TOWARDS A GENERAL THEORY FOR MODELLING QUALITATIVE SPACE

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Qualitative spatial representation and reasoning are techniques for modeling and manipulating objects and relationships in space. Finding ways for defining the complete and sound (physically plausible) set of relationships between spatial objects is a prerequisite for the development and realization of qualitative representation and reasoning formalisms. Establishing the set of sound relationships is a complicated task especially when complex objects are considered. Hence, current approaches to qualitative representation and reasoning are limited to handling simple spatial objects. In this paper, we introduce a constraint-based approach to qualitative representation of topological relationships by defining a set of general soundness rules. The rules reduce the combinatorial set of relations produced by the method to the complete and physically possible ones. The rules are general and apply to objects of arbitrary complexity and together with the representation and reasoning formalism form a theory for qualitative space.

1. Introduction

Development of representation models for qualitative spatial relationships is an important research topic of interest to many application domains including, Artificial Intelligence, image processing and physical and engineering applications and Geographic Information Systems [1,2,3,4]. The need for a mathematical basis to under-pin such representation formalisms is two-folds: to provide precise unambiguous definitions of relations and to facilitate a straightforward mapping into implementation algorithms or spatial query languages. Also, a formal theory for the representation of spatial relations provides the essential basis for spatial reasoning in spatial information systems.

Two main criteria for classification of representation formalisms are: *completeness*, which is the ability of the formalism to represent all possible spatial relationship of interest in the domain studied and *soundness*, which is the ability of the formalism to define only physically possible relationships. The definition of a complete and sound set of spatial relationships is a pre-requisite for devising any reasoning mechanisms over such representation schemes. Composition of spatial relationships is the process of deriving the possible relation(s) between any pair of objects A and C given the relationships $R_1(A,B)$ and $R_2(B,C)$. Non-deterministic results may occur which are usually expressed as a disjunction of possible relations, thus consequently, requiring the knowledge of the set of all possible relationships between the objects in the domain studied.

The problem is significant, especially in domains where objects of arbitrary complexity are considered and ontologies for spatial relations are not readily available. Deriving the sound set of spatial relationships has so far been limited to using visual reasoning where no guarantee of completeness can be established or, in few works [5] by proposing sets of rules for specific object representations. The later method is limited, as new rules need to be devised every time a new object type or shape is considered.

In this paper a constraint-based approach, proposed in earlier work [6], to the representation of spatial relationships between objects of arbitrary complexity is first introduced. The complete set of topological relationships is derivable by the method. The method is then used as a basis for the development of general soundness rules and constraints. The constraints are used to extract the physically possible set of relationships from the complete set. The automation of the above procedure is novel and provides new possibilities for the application of spatial reasoning formalisms in large spatial databases.

The rest of the paper is structured as follows. Section 2 provide an overview of the dimensions of the problem. Related work is presented in section 3. In section 4, the representation method is introduced and examples are provided to demonstrate its applicability to complex spatial objects. Soundness rules are given in section 5 which are then mapped to constraints that can be applied on the representation method. A method for calculating the number of sound spatial relations is also introduced, followed by conclusions and a view of future work in section 6.

2. Problem Definition

Qualitative spatial representation and reasoning (QSRR) is an application of the general field of qualitative modeling, where the main goal is to model the state and behaviour of a given system. In a qualitative space, the states to be modeled are the spatial objects and their inter-relationships, and the behaviour is either static or dynamic.

In the case of static behaviour, a space containing a set of three or more spatial

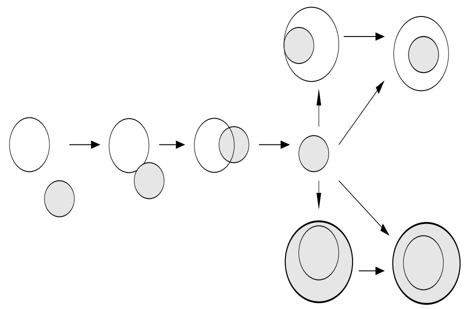


Figure 1: The possible states of transition between the various relations in the case of two simple regions.

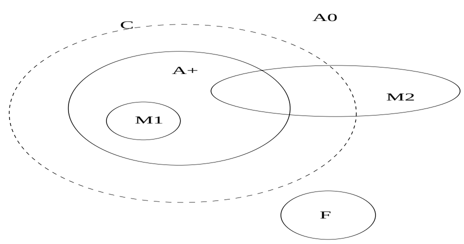


Figure 2: Different sets of spatial relations for characterising representation models.

objects is studied to identify the possible or feasible set of relationships between those objects. The basic problem in this case is the composition of spatial relations stated as follows: given a relation R_1 between objects A and B and a relation R_2 between objects B and C, find the corresponding set of possible relations between objects A and C. For example, if $contains(A, B)$ and $overlaps(B, C)$, it should be concluded that the relations between A and C is either $contain(A, C)$ or $overlap(A, C)$.

In the case of dynamic behaviour, the model needs to compute the possible sequence of transitions between different states (or spatial relations). For example, figure 1 is a model of the sequence of transition of relations between two simple regions.

