inequalities and operation of Bernstein functions

In [7], Proposition 4, it is shown that an ECF  $\theta$  on  $\mathcal{F}(T)$  satisfies the following bivariate inequalities for  $r, s, t \in T$ :

$$\begin{aligned} \theta(\{s, t\}) &\leq \theta(\{s, r\})\theta(\{r, t\}), \\ \theta(\{s, t\})^\alpha &\leq \theta(\{s, r\})^\alpha + \theta(\{r, t\})^\alpha - 1, & 0 < \alpha \leq 1, \\ \theta(\{s, t\})^\alpha &\geq \theta(\{s, r\})^\alpha + \theta(\{r, t\})^\alpha - 1, & \alpha \leq 0. \end{aligned}$$

These inequalities have in common, that they are in fact triangle inequalities of the form

$$g \circ \eta(\{s, t\}) \leq g \circ \eta(\{s, r\}) + g \circ \eta(\{r, t\}), \quad (17)$$

if we rewrite them in terms of  $\eta := \theta - 1$  and

$$\begin{aligned} g(x) &= \log(1 + x), \\ g(x) &= (1 + x)^\alpha - 1, & 0 < \alpha \leq 1, \\ g(x) &= 1 - (1 + x)^\alpha, & \alpha \leq 0. \end{aligned}$$

These functions  $g$  have in common that they are in fact *Bernstein functions*.

**Definition 19 (Bernstein function).** *A function  $g: [0, \infty) \rightarrow [0, \infty)$  is called a Bernstein function if one of the following equivalent conditions is satisfied (cf. [2], 4.4.3 and page 141)*

- (i) *The function  $g$  is of the form*

$$g(r) = c + br + \int_0^\infty (1 - e^{-\lambda r}) \nu(d\lambda),$$

*where  $c, b \geq 0$  and  $\nu$  is a positive Radon measure on  $(0, \infty)$  with  $\int_0^\infty \frac{\lambda}{1+\lambda} \nu(d\lambda) < \infty$ .*

- (ii) *The function  $g$  is continuous and  $g \in C^\infty((0, \infty))$  with  $g \geq 0$  and  $(-1)^n g^{(n+1)} \geq 0$  for all  $n \geq 0$ . (Here,  $g^{(n)}$  denotes the  $n$ th derivative of  $g$ .)*
- (iii) *The function  $g$  is continuous,  $g \geq 0$  and  $g$  is negative definite as a function on the semigroup  $([0, \infty), +, 0)$ .*

For a comprehensive treatise on Bernstein functions including a table of examples, see [24]. Bernstein functions play already an important role in the construction of advanced Gaussian processes by generating novel covariance functions from given ones, cf. [32] and [21]. Here, we see that they are equally useful for generating new ECFs from given ECFs and correspondingly new Tawn–Molchanov processes from given ones.



**Corollary 20.** *Let  $T$  be a set and  $\theta \in \Theta(T)$  an ECF. Let  $g$  be a Bernstein function which is not constant. Then the function on  $\mathcal{F}(T)$*

$$A \mapsto \frac{g(\theta(A)) - g(0)}{g(1) - g(0)}$$

*is again an ECF in  $\Theta(T)$ .*

**Proof.** The result is immediate from Theorem 8, since Bernstein functions operate on negative definite kernels (cf. [2], 3.2.9 and 4.4.3).  $\square$

For instance, if  $\theta$  is an ECF, then also  $\log(1+\theta)/\log(2)$  or  $((\theta+a)^q - a^q)/((1+a)^q - a^q)$  are ECFs for  $0 < q < 1$  and  $a \geq 0$ . Finally, we show that (17) holds true for arbitrary Bernstein functions. In fact, the result of [7], Proposition 4, can be generalized to the following extent as a corollary to Theorem 8.

**Corollary 21.** *Let  $\theta \in \Theta(T)$  be an ECF. Set  $\eta := \theta - 1$  and let  $g$  be a Bernstein function. Then we have for  $A, B, C \in \mathcal{F}(T) \setminus \{\emptyset\}$  that*

$$g \circ \eta(A \cup B) \leq g \circ \eta(C) + g \circ \eta(A \cup B) \leq g \circ \eta(A \cup C) + g \circ \eta(C \cup B).$$

**Proof.** Since  $\theta$  is an ECF, it is negative definite (cf. Theorem 8). Subtracting 1 does not change this property. Notice further that  $\theta$  takes values in  $\{0\} \cup [1, \infty)$ , where the value 0 is only attained for the empty set  $\emptyset$  (the neutral element of  $\mathcal{F}(T)$ ). Thus, the function  $\eta := \theta - 1 : \mathcal{F}(T) \setminus \{\emptyset\} \rightarrow \mathbb{R}$  is negative definite and takes values only in  $[0, \infty)$ . Applying a Bernstein function  $g$  does not change this property (cf. [2], 3.2.9 and 4.4.3). By [2], 8.2.7, this also means that  $f := g \circ \eta : \mathcal{F}(T) \setminus \{\emptyset\} \rightarrow \mathbb{R}$  is negative definite on  $\mathcal{F}(T) \setminus \{\emptyset\}$ . Since we have also  $f \geq 0$  on  $\mathcal{F}(T) \setminus \{\emptyset\}$ , we may derive for  $A, B, C \in \mathcal{F}(T) \setminus \{\emptyset\}$

