An update on matters discussed at Oberwolfach by Ian Macdonald in May 1977 and David Robbins in May 1982

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Oberwolfach Workshop 1820 on *Enumerative Combinatorics*, at which this talk was given, ran from 13 to 19 May 2018, almost exactly 41 years after Oberwolfach Workshop 7719 on *Kombinatorik*, which ran from 8 to 14 May 1977, and almost exactly 36 years after Oberwolfach Workshop 8219 on *Kombinatorik*, which ran from 9 to 15 May 1982. At the May 1977 workshop, a talk was given by Ian Macdonald, with the title *Plane partitions*, and at the May 1982 workshop, two talks were given by David Robbins, with the titles *Proof of the Macdonald conjecture* and *Alternating sign matrices and descending plane partitions*. The essential contents of these talks can be inferred from several sources, including the workshop reports and handwritten abstracts in the Oberwolfach Digital Archive [11, 12], citations to the talks (in particular, those by Andrews [1, 2] for the talk by Macdonald), papers on which the talks were based (in particular, those by Mills, Robbins and Rumsey [8, 9] for the talks by Robbins), and first-hand written accounts of the talks (such as those by Bressoud [4] and Zeilberger [14] for the talks by Robbins).

It is clear that the talks included detailed discussions of *cyclically symmetric plane partitions* (CSPPs), *descending plane partitions* (DPPs) and *alternating sign matrices* (ASMs). An n-CSPP is a plane partition whose 3-dimensional Ferrers diagram is contained in an $n \times n \times n$ box and is invariant under cyclic rotations of coordinates. For example, there are five 2-CSPPs: $\emptyset$, (1), $\left(\begin{smallmatrix} 2 & 1 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. An n-DPP is a column strict shifted plane partition in which the largest part is at most n, and the first part of any row is larger than the number of parts in that row but (except for the first row) no larger than the number of parts in the row above. For example, there are seven 3-DPPs: $\emptyset$, (2), (3), $(3\ 1)$, $(3\ 2)$, $(3\ 3)$, and $\left(\begin{smallmatrix} 3 & 3 \\ & 2 \end{smallmatrix}\right)$. An n-ASM is an $n \times n$ matrix for which each entry is 0, 1 or -1, the sum of entries in each row and column is 1, and the nonzero entries alternate in sign along each row and column. For example, there are seven 3-ASMs: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Note that every $n \times n$ permutation matrix is an n-ASM.

In Macdonald's May 1977 talk, he conjectured that the generating function for CSPPs is given by a simple product formula involving q-numbers and q-factorials:

$$(1) \qquad \sum_{n\text{-CSPPs } C} q^{\sum_{ij} C_{ij}} = \prod_{i=0}^{n-1} \frac{[3i+2]_q\, [3i+1]_{q^3}!}{[3i+1]_q\, [n+i]_{q^3}!}.$$

The $q = 1$ case of Macdonald's conjecture was proved in [1, 2] by Andrews. In [1, 2], Andrews also introduced DPPs, conjectured that their generating function is given by a product formula similar to that for CSPPs, i.e.,

$$(2) \qquad \sum_{n\text{-DPPs } D} q^{\sum_{ij} D_{ij}} = \prod_{i=0}^{n-1} \frac{[3i+1]_q!}{[n+i]_q!},$$

and proved the $q = 1$ case of this conjecture. Furthermore, in [1, 2], Andrews defined (in a slightly different form) an (n, k)-*column strict shifted plane partition* (CSSPP) to be a column strict shifted plane partition in which the largest part is at most $n + k$ and the first part of any row exceeds the number of parts in that row by exactly k, and he showed that

$$(3) \quad (\#\ \text{of}\ (n, k)\text{-CSSPPs}) = 2 \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{(2i+k+2)_i\,(2i+(k+3)/2)_{i-1}}{(i)_i\,(i+(k+3)/2)_{i-1}}$$
$$\times \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{(2i+k)_{i-1}\,(2i+(k+1)/2)_i}{(i)_i\,(i+(k+1)/2)_{i-1}},$$

where the Pochhammer symbol has been used. The definition of CSSPPs was motivated by the facts that $(n, 0)$-CSSPPs are in simple bijection with n-CSPPs, and that $(n-1, 2)$-CSSPPs are in simple bijection with n-DPPs. Accordingly, the $q = 1$ case of (1) is the $k = 0$ case of (3), and the $q = 1$ case of (2) is the $k = 2$ case, with n replaced by $n - 1$, of (3). It was subsequently shown by Ciucu and Krattenthaler [5] that (n, k)-CSSPPs are in simple bijection with cyclically symmetric rhombus tilings of a hexagon with alternating sides of lengths n and $n + k$, and a central equilateral triangular hole of side length k.