A basic requirement for a QSRR model to be complete and sound is to represent all the possible states (i.e. spatial relations) and to exclude any non-feasible ones. In this paper, this problem is addressed with spatial objects of arbitrary complexity and dimension. The approach is also valid in any space dimension.

Let A^+ be the set of sound relations between a pair of spatial objects, as shown in figure 2. A^+ is a finite set. If A_0 represents the (infinite) set of all invalid or non-sound states that can be modeled, then $A = A^+ \cup A_0$ is the set of all states that can be modeled.

The set A^+ is the complete (all) and sound (physically possible) set of relations between the two objects. Different categories of representation model can be identified.

1. An invalid representation model is a model which represents only invalid or non-sound relations. In figure 2, the set F is the set of invalid relations, where $F \not\subset A^+$ and $F \subset A$.
2. A sound, but incomplete model represents relations set M_1 in figure 2, where $M_1 \subset A^+$, $M_1 \not\subset A_0$ and $M_1 \neq A^+$.
3. A partially sound model represents relations set M_2 in figure 2, where $M_2 \subset A^+$ and $M_2 \subset A_0$.
4. A complete but not sound model represents relations set C in figure 2, where $A^+ \subset C$ and $C \subset A_0$.
5. A complete and sound model represents relations set I, where $I = A^+$ and $I \not\subset A_0$.

A QSRR formalism strives to be complete and sound. The above taxonomy is used in the following section to characterise the various approaches proposed in the literature

3. Related Work

Two main approaches can be classified for modeling spatial relations. One starts by identifying the set of sound relations in the domain studied and then model the relations using constraints that define the different distinct states. The other approach starts by identifying the set of constraints that govern the space studied and use them to define the states or relations using those constraints. We denote the first approach a *relation-driven* approach and the second approach a *constraint-driven* approach. The first approach is sound, but with no guarantee of completeness and the second approach aims to achieve completeness but does not guarantee soundness.

This paper is primarily concerned with topological relations. The spatial domain is rich with various possible types, dimensions and shapes of spatial objects. A vast number of possible relations may exist between those objects. It is therefore practically impossible for the relation-driven approaches to claim completeness. The model in this case can be represented by the set M_1 in figure 2, where M_1 is much smaller than A^+.

A primary method in this category is due to Randell et al [7]. Eight relations between simple convex regions are axiomatised using the concept of connection. Attempts have been made to extend the formalism by introducing different taxonomies of relationships between concave regions [8] and regions with in-determined boundaries [9]. The power of this logic-based formalism was investigated in [10] and [11,12] using different variations of doughnut-shaped regions. Bennett et al [13] used this connection logic to describe qualitative geometry. However, it was noted in [14] that the region connection method is limited to studying objects of similar dimension and can't handle objects with different dimensions. Theorem proving techniques were used for reasoning (static behaviour modeling) on the defined axioms. The difficulties and complexity of such task are reported by Cohn in [15].

In the constraint-driven approach, an object is represented in terms of the set of its components, and relationships are the result of the combinatorial intersection of those components. These group of approaches generally aim to satisfy completeness, but not soundness. The degree of completeness is dependent on the modeling strategy adopted for space and relations.

An intersection-based approach was proposed by Egenhofer et al [16,17] where point-set topology was used for the definition of the components of two simple regions as interior (A°), boundary (δA) and exterior (A^{-}). Spatial relationships between the regions considered are the result of the exhaustive combinatorial intersection of their components ($2^9 = 512$ possible relationships in this case). Only eight relations between the regions are possible. Special rules were introduced to reduce the combinatorial set and eliminate the non-sound relations. A set of 11 rules were used to reason about the relations between the regions. The rules are, however, specific and could be applied only between simple regions.

Various extensions of this approach has been proposed to represent relationships between lines [5], between regions with holes [18] and between regions with indetermined boundaries [19]. Clemintini and De Felice [20] have further extended the later model to handle complex objects with broad boundaries, where they identified 56 possible relations in this case.

All the identified sets of relations above can be categorised as set M_2 in figure 2. Egenhofer's approach and the various extensions thereof are limited, as specific soundness rules have to be devised to eliminate invalid relations on a case by case basis and whenever a new type of object is considered. Furthermore, the reasoning rules proposed were limited to the case of simple convex regions only.

Hence, while the constraint-driven approach guarantee completeness, it does not provide for soundness of representation, and the relation-driven approach is sound but not complete. Note that if the domain is restricted to a set of simple (regular) shapes then both approaches can be made complete and unique with respect to representation.

In a previous work [6], we proposed an intersection model that generealised the representation of objects and spatial relations. It was shown how the model can apply to objects with arbitrary complexity and to model their static behaviour. The model was extended to handle orientation and proximity relations [21,22]. The approach represents relations set C in figure 2. Although complete and general, the model had to assume that the set of sound relations between any of the considered objects were pre-defined.

The only practical method for reducing the set C to the complete and sound set A^+ is to devise general soundness rules that are applicable on spatial relations between any type of spatial objects. This paper addresses this problem by studying the characteristics of the underlying qualitative space as explained below.