$$\begin{aligned} & f(C) + f(A \cup B) - f(A \cup C) - f(C \cup B) \\ &= (f(C) - f(A \cup C) - f(C \cup B) + f(A \cup B \cup C)) + (f(A \cup B) - f(A \cup B \cup C)) \\ &= \Delta_A \Delta_B f(C) + \Delta_C f(A \cup B) \leq 0 \end{aligned}$$

as desired. This finishes the proof.  $\square$

#### 4.4. Stochastic continuity

In this section, we require  $T$  to be a metric space. We need to define the notion of continuity that we will use in connection with ECFs  $\theta : \mathcal{F}(T) \rightarrow [0, \infty)$ . Therefore, let  $f : \mathcal{F}(T) \rightarrow \mathbb{R}$  be a function on the finite subsets of  $T$ . Then  $f$  induces a family of functions  $\{f^{(m)}\}_{m \geq 0}$  where  $f^{(m)} : T^m \rightarrow \mathbb{R}$  is given by

$$f^{(m)}(t_1, \dots, t_m) = f(\{t_1, \dots, t_m\}).$$

**Definition 22.** Let  $f: \mathcal{F}(T) \rightarrow \mathbb{R}$  be a function on the finite subsets of a metric space  $T$ . We say that  $f$  is continuous if all induced functions  $f^{(m)}: T^m \rightarrow \mathbb{R}$  are continuous for all  $m \geq 0$ , where  $T^m$  is endowed with the product topology.

**Lemma 23.** Let  $X = \{X_t\}_{t \in T}$  be a simple max-stable process with ECF  $\theta$ . Then the following implication holds:

$$X \text{ is stochastically continuous} \implies \theta \text{ is continuous.}$$

**Proof.** Stochastic continuity of  $X$  means that for any  $\varepsilon > 0$ , for any  $t \in T$  and sequence  $t^{(n)} \rightarrow t$  we have  $\mathbb{P}(|X_{t^{(n)}} - X_t| > \varepsilon) \rightarrow 0$ . From this, we can easily derive that for any  $\varepsilon > 0$ , any  $m \in \mathbb{N}$ , any  $(t_1, \dots, t_m) \in T^m$  and a sequence  $(t_1^{(n)}, \dots, t_m^{(n)}) \rightarrow (t_1, \dots, t_m)$ , also  $\mathbb{P}(\|(X_{t_i^{(n)}} - X_{t_i})_{i=1}^m\| > \varepsilon) \rightarrow 0$  for any reference norm  $\|\cdot\|$  on  $\mathbb{R}^m$ . The latter implies the corresponding convergence in distribution:  $F_{(t_1^{(n)}, \dots, t_m^{(n)})} \rightarrow F_{(t_1, \dots, t_m)}$ . Since  $\log F_{(t_1, \dots, t_m)}: [0, \infty)^m \rightarrow \mathbb{R}$  is monotone and homogeneous, we have that for  $x > 0$  the point  $(x, \dots, x) \in (0, \infty)^m$  is a continuity point of  $F_{(t_1, \dots, t_m)}$  (cf. [22], page 277). Thus, the induced function  $\theta^{(m)}$  on  $T^m$  is continuous, since  $\theta^{(m)}(t_1, \dots, t_m) = -x \log F_{(t_1, \dots, t_m)}(x, \dots, x)$ . Hence,  $\theta$  is continuous.  $\square$

Second, we prove the following upper bound that shows that stochastic continuity of the TM process  $X^*$  is indeed controlled by the bivariate extremal coefficients.

**Lemma 24.** Let  $X^* = \{X_t^*\}_{t \in T}$  be the TM process with ECF  $\theta$ . Set  $\eta := \theta - 1$ . Then we have for any  $\varepsilon > 0$

$$\mathbb{P}(|X_s^* - X_t^*| > \varepsilon) \leq 2 \left( 1 - \exp\left(-\frac{\eta(\{s, t\})}{\varepsilon}\right) \right) \leq \frac{2}{\varepsilon} \eta(\{s, t\}).$$

**Proof.** Let  $\varepsilon > 0$ . We will prove the statement for  $2\varepsilon$  instead of  $\varepsilon$ . Therefore, consider the following disjoint events on a corresponding probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  for  $k = 0, 1, 2, \dots$

$$A_k := \{\omega \in \Omega : (X_s^*(\omega), X_t^*(\omega)) \in (k\varepsilon, (k+2)\varepsilon]^2 \setminus ((k+1)\varepsilon, (k+2)\varepsilon]^2\}.$$

The disjoint union  $\bigcup_{k=0}^{\infty} A_k$  is a subset of  $\{\omega \in \Omega : |X_s^*(\omega) - X_t^*(\omega)| \leq 2\varepsilon\}$  and so

$$\mathbb{P}(|X_s^* - X_t^*| \leq 2\varepsilon) \geq \mathbb{P}\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} \mathbb{P}(A_k) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \mathbb{P}(A_k).$$

From (12) and Theorem 8, we see that the bivariate distribution of the process  $X^*$  is given by

$$-\log \mathbb{P}(X_s^* \leq x, X_t^* \leq y) = \frac{\eta(\{s, t\})}{x \vee y} + \frac{1}{x \wedge y}. \quad (18)$$