The general cases (i.e., with q arbitrary) of (1) and (2) were proved in [8] by Mills, Robbins and Rumsey, with the proofs being discussed by Robbins in his first May 1982 talk. ASMs were introduced, and various conjectures for their enumeration were made, by Mills, Robbins and Rumsey in [8, 9], with these conjectures being discussed by Robbins in his second May 1982 talk. In particular, it was conjectured that

$$(4) \qquad\qquad (\#\ \text{of}\ n\text{-ASMs}) = (\#\ \text{of}\ n\text{-DPPs}),$$

and that a refinement of this equality, involving three ASM statistics and three DPP statistics, also holds.

In my talk, I outlined the matters above, and provided some updates (including a description of current work in progress), as follows. The conjecture (4) was first proved in 1996 by Zeilberger [13]. Shortly thereafter, a different proof was obtained by Kuperberg [7]. The three-statistic refinement of (4) was proved in 2012 by myself, Di Francesco and Zinn-Justin [3]. All of these proofs are nonbijective.

Despite the confirmation of the validity of (4), several difficult problems remain unresolved. The most important of these is probably that of finding a bijective proof of (4). Stanley [10, Prob. 226] described this (together with associated problems) as *"one of the most intriguing open problems in the area of bijective proofs"*,

while Krattenthaler [6] (in a Festschrift for Stanley) wrote that *"the greatest, still unsolved, mystery concerns the question of what plane partitions have to do with ASMs"*.

Another open problem is that of finding a natural definition of (n, k)-ASMs, such that $(n-1, 2)$-ASMs are the same as, or in simple bijection with, n-ASMs, and there is equality between the numbers of (n, k)-ASMs and (n, k)-CSSPPs. Hence, the case $k = 2$ would be the equality (4). Some work related to this problem is currently being done by myself and Ilse Fischer. In particular, we have defined an (n, k)-*alternating sign trapezoid* (AST) to be an array of $n(n + k)$ entries arranged in n vertically-centred rows of lengths $2n + k - 1$, $2n + k - 3$, ..., $k + 3$, $k + 1$ (from top to bottom) such that, for $k \geq 1$: (i) Each entry is 0, 1 or -1; (ii) Along each row, the nonzero entries alternate in sign; (iii) The sum of entries in each row is 1; (iv) Moving down each column, the nonzero entries (if there are any) alternate in sign, starting with a 1; and (v) The sum of entries in columns $n + 1$, ..., $n + k - 1$ is 0. For $k = 0$, (i), (ii) & (iv) should be satisfied, together with: (iii$_0$) The sum of entries in each row except row n is 1, while row n consists of 0 or 1. Our main results regarding ASTs, which are proved nonbijectively, are that

(5) $$(\# \text{ of } (n, k)\text{-ASTs}) \ = \ (\# \text{ of } (n, k)\text{-CSSPPs}),$$

and that a refinement of this equality, involving certain AST and CSSPP statistics, also holds. However, the problem stated above has not been completely solved, since we do not currently have a bijection between $(n-1, 2)$-ASTs and n-ASMs.

References

[1] G. Andrews, *Plane partitions. III. The weak Macdonald conjecture*, Invent. Math. **53** (1979), 193–225.

[2] G. Andrews, *Macdonald's conjecture and descending plane partitions*, Lecture Notes in Pure and Appl. Math., vol. 57, Dekker, New York (1980), 91–106.

[3] R. Behrend, P. Di Francesco and P. Zinn-Justin, *On the weighted enumeration of alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **119** (2012), 331–363.

[4] D. Bressoud, *Editor's preface*, Adv. in Appl. Math. **34** (2005), 645–646.

[5] M. Ciucu and C. Krattenthaler, *Plane partitions. II. 5 1/2 symmetry classes*, Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo (2000), 81–101.

[6] C. Krattenthaler, *Plane partitions in the work of Richard Stanley and his school*, The mathematical legacy of Richard P. Stanley, Amer. Math. Soc. (2016), 231–261.

[7] G. Kuperberg, *Another proof of the alternating-sign matrix conjecture*, Internat. Math. Res. Notices (1996), no. 3, 139–150.

[8] W. Mills, D. Robbins and H. Rumsey, *Proof of the Macdonald conjecture*, Invent. Math. **66** (1982), 73–87.

[9] W. Mills, D. Robbins and H. Rumsey, *Alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **34** (1983), 340–359.

[10] R. Stanley, *Bijective proof problems* (2009), http://www-math.mit.edu/~rstan/bij.pdf

[11] Oberwolfach Workshop 7719, https://www.mfo.de/occasion/7719/www_view

[12] Oberwolfach Workshop 8219, https://www.mfo.de/occasion/8219/www_view

[13] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. **3** (1996), no. 2, Research Paper 13, 84 pp.

[14] D. Zeilberger, *Dave Robbins' art of guessing*, Adv. in Appl. Math. **34** (2005), 939–954.