4. The Representation Approach

The first part of the paper addresses the problem of qualitative representation of objects with arbitrary spatial complexity and their topological relationships. The representation methodology is first described and examples are used to demonstrate how relationships between objects of random complexity can be represented.

4.1. *The General Representation*

Objects of interest and their embedding space are divided into components according to a required resolution. The connectivity of those components is explicitly represented. Spatial relations are represented by the intersection of object components [23] in a similar fashion to that described in [24] but with no restriction on object components to consist only of two parts (boundary and interior).

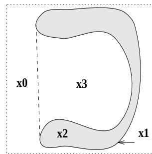
4.1.1. *The Underlying Representation of Object Topology*

Let S be the space in which the object is embedded. The object and its embedding space are assumed to be *dense* and *connected*. The embedding space is also assumed to be infinite. The object and its embedding space are decomposed into components which reflects the objects and space topology such that,

1. No overlap exists between any of the representative components.

2. The union of the components is equal to the embedding space.

The topology of the object and the embedding space can then be described by a matrix whose elements represent the connectivity relations between its components. This matrix shall be denoted *adjacency matrix*. In figure 3(a) a possible decomposition of a concave shaped object (for example an island with a bay) and its embedding space is shown and in 3(b) the adjacency matrix for its components is presented. The object is represented by two components a linear component x_1 (the shore line of the island) and an areal component x_2 and the rest of its embedding space is represented by a finite areal component x_3 (representing the bay of the island) and infinite areal component x_0 representing the surrounding area. The fact that two components are connected is represented by a (1) in the adjacency matrix and by a (0) otherwise. Since connectivity is a symmetric relation, the resulting matrix will be symmetric around the diagonal. Hence, only half the matrix is sufficient for the representation of the object's topology and the matrix can be collapsed to the structure in figure 3(c). In the decomposition strategy, the complement of the object in question shall be considered to be infinite. The suffix 0 (x_0) is used to represent this component.



(a)

	x_0	x_1	x_2	x_3
x_0	-	1	0	1
x_1	1	-	1	1
x_2	0	1	-	0
x_3	1	1	0	-

(b)

x_0			
1	x_1		
0	1	x_2	
1	1	0	x_3

(c)

Figure 3: (a) Possible decomposition of a concave-shaped object and its embedding space. (b) Adjacency matrix of the shape in (a). (c) Half the symmetric adjacency matrix is sufficient to capture the object representation.

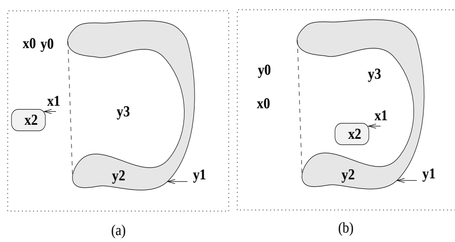


Figure 4: Different qualitative spatial relationships can be distinguished by identifying the appropriate components of the objects and the space.

4.1.2. *The Underlying Representation of Spatial Relations*

In this section, the representation of the topological relations through the intersection of their components [18] is adopted and generalized for objects of arbitrary complexity.

Distinction of topological relations is dependent on the strategy used in the decomposition of the objects and their related spaces. For example, in figure 4 different relationships between two objects representing a ship (x) and an island (y) are shown, where in 4(a) the ship is outside the bay and in 4(b) the ship is inside the bay. The concave region representing the island (y) is decomposed into two components y_1 and y_2 and the rest of the space associated with y is decomposed into two components (y_3 representing the bay and y_0 representing the rest of the ocean). Note that the component y_3 is a virtual component, i.e. with no physical boundary to delineate its spatial extension. It is the identification of this component that makes the distinction between the two relationships in the figure. The complete set of spatial relationships are represented by combinatorial intersection of the components of one space with those of the other space.

	y_0	y_1	y_2	y_3
x_0	1	1	1	1
x_1	1	0	0	0
x_2	1	0	0	0

(a)

	y_0	y_1	y_2	y_3
x_0	1	1	1	1
x_1	0	0	0	1
x_2	0	0	0	1

(b)

Figure 5: The corresponding **intersection matrices** for the relationships in figure 4 respectively.

If $R(x,y)$ is a relation of interest between object x and object y, and X and Y are the spaces associated with the objects respectively such that m is the number of components in X and l is the number of components in Y, then a spatial relation $R(x,y)$ can be represented by one state of the following equation:

$$
\begin{aligned}
R(x,y) &= X \cap Y \\
&= \left(\bigcup_{i=1}^{m} x_i\right) \cap \left(\bigcup_{j=1}^{l} y_j\right) \\
&= (x_1 \cap y_1, \cdots, x_1 \cap y_l, x_2 \cap y_1, \cdots, x_m \cap y_l)
\end{aligned}
$$

The intersection $x_i \cap y_j$ can be an empty or a non-empty intersection. The above set of intersections shall be represented by an intersection matrix, as follows,

$$
R(x,y) = \begin{array}{|c|c|c|c|c}
\hline
 & y_0 & y_1 & y_2 & \cdots \\
\hline
x_0 & & & & \\
\hline
x_1 & & & & \\
\hline
x_2 & & & & \\
\hline
\vdots & & & & \\
\end{array}
$$

For example, the intersection matrices corresponding to the spatial relationships in figure 4 are shown in figure 5. The components x_1 and x_2 have a non-empty intersection with y_0 in 5(a) and with y_3 in 5(b).