For further calculations, we abbreviate for  $p, q \in \mathbb{N} \cup \{0\}$

$$B(p, q) := \mathbb{P}(X_s^* \leq p \cdot \varepsilon, X_t^* \leq q \cdot \varepsilon).$$

Note that  $B(p, q) = B(q, p)$  and  $B(p, 0) = 0$ . With this notation, we rearrange

$$\sum_{k=0}^n \mathbb{P}(A_k) = -B(n+1, n+1) + 2 \sum_{k=0}^n [B(k+2, k+1) - B(k+2, k)].$$

For the second summand, we have (cf. (18))

$$\begin{aligned} & \sum_{k=0}^n [B(k+2, k+1) - B(k+2, k)] \\ & \stackrel{(18)}{=} \sum_{k=0}^n \left[ \exp\left(-\frac{1}{\varepsilon} \left[ \frac{\eta(\{s, t\})}{k+2} + \frac{1}{k+1} \right]\right) - \exp\left(-\frac{1}{\varepsilon} \left[ \frac{\eta(\{s, t\})}{k+2} + \frac{1}{k} \right]\right) \right] \\ & = \sum_{k=0}^n \exp\left(-\frac{1}{\varepsilon} \left[ \frac{\eta(\{s, t\})}{k+2} \right]\right) \left[ \exp\left(-\frac{1}{(k+1)\varepsilon}\right) - \exp\left(-\frac{1}{k\varepsilon}\right) \right] \\ & \geq \sum_{k=0}^n \exp\left(-\frac{\eta(\{s, t\})}{2\varepsilon}\right) \left[ \exp\left(-\frac{1}{(k+1)\varepsilon}\right) - \exp\left(-\frac{1}{k\varepsilon}\right) \right] \\ & = \exp\left(-\frac{\eta(\{s, t\})}{2\varepsilon}\right) \exp\left(-\frac{1}{(n+1)\varepsilon}\right). \end{aligned}$$

Finally,

$$\begin{aligned} & \mathbb{P}(|X_s^* - X_t^*| > 2\varepsilon) \\ & = 1 - \mathbb{P}(|X_s^* - X_t^*| \leq 2\varepsilon) \leq 1 - \lim_{n \rightarrow \infty} \sum_{k=0}^n \mathbb{P}(A_k) \\ & = 1 + \lim_{n \rightarrow \infty} B(n+1, n+1) - 2 \lim_{n \rightarrow \infty} \sum_{k=0}^n [B(k+2, k+1) - B(k+2, k)] \\ & \leq 1 + \lim_{n \rightarrow \infty} \exp\left(-\frac{\eta(\{s, t\}) + 1}{(n+1)\varepsilon}\right) - 2 \lim_{n \rightarrow \infty} \left( \exp\left(-\frac{\eta(\{s, t\})}{2\varepsilon}\right) \exp\left(-\frac{1}{(n+1)\varepsilon}\right) \right) \\ & = 2 - 2 \exp\left(-\frac{\eta(\{s, t\})}{2\varepsilon}\right) \leq \frac{2}{2\varepsilon} \eta(\{s, t\}). \end{aligned}$$

This finishes the proof.  $\square$

**Theorem 25.** *Let  $X^* = \{X_t^*\}_{t \in T}$  be the TM process with ECF  $\theta$ . Then the following statements are equivalent:*

- (i)  $X^*$  is stochastically continuous.
- (ii)  $\theta$  is continuous.
- (iii) The bivariate map  $(s, t) \mapsto \theta(\{s, t\})$  is continuous.
- (iv) The bivariate map  $(s, t) \mapsto \theta(\{s, t\})$  is continuous on the diagonal.

**Proof.** The implication (i)  $\Rightarrow$  (ii) follows from Lemma 23. Clearly, continuity of  $\theta$  implies continuity of the induced function  $\theta^{(2)}(s, t) := \theta(\{s, t\})$ , which implies continuity of  $\theta^{(2)}$  on the diagonal. This shows the implications (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (iv). Finally, the implication (iv)  $\Rightarrow$  (i) follows from Lemma 24, since  $\eta(\{t, t\}) = \theta(\{t\}) - 1 = 0$ .  $\square$

## 5. Dependency sets – the special role of TM processes

In this section, we show that the TM process  $X^*$  with ECF  $\theta$  is exceptional among all max-stable processes sharing the same ECF  $\theta$  as  $X^*$  in the sense that its dependency set  $\mathcal{K}^*$  (to be introduced below) is maximal w.r.t. inclusion.

Therefore, recall that for a finite non-empty subset  $M \subset T$  the dependency set  $\mathcal{K}_M$  of  $\{X_t\}_{t \in M}$  is the largest compact convex set  $\mathcal{K}_M \subset [0, \infty)^M$  satisfying (cf. (5))

$$\ell_M(x) = \sup\{\langle x, y \rangle : y \in \mathcal{K}_M\} \quad \forall x \in [0, \infty)^M.$$

The closed convex set  $\mathcal{K}_M$  may also be described as the following intersection of half spaces (cf. [27], Section 1.7):

$$\mathcal{K}_M = \bigcap_{x \in S_M} \{y \in [0, \infty)^M : \langle x, y \rangle \leq \ell_M(x)\}. \quad (19)$$

**Example 26 ([19], Example 1 and Proposition 2).** The simplest examples for dependency sets  $\mathcal{K}_M$  are the unit cube  $[0, 1]^M$  corresponding to a collection of independent random variables  $\{X_t\}_{t \in M}$  and the cross-polytope  $D^M := \{x \in [0, \infty)^M : \sum_{t \in M} x_t \leq 1\}$  corresponding to identical random variables  $\{X_t\}_{t \in M}$ . Any dependency set  $\mathcal{K}_M$  satisfies

$$D^M \subset \mathcal{K}_M \subset [0, 1]^M.$$

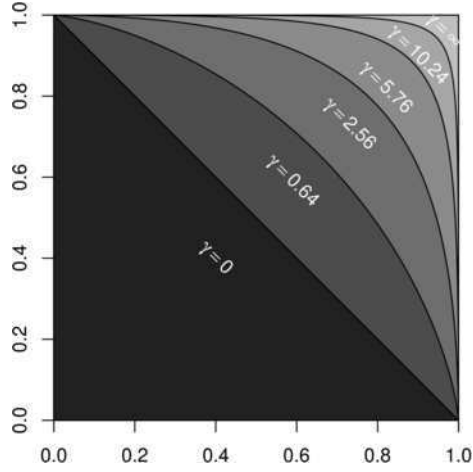
**Example 27 (Brown–Resnick process/Hüsler–Reiss distribution).** The f.d.d. of a Brown–Resnick process (cf. Example 3) are the multivariate Hüsler–Reiss distributions (cf. [14]). In the bivariate case, when  $M = \{1, 2\}$  consists of two points only, the distribution function of a Hüsler–Reiss distributed random vector  $(X_1, X_2)$ , standardized to unit Fréchet marginals, is

$$-\log \mathbb{P}_\gamma(X_1 \leq x_1, X_2 \leq x_2) = \frac{1}{x_1} \Phi\left(\frac{\sqrt{\gamma}}{2} + \frac{\log(x_2/x_1)}{\sqrt{\gamma}}\right) + \frac{1}{x_2} \Phi\left(\frac{\sqrt{\gamma}}{2} + \frac{\log(x_1/x_2)}{\sqrt{\gamma}}\right)$$

for  $x_1, x_2 \geq 0$ . Here  $\Phi$  denotes the distribution function of the standard normal distribution and the parameter  $\gamma$  is the value of the variogram between the two points (cf. Example 3). Figure 3 illustrates, how the corresponding dependency sets range between full dependence ( $\gamma = 0$ ) and independence ( $\gamma = \infty$ ).

In order to define a single dependency set for a simple max-stable process comprising all multivariate dependency sets, we write

$$\text{pr}_M : [0, \infty)^T \rightarrow [0, \infty)^M, \quad (x_t)_{t \in T} \mapsto (x_t)_{t \in M}$$



**Figure 3.** Nested dependency sets  $\mathcal{K}_M^{(\gamma)}$  of the bivariate Brown–Resnick (resp. Hüsler–Reiss) distribution where  $M = \{1, 2\}$  (cf. Example 27). The dependency sets grow as the parameter  $\gamma$  increases. They range between full dependence ( $\gamma = 0$ ) and independence ( $\gamma = \infty$ ).

for the natural projection.

**Definition 28.** Let  $X$  be a simple max-stable process  $X = \{X_t\}_{t \in T}$  and denote for finite  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$  the multivariate dependency set of the random vectors  $\{X_t\}_{t \in M}$  by  $\mathcal{K}_M$ . Then we define the dependency set  $\mathcal{K} \subset [0, \infty)^T$  of  $X$  as

$$\mathcal{K} := \bigcap_{M \in \mathcal{F}(T) \setminus \{\emptyset\}} \text{pr}_M^{-1}(\mathcal{K}_M).$$

Analogously to (5), the dependency set  $\mathcal{K}$  may be characterized as follows.

**Lemma 29.** The dependency set  $\mathcal{K}$  of a simple max-stable process  $X = \{X_t\}_{t \in T}$  is the largest compact convex set  $\mathcal{K} \subset [0, \infty)^T$  satisfying

$$\ell_M(x) = \sup \left\{ \sum_{t \in M} x_t y_t : y \in \mathcal{K} \right\} \quad \forall x \in [0, \infty)^M \forall \emptyset \neq M \in \mathcal{F}(T), \quad (20)$$

where  $\ell_M$  is the stable tail dependence function of  $\{X_t\}_{t \in M}$ .

**Proof.** Convexity of  $\mathcal{K}$  follows from the convexity of each  $\mathcal{K}_M$  and from the linearity of the projections  $\text{pr}_M$  for  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$ . Since  $\mathcal{K}_{\{t\}} = [0, 1]$  is the unit interval for each  $t \in T$ , the set  $\mathcal{K}$  is contained in the compact space  $[0, 1]^T$ . Moreover,  $\mathcal{K}$  is closed as the intersection of closed sets, hence  $\mathcal{K}$  is compact.