Different combinations in the intersection matrix can represent different qualitative relations. The set of valid or sound spatial relationships between objects is dependent on the particular domain studied.

In what follows the following subset notation is used. If x' is a set of components (set of point-sets) $\{x_1, \cdots, x_{m'}\}$ in a space X, and y_j is a component in space Y, then $\sqsubseteq$ denotes the following subset relationship.

- $y_j \sqsubseteq x'$ denotes the subset relationship such that: $\forall x_i \in x' (y_j \cap x_i \neq \phi) \wedge y_j \cap (X - x_1 - x_2 \cdots - x_m) = \phi \ where\ i = 1, \cdots m'$. Intuitively, this symbol indicates that the component y_j intersects with every set in the collection x' and does not intersect with any set outside of x'.

5. General Soundness Rules and Constraints

To reduce the set of complete relations in a domain to the set of complete and sound ones, soundness rules have to be devised which incorporate the physical properties or constraints of the topological space. Relations that do not conform to those rules would thus be filtered out. Physical properties of the topological space are constant under any topological mapping. Hence, properties such as size and shape are not considered. The set of soundness rules, denoted here as topological mapping rules, are then transformed to a set of constraints that can be directly applied to reduce the combinatorial intersection in the intersection matrices to only those representing physically possible or sound relationships.

The set of rules and constraints represents the properties most commonly used by humans in the process of topological visual reasoning. The representation formalism together with the constraints proposed here represent a major step towards developing a general theory for qualitative space.

5.1. *General Topological Mapping Rules*

In a topological space, object properties remain invariant under topological transformations such as, stretching or rotation. The following set of rules captures the main characteristics of the qualitative space and govern the process of space and object decomposition in that space.

Connectivity Rule: A connected component x will preserve its connectivity under any topological transformation.

Component Dimension Rule: A component x with dimension n, where $n = 0 \vee 1 \vee 2 \vee 3$ will preserve its dimension under any topological transformation.

Closed Component Rule: A closed component, e.g. a line forming a closed curve or an area forming a closed surface, will preserve its closure under any topological transformation.

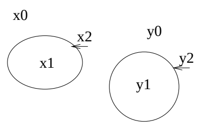
Closure Rule: An open set will remain open under any topological transformation and a closed point set will remain closed under any topological transformation.

In addition, the two assumptions used in our model, those of infinity and equality of spaces have to be preserved under topological transformations. The following two rules captures both assumptions.

Infinite Component Rule: Infinite components of a space will remain infinite under any topological transformation.

Space Equality Rule: Any two infinite and equal spaces will remain equal under any topological transformation.

5.2. *Mapping Rules to Soundness Constraints*



(a)

	y_1	y_2	y_0
x_1	1	-	-
x_2	0	-	-
x_0	1	-	-

(b)

x_0		
1	x_2	
0	1	x_1

(c)

y_0		
1	y_2	
0	1	y_1

(d)

Figure 6: (a) Simple regions x and y. (b) An impossible intersection matrix for the regions. A $-$ in the matrix represents $0 \vee 1$.

The above rules must apply when objects interact in any possible topological relationship. Hence, any intersection relation that violates one or more of the above rules is not a physically possible relation and can be omitted. The following constraints are the interpretation of the above rules on the intersection relations. The simple case of the intersection relations between two simple regions x and y in 2D space is used here to illustrate the concepts.

1. **Connectivity constraint:** If y_j is a connected component of space Y and x' is a subset of space X, where $x' = \{x_1, x_2, \cdots, x_{m'}\}$ and $m' \leq m$ and m is the total number of components of space x, then if $y_j \sqsubseteq x'$, x' must be connected. I.e. each component in the set x' must be adjacent to one or more components of the set x'. If x' is not connected, then the corresponding intersection relation is false.

   Hence, in the case of simple regions x and y, the matrix in figure 6(b) is false and can be eliminated. In the matrix $y_1 \sqsubseteq \{x_1, x_0\}$ and x_1 is not connected to x_0.

   This condition can be generalised as follows. Let $y' \sqsubseteq x'$ and $y' = \{y_1, y_2, \cdots, y_{n'}\}$ and $x' = \{x_1, x_2, \cdots, x_{m'}\}$. If x' is connected, then y' must also be connected. For example, the following matrix is not valid.

	y_1	y_2	y_0
x_1	1	0	-
x_2	0	0	-
x_0	0	1	-

   In the matrix, let $y' = \{y_1, y_2\}$ and $x' = \{x_1, x_0\}$. In this case y' is connected but x' is not.

2. **Dimension constraint:**

   If y_j is a component with dimension p in space Y, and $x' \subseteq X$ where $max(dim(x')) = q$, then $y_j \sqsubseteq x' \rightarrow q \geq p$.