Next, we prove that  $\mathcal{K}_M = \text{pr}_M(\mathcal{K})$ . By definition of  $\mathcal{K}$  it is clear that  $\text{pr}_M(\mathcal{K}) \subset \mathcal{K}_M$  for  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$ . To prove the reverse inclusion, let  $y_M$  be an element of  $\mathcal{K}_M$  and set

$V(y_M) := \text{pr}_M^{-1}(\{y_M\}) \cap \mathcal{K} = \text{pr}_M^{-1}(\{y_M\}) \cap \mathcal{K} \cap [0, 1]^T$ . We need to show that  $V(y_M) \neq \emptyset$ . Denoting  $V(y_M, A) := \text{pr}_M^{-1}(\{y_M\}) \cap \text{pr}_A^{-1}(\mathcal{K}_A) \cap [0, 1]^T$ , we see that

$$V(y_M) = \bigcap_{A \in \mathcal{F}(T) \setminus \{\emptyset\}} V(y_M, A).$$

Note that each  $V(y_M, A)$  is a closed subset of the compact Hausdorff space  $[0, 1]^T$ . Therefore, it suffices to verify the finite intersection property for the system of sets  $\{V(y_M, A)\}_{A \in \mathcal{F}(T) \setminus \{\emptyset\}}$  in order to show that  $V(y_M) \neq \emptyset$ . But this follows from the consistency of the finite dimensional dependency sets  $\{\mathcal{K}_A\}_{A \in \mathcal{F}(T) \setminus \{\emptyset\}}$  as follows: As [19], Section 7, Proposition 8, essentially says, we have that if  $A$  and  $B$  are non-empty finite subsets of  $T$  with  $A \subset B$ , then  $\mathcal{K}_A$  is the projection of  $\mathcal{K}_B$  onto the respective coordinate space. In particular,  $\text{pr}_B^{-1}(\mathcal{K}_B) \subset \text{pr}_A^{-1}(\mathcal{K}_A)$  and  $\text{pr}_A^{-1}(\{y_A\}) \cap \text{pr}_B^{-1}(\mathcal{K}_B) \cap [0, 1]^T \neq \emptyset$  for  $y_A \in \mathcal{K}_A$ . Now, let  $A_1, \dots, A_k$  be non-empty finite subsets of  $T$ . Then

$$\begin{aligned} \emptyset &\neq \text{pr}_M^{-1}(\{y_M\}) \cap \text{pr}_{M \cup \bigcup_{i=1}^k A_i}^{-1}(\mathcal{K}_{M \cup \bigcup_{i=1}^k A_i}) \cap [0, 1]^T \\ &\subset \text{pr}_M^{-1}(\{y_M\}) \cap \bigcap_{i=1}^k \text{pr}_{A_i}^{-1}(\mathcal{K}_{A_i}) \cap [0, 1]^T = \bigcap_{i=1}^k V(y_M, A_i), \end{aligned}$$

as desired and we have shown that  $\mathcal{K}_M \subset \text{pr}_M(\mathcal{K})$ . Both inclusions give  $\mathcal{K}_M = \text{pr}_M(\mathcal{K})$ .

By definition, we have  $\ell_M(x) = \sup\{x, y\} : y \in \mathcal{K}_M\}$  for  $x \in [0, \infty)^M$ . Thus, (20) follows from  $\mathcal{K}_M = \text{pr}_M(\mathcal{K})$ .

Finally, let  $\mathcal{L} \subset [0, \infty)^T$  be also convex compact and satisfying (20) with  $\mathcal{K}$  replaced by  $\mathcal{L}$ . Then it follows immediately that  $\text{pr}_M(\mathcal{L}) = \mathcal{K}_M$  for any non-empty finite subset  $M \subset T$ . We conclude that  $\mathcal{L} \subset \mathcal{K}$  by definition of  $\mathcal{K}$ . This finishes the proof.  $\square$

In particular, the ECF  $\theta$  of a simple max-stable process  $X = \{X_t\}_{t \in T}$  can be expressed in terms of the dependency set  $\mathcal{K}$  of  $X$  as

$$\theta(A) = \sup \left\{ \sum_{t \in A} x_t : x \in \mathcal{K} \right\}. \quad (21)$$

In order to make statements about the dependency sets  $\mathcal{K}$  of processes  $X = \{X_t\}_{t \in T}$  in terms of the ECF  $\theta$ , we introduce the following notation: For any non-empty finite subsets  $A$  of  $T$ , we set the halfspace

$$\mathcal{H}_A(\theta) := \left\{ x \in [0, \infty)^T : \sum_{t \in A} x_t \leq \theta(A) \right\}$$

that is bounded by the hyperplane

$$\mathcal{E}_A(\theta) := \left\{ x \in [0, \infty)^T : \sum_{t \in A} x_t = \theta(A) \right\}.$$

**Lemma 30.** *Let  $\mathcal{K}$  be the dependency set of a simple max-stable process  $X = \{X_t\}_{t \in T}$  with ECF  $\theta$ . Then the following inclusion holds*

$$\mathcal{K} \subset \bigcap_{A \in \mathcal{F}(T) \setminus \{\emptyset\}} \mathcal{H}_A(\theta).$$

On the other hand for each  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$  there is at least one point  $\mathbf{x}^A$  in the intersection

$$\mathbf{x}^A \in \mathcal{K} \cap \mathcal{E}_A(\theta).$$

**Proof.** Let  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$  and  $x \in \mathcal{K}$ . Then the assumption  $\sum_{t \in A} x_t > \theta(A)$  contradicts  $\theta(A) = \sup\{\sum_{t \in A} x_t : x \in \mathcal{K}\} > \theta(A)$  (cf. (21)). So  $\sum_{t \in A} x_t \leq \theta(A)$ . This proves the inclusion. Second, since  $\mathcal{K}$  is compact and the map  $[0, \infty)^T \ni x \rightarrow \sum_{t \in A} x_t$  is continuous, we know that it attains its supremum at some  $\mathbf{x}^A \in \mathcal{K}$ .  $\square$