   This constraint states that an object component of a certain dimension can't be a subset of another object component of lower dimension. Hence, the following matrix is false in the case of two simple regions, since $y_1 \sqsubseteq x_2$ and $dim(y_1) > dim(x_2)$. The object components are as shown in figure 6(a).

	y_1	y_2	y_0
x_1	0	1	0
x_2	1	1	0
x_0	0	1	1

Accordingly, an object component of $dim = 0$, i.e. a point, can't have a positive intersection with more than one component of the other space of any dimension.

3. **Closure constraint:** If y_j is a closed component of space Y, i.e. a closed line or an area, and if x_i is a non-closed component of space X of the same type (i.e. x_i is a line if y_j is a line and an area if y_j is an area), then $y_j \sqsubseteq x_i$ represents a false relation.

   The reason being, if $y_j \sqsubseteq x_i$ then x_i must be either closed (a contradiction) or intersects itself (excluded case by assumption) or of higher dimension. An example of this constraint is in the case of line-region relations where if the region boundary have a positive intersection with the line, it must also intersects its embedding space as well.

4. **Open and closed set constraint:** Let y_j be a component of space Y and $x' \subseteq X$. If y_j is an open set and $y_j \sqsubseteq x'$, then x' must also be an open set. Remember that $y_j \sqsubseteq x' \rightarrow y_j$ intersects only with every member of the set x'.

   Consider the example of the two simple regions x and y. The components of space X are x_1, x_2 and x_0 as shown in figure 6. Either the components $x_1 \cup x_2$ are closed and x_0 is open or the components $x_0 \cup x_2$ are closed and x_1 is open. Hence, if y_1 (an open set) intersects with both x_1 and x_2, the result of the intersection must be an open set which has to consequently intersect with x_0 as well. The same is true if y_1 (or y_0) intersects x_0 and x_2. Hence, $y_1 \sqsubseteq (x_1 \cup x_2)$ is not valid, since y_1 is an open set and $x_1 \cup x_2$ is a closed set. Intuitively, this means that y_1 can't intersect with both x_1 and x_2 without part of y_1 intersecting part of x_2. Hence, any relation conforming with the following matrix is not valid.

	y_1	y_2	y_0
x_1	1	-	-
x_2	1	-	-
x_0	0	-	-

   Another implication of this constraint can be stated as follows. If a component y_j of the same dimension of the embedding space intersects with a component of dimension 0, i.e. a point, it must intersect with all its adjacent components. Otherwise, it intersects its boundary whose dimension is less than the embedding space.

5. **Infinity constraint:** If y_0 is an infinite component of space X and if $y_0 \sqsubseteq x'$, then x' must contain at least one infinite component.

   Intuitively this constraint says that it is impossible for an infinite component in the space to only have an intersection with finite component(s). In this case the infinite component becomes a subset of the finite component(s) which is not possible. Hence, any relation conforming with the following matrix is not valid.

	y_1	y_2	y_0
x_1	-	-	-
x_2	-	-	-
x_0	-	-	0

6. **Space equality constraint:** Every component from one space must intersect with at least one component from the other space.

   If one component of one space does not intersect with any component of the other space, either the two spaces are not equal or the spaces are not dense or *connected.* Both conditions are excluded by the initial assumptions. This implies that there cannot exist a row or a column in the intersection matrix whose elements are all empty intersections. Hence the combinatorial cases in the matrix where this case exists can be ignored. For example, relations represented by the following matrix are not valid.

	y_1	y_2	y_0
x_1	0	-	-
x_2	0	-	-
x_0	0	-	-

5.3. *Non Topological or Domain Specific Constraints*

In studying specific problem domains, more specific constraints, in addition to the general ones, need to be identified and applied to filter out non-sound relations. In some cases, quantitative as opposed to qualitative properties need to be considered. Four general types of domain specific constraints can be identified.

- **Component Size/dimensions Constraint:** The size of an object component, measured by its length, width, area or volume plays a role in filtering out invalid relations where components of larger size cannot be subsets of components of smaller (shorter, narrower) size. If $y_j > x_i$ then any relation where $y_j \sqsubseteq x_i$ is an invalid relation.

- **Component Shape Constraint:** This constraint excludes the cases where two components of different shapes intersect only with each other. If x_i and y_j are two components of spaces x and y respectively, then if the shape of y_j is not equal to the shape of x_i, y_j can't be equal to x_i. I.e. $y_j \sqsubseteq x_i$ and $x_i \sqsubseteq y_j$ are false relations.

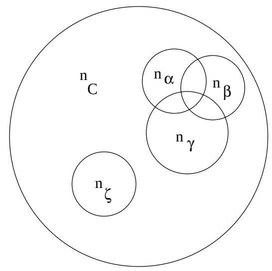


Figure 7: Overlapping effect of the soundness rules.

- **Physical Properties Constraint:** Many different types of constraints related to the physical properties of the objects studied may be used, such as permeability, rigidity, elasticity, deformability, etc. Considering those constraints may lead to the elimination of cases where some interaction between the components of the different spaces are not allowed. For example, a rigid component of one space can only intersect with the complement of the surrounding space of a non-permeable object.

- **Spatial Arrangement Constraint:** This constraint involves the identification of sound relations based on the allowable spatial arrangements of different object components, using orientation and relative distance relations.