**Example 31.** We give a simple multivariate example for Lemma 30 (as illustrated in Figure 1 in the introduction for the trivariate case): The Euclidean norm  $\ell_M(x) = \|x\|_2$  is a stable tail dependence function on  $[0, \infty)^M$  (cf. [19], Example 2) and defines a simple max-stable distribution (cf. (4)) with ECF  $\theta(A) = \sqrt{|A|}$  for  $A \subset M$ , such that

$$\begin{aligned} \mathcal{H}_A(\theta) &= \{x \in [0, \infty)^M : \langle x, \mathbf{1}_A \rangle \leq \sqrt{|A|}\}, \\ \mathcal{E}_A(\theta) &= \{x \in [0, \infty)^M : \langle x, \mathbf{1}_A \rangle = \sqrt{|A|}\} \end{aligned}$$

for  $\emptyset \neq A \subset M$ . It can be easily seen that for  $x \in [0, \infty)^M \setminus \{\mathbf{1}_\emptyset\}$

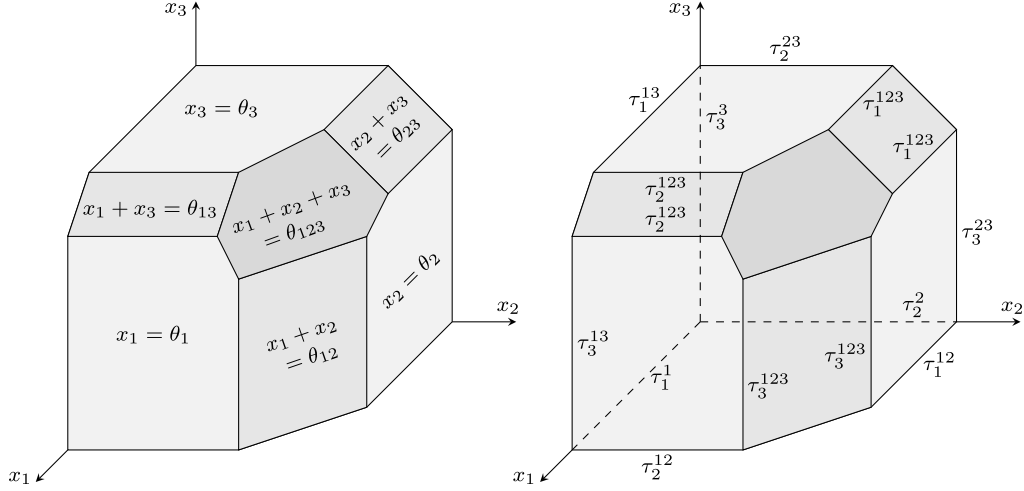
$$\ell_M(x) = \|x\|_2 = \langle x, x/\|x\|_2 \rangle = \sup\{\langle x, y \rangle : y \in B^+\},$$

where  $B^+ := \{y \in [0, \infty)^M : \|y\|_2 \leq 1\}$  denotes the positive part of the (Euclidean) unit ball. So, the dependency set  $\mathcal{K}$  is clearly  $B^+$  in this case. Now, the planes  $\mathcal{E}_A(\theta)$  are tangent to the boundary of  $B^+$  with common points  $\mathbf{x}^A = \mathbf{1}_A/\sqrt{|A|}$  for  $\emptyset \neq A \subset M$ , which makes it easy to see that Lemma 30 holds true in this example. Figure 1 shows the dependency set  $\mathcal{K} = B^+$  (left) and the intersection of halfspaces bounded by the planes  $\mathcal{E}_A(\theta)$  (right). In the middle it is illustrated that this intersection contains  $B^+$  and the points  $\mathbf{x}^A$  are marked.

The following theorem shows that the inclusion from Lemma 30 is sharp and attained by TM processes. Figure 4 illustrates the dependency set of a trivariate distribution of a TM process.

**Theorem 32.** *Let  $\mathcal{K}^*$  be the dependency set of the TM process  $X^* = \{X_t^*\}_{t \in T}$  with ECF  $\theta$ . Then*

$$\mathcal{K}^* = \bigcap_{A \in \mathcal{F}(T) \setminus \{\emptyset\}} \mathcal{H}_A(\theta).$$



**Figure 4.** Dependency set  $\mathcal{K}^*$  of the random vector  $\{X_t^*\}_{t \in M}$  for  $M = \{1, 2, 3\}$ . The dependency set  $\mathcal{K}^*$  is bounded by the hyperplanes  $\mathcal{E}_A(\theta)$  that are given by the equations  $\sum_{t \in A} x_t = \theta(A)$ , where  $\theta$  denotes the ECF of  $X^*$ . The coefficients  $\tau_{\{t\}}^L$  for  $L \in \mathcal{F}(M) \setminus \{\emptyset\}$  and  $t \in L$  turn up as lengths of the resulting polytope  $\mathcal{K}^*$  (cf. Theorem 8 (b) and Theorem 32).