The computation is usually simpler and more effective if domain-specific rules were applied first to eliminate some non-sound relations. Those constraints can significantly reduce the number of possibilities studied. General soundness constraints can be applied later to produce the set of complete and sound set of relations.

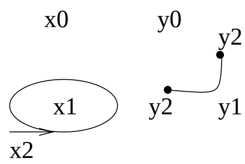
5.4. *Calculating the number of Complete and Sound Relationships*

Let the set $R = \{\alpha, \beta, \gamma, \zeta\}$ be the set of soundness constraints that will be applied to an intersection matrix of N elements. The total number of complete relations n_C is 2^N relations.

Let n_f be the total number of false relations excluded by the set of constraints R. Hence, the set of sound relations n_S is defined as: $n_S = n_C - n_f$. If $n_\alpha, n_\beta, n_\gamma$ and n_ζ are the sets of relations eliminated directly by the application of the constraints $\alpha, \beta, \gamma, \zeta$ respectively, there is no guarantee that there will be no overlap between the sets of relations excluded by each constraint. In this case, the overlap between constraints has to be accounted for to ensure that some combinations are not excluded more than once from the whole set n_C.

The number of false relations can therefore be calculated as follows:

$$\begin{aligned} n_f &= (n_\alpha + n_\beta + n_\gamma + n_\zeta) - (n_{\alpha\beta} + n_{\alpha\gamma} + n_{\alpha\zeta} + n_{\beta\gamma} + n_{\beta\zeta} + n_{\gamma\zeta}) \\ &+ (n_{\alpha\beta\gamma} + n_{\alpha\beta\zeta} + n_{\alpha\gamma\zeta} + n_{\beta\gamma\zeta}) - n_{\alpha\beta\gamma\zeta} \end{aligned}$$



(a)

	y_1	y_2	y_0
x_1	-	-	-
x_2	-	-	-
x_0	-	-	-

(b)

Figure 8: (a) Calculating the number of sound relationships between a region and a line. (b) Combinatorial intersections A $-$ in the matrix represents $0 \vee 1$.

$$= \quad A - B + C - D$$

where
• A = the total number of relations in the set α, β, γ and ζ.
• B = the number of relations resulting from the intersection of each two sets of α, β, γ and ζ.
• C = the number of relations resulting from the intersection of each three sets of α, β, γ and ζ.
• D = the number of relations resulting from the intersection of all the sets α, β, γ and ζ.

Two constraints sets overlap if their common elements in the intersection matrix had the same entry of 0 or 1, (e.g. $x_i \cap y_j = 0$). If their corresponding intersection result differs in one ore more element in the matrix, then they do not overlap.

5.5. *Example: Determining the set of complete and sound relations between a region and a line*

Consider objects x and y in figure 8 The number of possible instances of different intersection matrices for those objects is equal to 2^{n+m}, i.e. 2^9.

The application of constraints 2, 5 and 6 above results in the intersection of the component y_0 with all the components of X and consequently reduces the number of possible combinations to 2^6.

	y_1	y_2	y_0
x_1	-	-	-
x_2	-	-	-
x_0	1	1	1

The rest of the soundness constraints apply to the matrix as follows.

$I(x, y) =$

	y_2		y_1	
	0	1	0	1
x_1	(a)	(c) (e)	(b) (c)	(f)
x_2	(a)	(e)	(b) (f)	(-)
x_0	(a)	(e) (d)	(b) (d)	(f)

- a : y_2 has no intersection with space X (constraint 6)
- b : y_1 has no intersection with space X (constraint 6)
- c : $y_2 \cap x_1 = 1$ and $y_2 \cap x_2 = 1$ and $y_2 \cap x_0 = 1$. Since y_2 is a set of two separate points and x_1 is an area, x_1 must intersect with the adjacent component to y_2 which is y_1 in this case (constraint 4)
- d : $x_0 \cap y_2 = 1$ and $x_0 \cap y_1 = 0$. Similar to c (constraint 4)
- e : $x_1 \cap y_2 = 1$ and $x_1 \cap y_1 = 0$.

This is impossible since y_2 consists of two points and a point can have a positive intersection with only one component (constraint 4).

- f : $y_1 \sqsubseteq \{x_1, x_0\}$. x_1 and x_0 are not connected (constraint 1).

Taking into account the overlapping elements in the result of application of the different constraints, and noting that if two constraints imply both positive and negative intersection for the same components, then they do not overlap and can be excluded. The set of sound relations can be calculated as follows:

$$
\begin{aligned}
n_S &= n_C - (n_a + n_b + n_c + n_d + n_e + n_f) + \\
&\quad (n_{ab} + n_{af} + n_{bc} + n_{bd} + n_{be} + n_{cd} + n_{ce} + \\
&\quad n_{de} + n_{ef}) - (n_{abf} + n_{bcd} + n_{bce} + n_{bde} + n_{cde}) + \\
&\quad (n_{bcde}) \\
&= 64 - (2^3 + 2^3 + 2^4 + 2^4 + 2^3 + 2^3) + \\
&\quad (2^0 + 2^1 + 2^2 + 2^2 + 2^0 + 2^2 + 2^2 + 2^2 + 2^0) - \\
&\quad (2^0 + 2^1 + 2^0 + 2^0 + 2^1) + \\
&\quad 2^0 \\
&= 64 - (64) + (25) - (7) + (1) = 19
\end{aligned}
$$

The sound and complete set of 19 relationships are shown in tables 9 and 10.