**Proof.** First, we prove the theorem in the case, when  $T = M$  is finite and  $\mathcal{K}^* = \mathcal{K}_M^*$ : Therefore, write

$$\mathcal{L}_M := \bigcap_{\emptyset \neq A \subset M} \mathcal{H}_A(\theta) = \{x \in [0, \infty)^M : \langle x, \mathbf{1}_A \rangle \leq \theta(A) \text{ for all } \emptyset \neq A \subset M\}.$$

The inclusion  $\mathcal{K}_M^* \subset \mathcal{L}_M$  is proven in Lemma 30. So, it remains to show the other inclusion  $\mathcal{L}_M \subset \mathcal{K}_M^*$ . Due to (19), we have that

$$\mathcal{K}_M^* = \bigcap_{x \in S_M} \{y \in [0, \infty)^M : \langle x, y \rangle \leq \ell_M^*(x)\},$$

where

$$\ell_M^*(x) = \sum_{\emptyset \neq L \subset M} \tau_L^M \bigvee_{t \in L} x_t$$

is the stable tail dependence function of  $\{X_t^*\}_{t \in M}$ , here expressed in terms of the coefficients  $\tau_L^M$  from Theorem 8 (b) (cf. (12)). Thus, it suffices to show the following implication in order to prove  $\mathcal{L}_M \subset \mathcal{K}_M^*$ :

$$x \in S_M \quad \text{and} \quad y \in \mathcal{L}_M \implies \langle x, y \rangle \leq \ell_M^*(x).$$

We now prove this implication: Without loss of generality, we may label the elements of  $M = \{t_1, \dots, t_m\}$  such that  $x_{t_1} \geq x_{t_2} \geq \dots \geq x_{t_m}$ . Then we may write  $x = (x_t)_{t \in M} \in$



$S_M \subset [0, \infty)^M$  as

$$x = \underbrace{x_{t_m}}_{\geq 0} \mathbf{1}_M + \underbrace{(x_{t_{n-1}} - x_{t_m})}_{\geq 0} \mathbf{1}_{M \setminus \{t_m\}} + \cdots + \underbrace{(x_{t_2} - x_{t_3})}_{\geq 0} \mathbf{1}_{\{t_1, t_2\}} + \underbrace{(x_{t_1} - x_{t_2})}_{\geq 0} \mathbf{1}_{\{t_1\}}.$$

Taking the scalar product with  $y \in \mathcal{L}_M$ , we conclude

$$\begin{aligned} \langle x, y \rangle &\leq x_{t_m} \theta(M) + (x_{t_{n-1}} - x_{t_m}) \theta(M \setminus \{t_m\}) \\ &\quad + \cdots + (x_{t_2} - x_{t_3}) \theta(\{t_1, t_2\}) + (x_{t_1} - x_{t_2}) \theta(\{t_1\}) \\ &= x_{t_m} (\theta(M) - \theta(M \setminus \{t_m\})) + \cdots + x_{t_2} (\theta(\{t_1, t_2\}) - \theta(\{t_1\})) + x_{t_1} \theta(\{t_1\}). \end{aligned} \quad (22)$$

On the other hand the stable tail dependence function  $\ell_M^*$  is by this ordering of the components of  $x$  given as

$$\ell_M^*(x) = \sum_{\emptyset \neq L \subset M} \tau_L^M \bigvee_{t \in L} x_t = \sum_{i=1}^m x_{t_i} \left( \sum_{L \subset M: t_1, \dots, t_{i-1} \notin L, t_i \in L} \tau_L^M \right).$$

From (15), we see that this expression coincides with the r.h.s. of (22). Thus, we have our desired inequality  $\langle x, y \rangle \leq \ell_M^*(x)$ . This finishes the proof in the case, when  $T = M$  is finite.

Otherwise, the definition of the dependency set  $\mathcal{K}^*$  and the result for finite  $M$  give

$$\mathcal{K}^* = \bigcap_{M \in \mathcal{F}(T) \setminus \{\emptyset\}} \text{pr}_M^{-1}(\mathcal{K}_M^*) = \bigcap_{M \in \mathcal{F}(T) \setminus \{\emptyset\}} \bigcap_{\emptyset \neq A \subset M} \text{pr}_M^{-1}(\mathcal{H}_A^M(\theta)),$$

where  $\mathcal{H}_A^M(\theta) = \{x \in [0, \infty)^M : \sum_{t \in A} x_t \leq \theta(A)\}$ . Since  $\text{pr}_M^{-1}(\mathcal{H}_A^M(\theta)) = \mathcal{H}_A(\theta)$  for  $\emptyset \neq A \subset M$ , the claim follows.  $\square$

So, if we fix the ECF  $\theta$  of a simple max-stable process on  $T$ , then the TM process yields a maximal dependency set  $\mathcal{K}^*$  w.r.t. inclusion, that is

$$\mathcal{K}^* = \bigcup_{\substack{\mathcal{K} \text{ dependency set} \\ \text{with the same ECF as } \mathcal{K}^*}} \mathcal{K}. \quad (23)$$

Now, inclusion of dependency sets corresponds to stochastic ordering in the following sense (cf. [19], page 242): If  $\mathcal{K}'$  and  $\mathcal{K}''$  denote the dependency sets of the simple max-stable processes  $X'$  and  $X''$  respectively, then  $\mathcal{K}' \subset \mathcal{K}''$  implies

$$\mathbb{P}(X'_t \leq x_t, t \in M) \geq \mathbb{P}(X''_t \leq x_t, t \in M) \quad \forall x \in [0, \infty)^M$$

for all  $M \in \mathcal{F}(T) \setminus \{\emptyset\}$ . This leads to the following sharp inequality.