5.6. *Deriving Sound Relations using Domain Specific Constraints*

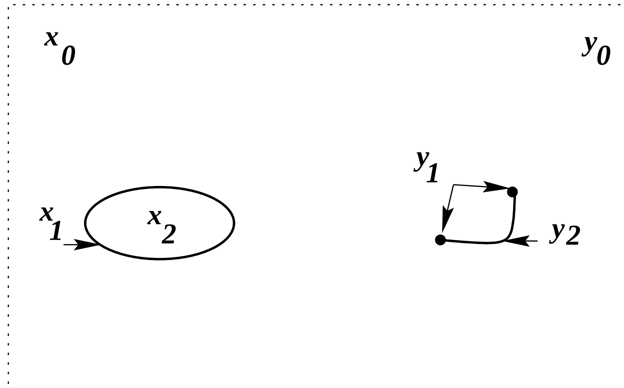
Consider the example shown in figure 11, where the relations between tennis racket and a tennis ball are considered.

Three components are used to represent the ball and a simplified representation of four components were used to define the racket. The three dimensional problem can be reduced to 2D space using the following assumptions.

- The tennis ball is solid (not permeable) and hence can be represented as a 2D region consisting of only two components, x_0 and x_1, as shown in figure 11 (c).

  Note that the tennis ball shape as a sphere, combined with its rigidity implies point contact of the component x_1 with any component y_1, y_2 or y_3. However, as the ball is elastic, its intersection with components of space y will be assumed to be an area.

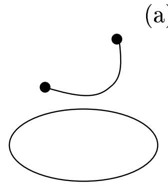
- No level difference exist between the frame of the tennis racket and the racket net which eliminates the need for further components.



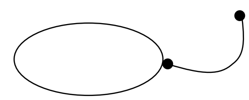
$$\begin{bmatrix} x_1 \cap y_1 & x_1 \cap y_2 & x_1 \cap y_0 \\ x_2 \cap y_1 & x_2 \cap y_2 & x_2 \cap y_0 \\ x_0 \cap y_1 & x_0 \cap y_2 & x_0 \cap y_0 \end{bmatrix}$$

(a)

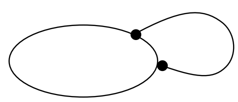
$$R_1(x,y) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



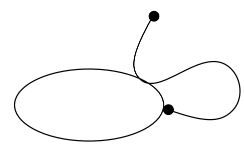
$$R_2(x,y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



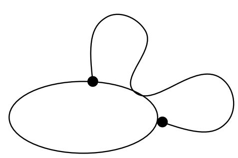
$$R_3(x,y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



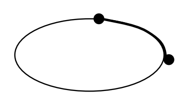
$$R_4(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



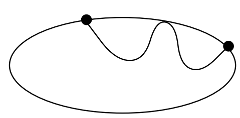
$$R_5(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



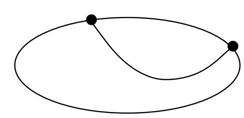
$$R_6(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



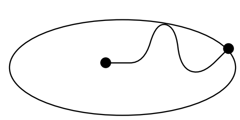
$$R_7(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_8(x,y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_9(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_{10}(x,y) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

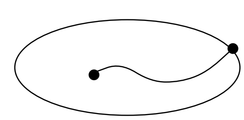
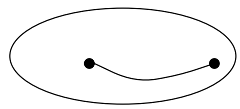
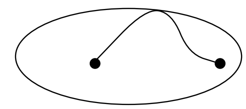


Figure 9: Part of the set of sound relations between a region and a line.

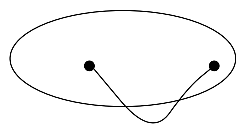
$$R_{11}(x,y) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



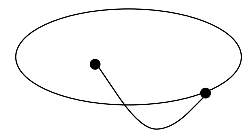
$$R_{12}(x,y) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



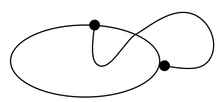
$$R_{13}(x,y) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



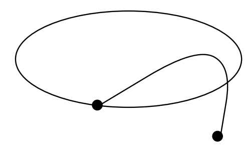
$$R_{14}(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



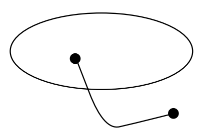
$$R_{15}(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



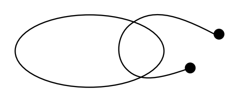
$$R_{16}(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



$$R_{17}(x,y) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



$$R_{18}(x,y) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



$$R_{19}(x,y) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

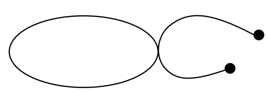


Figure 10: The rest of the set of sound relations between a region and a line.

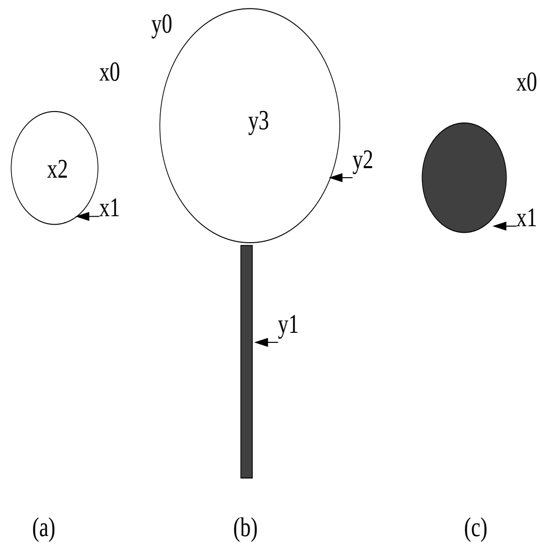


Figure 11: An example of using domain-specific constraints. (a) A tennis ball. (b) A tennis racket.

- The racket is symmetrical around the proposed 2D plane.

The intersection matrix representing the possible relations is defined as follows, giving rise to $8^2 = 256$ possible relations.

	y_1	y_2	y_3	y_0
x_1	-	-	-	-
x_0	-	-	-	-

A second domain-specific constraint Φ is that the size of the tennis ball is smaller than the size of the racket net. Hence, $y_3 \not\sqsubseteq x_1 \rightarrow x_0 \cap y_3 = 1$. From constraint number 6, we have $x_0 \cap y_0 = 1$. Hence, $x_0 \sqsubseteq \{y_3, y_0\}$. Since, y_3 and y_0 are not connected, hence $x_0 \cap y_2 = 1$, i.e. x_0 must intersect with y_2 as well.

A further constraint is that of size, where the diameter of the tennis ball is smaller than the length of the tennis racket handle y_1. Hence, $y_1 \not\sqsubseteq x_1 \rightarrow y_1 \cap x_0 = 1$.

Accordingly, the intersection relation of the component x_0 with all the components of space y are positive. The remaining intersection relations are between x_1 and space Y, namely, $4^2 = 16$ relations.

The significant reduction in the number of cases considered demonstrates the benefits of prior application of domain-specific constraints in the process of eliminating invalid relations.

The general soundness constraints apply to the matrix as follows.

$I(x, y) =$

	y_1		y_2		y_3		y_0	
	0	1	0	1	0	1	0	1
x_1	(b) (e)	(c) (d)	(a) (c) (e)	(b)	(b) (e)	(a) (c)	(e) (b) (d)	(a)

- a : x_1 intersects with y_3 and y_0 but not with y_2 (connectivity constraint)
- b : x_1 intersects only with y_2 (dimension constraint)

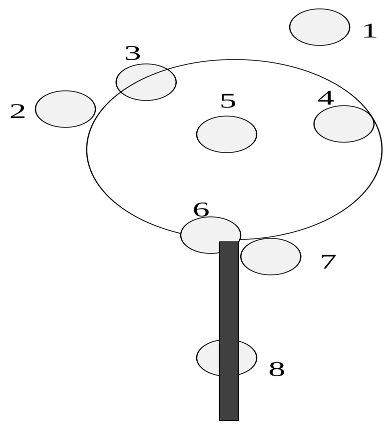


Figure 12: The set of eight sound relations between a tennis ball and a tennis racket.

- c : x_1 intersects with y_1 and y_3 but not with y_2 (connectivity constraint)
- d : x_1 intersects with y_1 but not with y_0 (dimension constraint)
- e : x_1 has no intersections (space equality constraint)

The set of sound relations can be calculated as follows:

$$\begin{aligned} n_S &= n_C - (n_a + n_b + n_c + n_d + n_e) + \\ &\quad (n_{ac} + n_{cd}) \\ &= 16 - 10 + 2 = 8 \end{aligned}$$

The eight relations are shown in figure 12. The corresponding possible intersections of the component x_1 with the space Y is as follows: 1) $(0,0,0,1)$, 2)$(0,1,0,1)$, 3)$(0,1,1,1)$, 4) $(0,1,1,0)$, 5) $(0,0,1,0)$, 6) $(1,1,1,1)$, 7) $(1,1,0,1)$ and 8)$(1,0,0,1)$.

6. Conclusions

General rules are proposed for the derivation of the set of sound qualitative spatial relations between objects of arbitrary complexity. They are based on the topologically invariant aspects of space. The rules are mapped into general constraints governing different aspects of space and object representation, including connectivity, component dimension, closed components, open sets, infinite sets and space equality. The constraints proposed complements a general formalism for qualitative spatial representation and reasoning proposed earlier [6] and together provide means for the automation of spatial reasoning techniques and their implementation. Domain specific constraints can be specified to reflect the characteristics of different problems. A method of calculating the number of sound relationships is also presented. This work is part of ongoing research that aims to develop a general theory for the treatment of qualitative space and time. Future work include realizing the theory by implementation in systems, such as GIS, and finding ways of using both quantitative and qualitative aspects and also applying the constraints in the temporal domain to facilitate the description of complex temporal relations between non-convex intervals.

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