**Corollary 33.** Let  $X = \{X_t\}_{t \in T}$  be a simple max-stable process with ECF  $\theta$ . Let  $M$  be a non-empty finite subset of  $T$ . Then

$$\mathbb{P}(X_t \leq x_t, t \in M) \geq \exp\left(-\sum_{\emptyset \neq L \subset M} \tau_L^M \bigvee_{t \in L} \frac{1}{x_t}\right) \quad \forall x \in [0, \infty)^M, \quad (24)$$

where the coefficients  $\tau_L^M$  depend only on  $\theta$  and can be computed as in Theorem 8(b). Equality holds for the TM process  $X^*$ .

**Example 34.** Let us abbreviate  $\eta_A := \theta(A) - 1$ . In the bivariate case, the inequality (24) reads as

$$\begin{aligned} \mathbb{P}(X_s \leq x_s, X_t \leq x_t) &\geq \exp\left(-\left[\frac{\eta_{st}}{x_s \vee x_t} + \frac{1}{x_s \wedge x_t}\right]\right) \\ &= \exp\left(-\frac{\eta_{st} + 1}{x_s \wedge x_t}\right) \exp\left(\eta_{st} \left|\frac{1}{x_s} - \frac{1}{x_t}\right|\right). \end{aligned}$$

Indeed this inequality is much better than the trivial inequality  $\mathbb{P}(X_s \leq x_s, X_t \leq x_t) \geq \mathbb{P}(X_s \leq x_s \wedge x_t, X_t \leq x_s \wedge x_t)$ , which can be written in the above terms as

$$\mathbb{P}(X_s \leq x_s, X_t \leq x_t) \geq \exp\left(-\frac{\eta_{st} + 1}{x_s \wedge x_t}\right).$$

Further note that  $\eta_{st} = \theta(\{s, t\}) - 1$  can be interpreted as a normalized madogram:

$$\eta_{st} \stackrel{(14)}{=} \lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_s \geq x \text{ or } X_t \geq x)}{\mathbb{P}(X_t \geq x)} - 1 = \lim_{x \rightarrow \infty} \frac{\mathbb{E}|\mathbb{1}_{X_s \geq x} - \mathbb{1}_{X_t \geq x}|}{2\mathbb{E}\mathbb{1}_{X_t \geq x}}.$$

If we additionally take into account that (cf. [26], inequality (13))

$$\eta_{rs} \vee \eta_{st} \vee \eta_{rt} \vee (\eta_{rs} + \eta_{st} + \eta_{rt} - 1) \leq \eta_{rst} \leq (\eta_{rs} + \eta_{st}) \wedge (\eta_{st} + \eta_{rt}) \wedge (\eta_{rt} + \eta_{rs}),$$

we obtain from (24) the following (sharp) inequality for the trivariate distribution of a simple max-stable random vector  $(X_r, X_s, X_t)$  from bivariate quantities:

$$\begin{aligned} &\mathbb{P}(X_r \leq x_r, X_s \leq x_s, X_t \leq x_t) \\ &\geq \exp\left(-\left[\frac{1 - \eta_{rs} \vee \eta_{st} \vee \eta_{rt}}{x_r \wedge x_s \wedge x_t} + (a_{rst} \wedge 1) \left(\frac{1}{x_r \wedge x_s} + \frac{1}{x_s \wedge x_t} + \frac{1}{x_r \wedge x_t}\right) \right. \right. \\ &\quad \left. \left. - \left(\frac{\eta_{rs}}{x_r \wedge x_s} + \frac{\eta_{st}}{x_s \wedge x_t} + \frac{\eta_{rt}}{x_r \wedge x_t}\right) + a_{rst} \left(\frac{1}{x_r} + \frac{1}{x_s} + \frac{1}{x_t}\right) - \left(\frac{\eta_{st}}{x_r} + \frac{\eta_{rt}}{x_s} + \frac{\eta_{rs}}{x_t}\right)\right]\right), \end{aligned}$$

where  $a_{rst} := (\eta_{rs} + \eta_{st}) \wedge (\eta_{rs} + \eta_{rt}) \wedge (\eta_{st} + \eta_{rt})$ .

Thus, if one can handle the ECF of a max-stable process, sharp lower bounds for its f.d.d. are available. However, beware that higher variate cases of these inequalities will

be numerically unstable.

**Remark 35.** It is an open problem and it would be interesting to know whether there exist also minimal dependency sets in the sense of (23) and if they would help to better understand the classification of all dependency structures. In view of Lemma 30 and Theorem 32 a very naive idea would be to take one point from each of the sets  $\mathcal{K}^* \cap \mathcal{E}_A$  where  $A \in \mathcal{F}(T) \setminus \{\emptyset\}$  and then to take the convex hull with 0 included. However, this fails to be a dependency set in dimensions  $|T| \geq 3$ , since it is not even a zonoid, which would be necessary (cf. [19]).

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